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CHARACTERIZATION OF GCR-LIGHTLIKE WARPED PRODUCT OF INDEFINITE COSYMPLECTIC MANIFOLDS

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Abstract. In this paper we prove that there do not exist warped product GCR-lightlike submanifolds in the form $M = N_\perp \times_\lambda N_T$ such that $N_\perp$ is an anti-invariant submanifold tangent to V and N_T an invariant submanifold of $\bar{M}$, other than GCR-lightlike product in an indefinite cosymplectic manifold. We also obtain some characterizations for a GCR-lightlike submanifold to be locally a GCR-lightlike warped product.

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1. Introduction

Cauchy-Riemann (CR)-submanifolds of Kaehler manifolds were introduced by Bejancu [2] as a generalization of holomorphic and totally real submanifolds of Kaehler manifolds and further investigated [3], [4], [5], [8], [9] etc. Contact CR-submanifolds of Sasakian manifolds were introduced by Yano and Kon [24]. They studied the geometry of CR-submanifolds with positive definite metric. Therefore, this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. Thus the geometry of CR-submanifolds with indefinite metric has become a topic of intensive discussion. Duggal and Bejancu [12] played a crucial role in this study by introducing the notion of CR-lightlike submanifolds of indefinite Kaehler manifolds. Since there is a significant use of the contact geometry in differential equations, optics, and phase spaces of a dynamical system (see Arnold [1], Maclane [19], Nazaikinskii et al. [20]), therefore contact geometry with definite and indefinite metric becomes the topic of main discussion. Thus, Duggal and Sahin [14] introduced contact CR-lightlike submanifolds and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds. But there do not exist any inclusion relations between invariant and screen real submanifolds. A new class of submanifolds, called Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Sasakian manifolds (which is an

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umbrella of invariant, screen real, contact CR-lightlike submanifolds) were derived by Duggal and Sahin [15]. In [7], the notion of warped product manifolds was introduced by Bishop and O' Neill in 1969, and it was further studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds. This generalized product metric appears in differential geometric studies in a natural way. For instance, a surface of revolution is a warped product manifold. Moreover, many important submanifolds in real and complex space forms are expressed as warped product submanifolds. In view of its physical applications, many research articles have recently appeared exploring existence (or non-existence) of warped product submanifolds in known spaces ([21]). Chen [10] introduced warped product CR-submanifolds and showed that there does not exist a warped product CR-submanifold in the form $M = N_{\perp} \times_{\lambda} N_T$ in a Kaehler manifold, where $N_{\perp}$ is a totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$. He proved that if $M = N_{\perp} \times_{\lambda} N_T$ is a warped product CR-submanifold of a Kaehler manifold $\bar{M}$, then M is a CR-product, that is, there do not exist warped product CR-submanifolds of the form $M = N_{\perp} \times_{\lambda} N_T$ other than CR-product. Therefore, he called a warped product CR-submanifold in the form $M = N_T \times_{\lambda} N_{\perp}$ a CR-warped product. Chen also obtained a characterization for CR-submanifold of a Kaehler manifold to be locally a warped product submanifold. He showed that a CR-submanifold M of a Kaehler manifold $\bar{M}$ is a CR-warped product if and only if $A_{JZ}X = JX(\mu)Z$ for each $X \in \Gamma(D), Z \in \Gamma(D')$, μ a C^{∞}-function on M such that $Z\mu = 0$ for all $Z \in \Gamma(D')$.

In this paper we prove that there do not exist warped product GCR-lightlike submanifolds in the form $M = N_{\perp} \times_{\lambda} N_T$ such that $N_{\perp}$ is an anti-invariant submanifold tangent to V and N_T an invariant submanifold of $\bar{M}$, other than GCR-lightlike product in an indefinite cosymplectic manifold. We also obtain some characterizations for a GCR-lightlike submanifold to be locally a GCR-lightlike warped product.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to Duggal and Bejancu [12] .

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}, \tag{1}$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM, \tag{2}$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = RadTM \perp S(TM), \tag{3}$$

$S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{4}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{5}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as

$$\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\},$$

where $\{\xi_1, ..., \xi_r\}$, $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(RadTM|_u)$, $\Gamma(ltr(TM)|_u)$ and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1. *[12] Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{6}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to the decomposition (5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. Here ∇ is a torsion-free linear connection on M; h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form; A_U is a linear operator on M and known as shape operator.

According to (4), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (7) and (8) give

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{10}$$

where we put $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l_X U = L(\nabla^\perp_X U)$, $D^s_X U = S(\nabla^\perp_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{12}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Using (4)-(5) and (9)-(12), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{13}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{14}$$

for any $\xi \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^\perp))$.

Next, we recall some basic definitions and results of indefinite cosymplectic manifolds (see [6]). An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an ϵ-contact metric manifold, if there is a $(1,1)$ tensor field ϕ, a vector field V, called characteristic vector field and a 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \epsilon\eta(X)\eta(Y), \quad \bar{g}(V,V) = \epsilon, \tag{15}$$

$$\phi^2(X) = -X + \eta(X)V, \quad \bar{g}(X,V) = \epsilon\eta(X), \tag{16}$$

$$d\eta(X,Y) = \bar{g}(X, \phi Y), \tag{17}$$

for any $X, Y \in \Gamma(TM)$, where $\epsilon = \pm 1$ then it follows that

$$\phi V = 0, \tag{18}$$

$$\eta o \phi = 0, \quad \eta(V) = \epsilon. \tag{19}$$

Then $(\phi, V, \eta, \bar{g})$ is called an ϵ-contact metric structure of $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is Nijenhuis tensor field of ϕ. A normal ϵ -contact metric manifold is called indefinite-cosymplectic manifold and for this we have

$$\bar{\nabla}_X V = 0. \tag{20}$$

$$(\bar{\nabla}_X \phi)Y = 0. \tag{21}$$

3. Genralized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite cosymplectic manifolds

Calin [11], proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$ then it belongs to $S(TM)$. Throughout this paper we assume the characteristic vector V is tangent to M.

Definition 3.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM). \tag{22}$$

(B) There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S. \tag{23}$$

where D_0 is invariant non-degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V and L, S are vector subbundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then, the tangent bundle TM of M is decomposed as

$$TM = D \oplus \bar{D} \oplus \{V\}, \quad \text{where} \quad D = Rad(TM) \oplus D_0 \oplus \phi(D_2). \tag{24}$$

Let Q, P_1 and P_2 be the projection morphisms on D, $\phi S = M_2$ and $\phi L = M_1$ respectively, therefore any $X \in \Gamma(TM)$ can be written as

$$X = QX + V + P_1X + P_2X, \tag{25}$$

or

$$X = QX + V + PX, \tag{26}$$

where P is a projection morphism on $\bar{D}$. Applying ϕ to (26), we obtain

$$\phi X = fX + \omega P_1X + \omega P_2X, \tag{27}$$

where $fX \in \Gamma(D)$, $\omega P_1X \in \Gamma(S)$ and $\omega P_2X \in \Gamma(L)$, or, we can write (27), as

$$\phi X = fX + \omega X, \tag{28}$$

where fX and ωX are the tangential and transversal components of ϕX, respectively.

Similarly, for any $U \in \Gamma(tr(TM))$, we have

$$\phi U = BU + CU, \tag{29}$$

where BU and CU are the sections of TM and $tr(TM)$, respectively.

Differentiating (27) and using (9)-(12) and (29), we have

$$D^l(X, \omega P_1 Y) = -\nabla^l_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y), \tag{30}$$

$$D^s(X, \omega P_2 Y) = -\nabla^s_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y), \tag{31}$$

for all $X, Y \in \Gamma(TM)$. By using cosymplectic property of $\bar{\nabla}$ with (7) and (8), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$, then we have*

$$(\nabla_X f)Y = A_{\omega Y} X + Bh(X, Y), \tag{32}$$

and

$$(\nabla^t_X \omega)Y = Ch(X, Y) - h(X, fY), \tag{33}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \tag{34}$$

$$(\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y. \tag{35}$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$, then we have*

$$(\nabla_X B)U = A_{CU} X - fA_U X, \tag{36}$$

and

$$(\nabla^t_X C)U = -\omega A_U X - h(X, BU), \tag{37}$$

where $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \tag{38}$$

$$(\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U. \tag{39}$$

Theorem 3.4. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$, then*

(A) The distribution $D \oplus \{V\}$ is integrable, if and only if,

$$h(X, \phi Y) = h(Y, \phi X), \quad \forall \quad X, Y \in \Gamma(D \oplus \{V\}). \tag{40}$$

(B) The distribution $\bar{D}$ is integrable, if and only if,

$$A_{\phi Z} U = A_{\phi U} Z, \quad \forall \quad Z, U \in \Gamma(\bar{D}). \tag{41}$$

Proof: Using (30) and (31), we have

$$wP\nabla_X Y = h(X, fY) - Ch(X, Y),$$

for any $X, Y \in \Gamma(D \oplus \{V\})$. Hence $wP[X, Y] = h(X, fY) - h(Y, fX)$, which proves (A). Next, using (32) and (34), we have

$$f\nabla_Z U = -A_{wU} Z - Bh(Z, U),$$

for any $Z, U \in \Gamma(\bar{D})$. Then we obtain $f[Z, U] = A_{wZ} U - A_{wU} Z$, which completes the proof.

Theorem 3.5. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. Then the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $Bh(X, \phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

Proof. From the definition of GCR-lightlike submanifolds of an indefinite cosymplectic manifold, it is clear that $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $g(\nabla_X Y, \phi\xi) = g(\nabla_X Y, \phi W) = 0$, for any $X, Y \in \Gamma(D \oplus \{V\})$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$. Using (21) and (9), we have

$$g(\nabla_X Y, \phi\xi) = -g(\phi\bar{\nabla}_X Y, \xi) = -g(h^l(X, \phi Y), \xi),$$

similarly

$$g(\nabla_X Y, \phi W) = -g(\phi\bar{\nabla}_X Y, W) = -g(h^s(X, \phi Y), W).$$

Therefore, it is clear from above equations that the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $h^l(X, \phi Y)$ and $h^s(X, \phi Y)$ have no components in L and S, respectively, that is, if and only if, $Bh^l(X, \phi Y) = 0$ and $Bh^s(X, \phi Y) = 0$. Hence the assertion follows. □

4. GCR-Lightlike Warped Product

Warped Product: Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and $\lambda > 0$ a differentiable function on B. Assume the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_\lambda F$ is the manifold $B \times F$ equipped with the Riemannian metric g, where

$$g = g_B + \lambda^2 g_F. \tag{42}$$

If X is tangent to $M = B \times_\lambda F$ at (p, q) then using (42), we have

$$\|X\|^2 = \|\pi_* X\|^2 + \lambda^2(\pi(X))\|\eta_* X\|^2. \tag{43}$$

The function λ is called the warping function of the warped product. For a differentiable function λ on M, the gradient $\nabla\lambda$ is defined by $g(\nabla\lambda, X) = X\lambda$, for all $X \in T(M)$.

Lemma 4.1. *([7]) Let $M = B \times_\lambda F$ be a warped product manifold. If $X, Y \in T(B)$ and $U, Z \in T(F)$ then*

$$\nabla_X Y \in T(B). \tag{44}$$

$$\nabla_X U = \nabla_U X = \frac{X\lambda}{\lambda} U. \tag{45}$$

$$\nabla_U Z = -\frac{g(U,Z)}{\lambda} \nabla\lambda. \tag{46}$$

Corollary 4.2. *On a warped product manifold $M = B \times_\lambda F$ we have*

(i) B is totally geodesic in M.

(ii) F is totally umbilical in M.

Definition 4.3. ([13]) A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h(X, Y) = Hg(X, Y), \tag{47}$$

for all $X, Y \in \Gamma(TM)$, it is easy to see that M is a totally umbilical if and only if on each coordinate neighborhood u, there exists smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$, such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y) \quad D^l(X, W) = 0, \tag{48}$$

for any $W \in \Gamma(S(TM^\perp))$.

Lemma 4.4. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$, then the distribution $\bar{D}$ defines a totally geodesic foliation in M.*

Proof. Let $X, Y \in \Gamma(\bar{D})$ then using (32) and (34) we have $f\nabla_X Y = -A_{wY}X - Bh(X, Y)$. Let $Z \in \Gamma(D_0)$ then using (21) we obtain

$$\begin{aligned} g(f\nabla_X Y, Z) &= -g(A_{wY}X, Z) = \bar{g}(\bar{\nabla}_X \phi Y, Z) \\ &= -\bar{g}(\bar{\nabla}_X Y, \phi Z) = -\bar{g}(\bar{\nabla}_X Y, Z') = g(Y, \nabla_X Z'), \end{aligned} \tag{49}$$

where $Z' = \phi Z \in \Gamma(D_0)$. Since $X \in \Gamma(\bar{D})$ and $Z \in \Gamma(D_0)$ then using (33), (35) and the hypothesis that M is a totally umbilical GCR-lightlike submanifold, we get $w\nabla_X Z = h(X, fZ) - Ch(X, Z) = Hg(X, fZ) - CHg(X, Z) = 0$, this implies that $\nabla_X Z \in \Gamma(D)$, then (49) implies that $g(f\nabla_X Y, Z) = 0$ then the non-degeneracy of the distribution D_0 implies that $f\nabla_X Y = 0$, this gives $\nabla_X Y \in \Gamma(\bar{D})$ for any $X, Y \in \Gamma(\bar{D})$. Hence the proof is complete. □

Theorem 4.5. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite cosymplectic manifold, then the totally real distribution $\bar{D}$ is integrable.*

Proof. Using (32) and (34) with above lemma, we get

$$A_{wY}X = -Bh(X,Y), \tag{50}$$

for any $X, Y \in \Gamma(\bar{D})$. Then using the symmetric property of h, we get $A_{wY}X = A_{wX}Y$, for any $X, Y \in \Gamma(\bar{D})$. This implies that the distribution $\bar{D}$ is integrable. □

Definition 4.6. A GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\bar{M}$ is called a GCR-lightlike product if both the distribution $D \oplus \{V\}$ and $\bar{D}$ define totally geodesic foliations in M.

Let $M = N_\perp \times_\lambda N_T$ be a warped product GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. Such submanifolds are always tangent to the structure vector field V. We distinguish two cases

(i) V is tangent to N_T.

(ii) V is tangent to $N_\perp$.

In this paper we consider the case when V is tangent to N_T.

Theorem 4.7. *Let M be a totally umbilical GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\bar{M}$. If $M = N_\perp \times_\lambda N_T$ be a warped product GCR-lightlike submanifold such that $N_\perp$ is an anti-invariant submanifold and N_T is an invariant submanifold of $\bar{M}$ tangent to V, then it is a GCR-lightlike product.*

Proof. Since M is a totally umbilical GCR-lightlike submanifold of an indefinite Cosymplectic manifold then using Lemma 4.4, the distribution $\bar{D}$ defines a totally geodesic foliation in M.

Let h^T and A^T be the second fundamental form and the shape operator of N_T in M then for $X, Y \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(\phi S) \subset \Gamma(\bar{D})$, we have $g(h^T(X,Y), Z) = g(\nabla_X Y, Z) = -\bar{g}(Y, \bar{\nabla}_X Z) = -g(Y, \nabla_X Z) - g(Y, h^l(X,Z)) = -g(Y, \nabla_X Z) - g(Y, H^l)g(X,Z) = -g(Y, \nabla_X Z)$. Using (45) for $M = N_\perp \times_\lambda N_T$, we get

$$g(h^T(X,Y), Z) = -(Zln\lambda)g(X,Y). \tag{51}$$

Now, let $\hat{h}$ be the second fundamental form of N_T in $\bar{M}$ then

$$\hat{h}(X,Y) = h^T(X,Y) + h^s(X,Y) + h^l(X,Y), \tag{52}$$

for any X, Y tangent to N_T then using (51), we get

$$g(\hat{h}(X,Y), Z) = g(h^T(X,Y), Z) = -(Zln\lambda)g(X,Y). \tag{53}$$

Since N_T is a holomorphic submanifold of $\bar{M}$ then we have

$$\hat{h}(X, \phi Y) = \hat{h}(\phi X, Y) = \phi\hat{h}(X,Y)$$

and therefore

$$g(\hat{h}(X,Y),Z) = -g(\hat{h}(\phi X,\phi Y),Z) = (Zln\lambda)g(X,Y). \tag{54}$$

Adding (53) and (54) we get

$$g(\hat{h}(X,Y),Z) = 0. \tag{55}$$

Using (52) and (55), we have

$$\begin{aligned} g(h(X,Y),\phi Z) &= g(\hat{h}(X,Y),\phi Z) - g(h^T(X,Y),\phi Z) \\ &= g(\hat{h}(X,Y),\phi Z) \\ &= -g(\phi\hat{h}(X,Y),Z) \\ &= -g(\hat{h}(X,\phi Y),Z) \\ &= 0. \end{aligned}$$

Thus $g(h(X,Y),\phi Z) = 0$ implies that $h(X,Y)$ has no components in $L_1 \perp L_2$ for any $X,Y \in \Gamma(D \oplus \{V\})$. In other words, we can say that $Bh(X,Y) = 0$, for any $X,Y \in \Gamma(D \oplus \{V\})$. Therefore using Theorem 3.5, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M. Hence M is a GCR-lightlike product. Thus the proof is complete. □

After proving Theorem 4.7, it is important to mention the theorems by Hasegawa and Mihai [16], Khan et. al. [18] and Siraj Uddin and Khan [23] respectively.

Theorem 4.8. *Let $\bar{M}$ be a $(2m+1)$-dimensional Sasakian manifold. Then there do not exist warped product submanifolds $M = M_1 \times_\lambda M_2$ such that M_1 is an anti-invariant submanifold tangent to V and M_2 an invariant submanifold of $\bar{M}$.*

Theorem 4.9. *Let $\bar{M}$ be a $(2m+1)$-dimensional Kenmotsu manifold. Then there do not exist warped product submanifolds $M = N_\perp \times_\lambda N_T$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ is anti-invariant submanifold of $\bar{M}$.*

Theorem 4.10. *There does not exist a proper warped product CR-submanifold $N_\perp \times_\lambda N_T$ of a cosymplectic manifold $\bar{M}$ such that V is tangent to $N_\perp$, where $N_\perp$ is an anti-invariant submanifold and N_T is an invariant submanifold of $\bar{M}$.*

In this paper, Theorem 4.7 also shows that there do not exist warped product GCR-lightlike submanifolds of the form $M = N_\perp \times_\lambda N_T$ such that $N_\perp$ is an anti-invariant submanifold and N_T an invariant submanifold tangent to V of $\bar{M}$, other than GCR-lightlike product. Thus for simplicity we call a warped product GCR-lightlike submanifold in the form $M = N_T \times_\lambda N_\perp$ such that $N_\perp$ is an anti-invariant submanifold and N_T is an invariant submanifold of $\bar{M}$ tangent to V, a GCR-lightlike warped product.

Lemma 4.11. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. Let $M = N_T \times_\lambda N_\perp$ be a proper GCR-lightlike warped product of an indefinite cosymplectic manifold $\bar{M}$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\bar{M}$, then N_T is totally geodesic in M.*

Proof. Let $X, Y \in N_T$ and $Z \in N_\perp$ then we have $g(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z) = -g(Y, \nabla_X Z) - g(Y, h^l(X, Z))$, using (45) we get $g(\nabla_X Y, Z) = -g(Y, h^l(X, Z))$. Since M is a totally umbilical GCR-lightlike submanifold therefore $h^l(X, Z) = h^s(X, Z) = 0$. Hence $g(\nabla_X Y, Z) = 0$ implies that N_T is totally geodesic in M. □

Theorem 4.12. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. If the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M then it is integrable.*

Proof. Let $X, Y \in \Gamma(D \oplus \{V\})$ then using (33) and (35), we have $h(X, fY) = Ch(X, Y) + \omega\nabla_X Y$. Since the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M therefore $\omega\nabla_X Y = 0$ and we get $h(X, fY) = Ch(X, Y)$, then taking into account that h is symmetric we obtain $h(X, fY) = h(fX, Y)$, for all $X, Y \in \Gamma(D \oplus \{V\})$. This proves the assertion. □

Theorem 4.13. *Let M be a totally umbilical proper GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$ then $H^l = 0$.*

Proof. Let M be a totally umbilical proper GCR-lightlike submanifold then using (32) and (34), we have $A_{wZ}Z = -f\nabla_Z Z - Bh^l(Z, Z) - Bh^s(Z, Z)$, for $Z \in \Gamma(\phi S)$. Taking inner product with $\phi\xi$, for any $\xi \in \Gamma(D_2)$ we obtain $g(A_{wZ}Z, \phi\xi) = -g(Bh^l(Z, Z), \phi\xi)$. Using (13) and the hypothesis we obtain $g(Z, Z)g(H^l, \xi) = 0$, then using the non-degeneracy of M_2, the result follows. □

5. Characterization of GCR-Lightlike Warped Products

For a GCR-lightlike warped product in indefinite cosymplectic manifolds, we have

Lemma 5.1. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$ then for a GCR-lightlike warped product $M = N_T \times_\lambda N_\perp$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\bar{M}$, we have*

$$\bar{g}(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0. \tag{56}$$

Proof. Since $\bar{M}$ is a cosymplectic manifold therefore for $X \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(M_2)$ using (21), we have $\phi\bar{\nabla}_X Z = \bar{\nabla}_X \phi Z$. Since M is a totally umbilical we have $\phi(\nabla_X Z) = -A_{wZ}X + \nabla^s_X wZ$, then taking inner product with ϕY where $Y \in \Gamma(D \oplus \{V\})$, we get $g(\phi\nabla_X Z, \phi Y) = -g(A_{wZ}X, \phi Y)$. Using (15) and (45), we obtain $g(A_{wZ}X, \phi Y) = 0$ then using (13) we get $\bar{g}(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0$. Hence the proof is complete. □

Corollary 5.2. *Let $Z \in \Gamma(M_1) \subset \Gamma(\bar{D})$ then clearly $g(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi Z) = 0$ and also $g(h^l(D \oplus \{V\}, D \oplus \{V\}), \phi Z) = 0$ for any $Z \in \Gamma(\bar{D})$. Thus $g(h(D \oplus \{V\}, D \oplus \{V\}), \phi \bar{D}) = 0$, this implies that $h(D \oplus \{V\}, D \oplus \{V\})$ has no component in $L_1 \perp L_2$, that is, $Bh(D \oplus \{V\}, D \oplus \{V\}) = 0$ therefore using Theorem 3.5 the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M.*

Next, we have the following characterizations of GCR-lightlike warped products.

Theorem 5.3. *A proper totally umbilical GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\bar{M}$ is locally a GCR-lightlike warped product $M = N_T \times_\lambda N_\perp$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\bar{M}$ if and only if*

$$A_{\phi Z} X = ((\phi X)\mu) Z, \tag{57}$$

for $X \in \Gamma(D \oplus \{V\})$, $Z \in \Gamma(\bar{D})$ and for some function μ on M satisfying $U\mu = 0, U \in \Gamma(\bar{D})$.

Proof. Assume that M be a proper GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$ satisfying (57). Let $X, Y \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(M_2) \subset \Gamma(\bar{D})$, we have

$$g(A_{\phi Z} X, \phi Y) = g(((\phi X)\mu) Z, \phi Y) = ((\phi X)\mu) g(Z, \phi Y) = 0,$$

then using (13) we get $g(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0$. Then, as done in above corollary, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M and consequently it is totally geodesic in M and using Theorem 4.12 the distribution $D \oplus \{V\}$ is integrable.

Now, taking inner product of (56) with $U \in \Gamma(M_2) \subset \bar{D}$ and using (15), (21), (45) and that M is a totally umbilical submanifold, we get

$$\begin{aligned} g(((\phi X)\mu) Z, U) = g(A_{\phi Z} X, U) = g(\phi Z, \nabla_X U) = g(\phi Z, \nabla_U X) = \\ - \bar{g}(\bar{\nabla}_U \phi Z, X) = g(\nabla_U Z, \phi X) + \bar{g}(h^l(U, Z), \phi X), \end{aligned}$$

then using the definition of gradient $g(\nabla \mu, X) = X\mu$ we get

$$g(\nabla_U Z, \phi X) = g(\nabla \mu, \phi X) g(Z, U) - \bar{g}(h^l(U, Z), \phi X). \tag{58}$$

Let h' and ∇' be the second fundamental form and the metric connection of $\bar{D}$, respectively in M, then we have

$$g(h'(U, Z), \phi X) = g(\nabla_U Z - \nabla'_U Z, \phi X). \tag{59}$$

Therefore, from (58) and (59), particularly for $X \in \Gamma(D_0)$, we get

$$g(h'(U, Z), \phi X) = g(\nabla_U Z, \phi X) = g(\nabla \mu, \phi X) g(Z, U)$$

this further implies that

$$h'(U, Z) = \nabla \mu g(Z, U), \tag{60}$$

this implies that the distribution $\bar{D}$ is totally umbilical in M. Using Theorem 4.5, the totally real distribution $\bar{D}$ is also integrable. Hence, using (60) and the condition $U\mu = 0$ for $U \in \bar{D}$ we obtain that each leaf of $\bar{D}$ is an extrinsic sphere in M. Hence, by a result of ([17]) which sais that "If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non-trivial vector subbundles such that E_1 is spherical and its orthogonal complement E_0 is autoparallel, then the manifold M is locally isometric to a warped product $M_0 \times_\lambda M_1$", therefore we can conclude that M is locally a GCR-lightlike warped product $N_T \times_\lambda N_\perp$ of $\bar{M}$ where $\lambda = e^\mu$.

Conversely, let $X \in \Gamma(N_T)$ and $Z \in \Gamma(N_\perp)$, since $\bar{M}$ is a cosymplectic manifold so we have $\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z$, which further becomes $-A_{\phi Z} X + \nabla^t_X \phi Z = ((\phi X) ln\lambda) Z$, comparing the tangential components, with $A_{\phi Z} X = -((\phi X) ln\lambda) Z$ for each $X \in \Gamma(D \oplus \{V\})$ and $Z \in (\bar{D})$. Since λ is a function on N_T so we also have $U(ln\lambda) = 0$ for all $U \in \Gamma(\bar{D})$. Hence the proof is complete. □

Lemma 5.4. *Let $M = N_T \times_\lambda N_\perp$ be a GCR-lightlike warped product of an indefinite cosymplectic manifold such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\bar{M}$ then*

$$(\nabla_Z f) X = f X (ln\lambda) Z. \tag{61}$$

$$(\nabla_U f) Z = g(Z, U) f(\nabla ln\lambda), \tag{62}$$

for any $U \in \Gamma(TM), X \in \Gamma(N_T)$ and $Z \in \Gamma(N_\perp)$.

Proof. For any $X \in \Gamma(N_T)$ and $Z \in \Gamma(N_\perp)$, using (34) and (45), we have $(\nabla_Z f) X = \nabla_Z f X - f((\frac{Xf}{f}) Z) = \nabla_Z f X - \frac{Xf}{f} f Z = \nabla_Z f X = f X (ln\lambda) Z$. Next, again using (34) we get $(\nabla_U f) Z = -f \nabla_U Z$ this implies that $(\nabla_U f) Z \in \Gamma(N_T)$, therefore for any $X \in \Gamma(D_0)$ we have

$$\begin{aligned} g((\nabla_U f) Z, X) &= -g(f \nabla_U Z, X) = g(\nabla_U Z, f X) = \bar{g}(\bar{\nabla}_U Z, f X) \\ &= -g(Z, \nabla_U f X) = -f X (ln\lambda) g(Z, U). \end{aligned}$$

Hence, using the definition of the gradient of λ and the non-degeneracy of the distribution D_0, the result follows. □

Theorem 5.5. *A proper totally umbilical GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\bar{M}$ is locally a GCR-lightlike warped product $M = N_T \times_\lambda N_\perp$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\bar{M}$ if*

$$(\nabla_X f) Y = (f Y(\mu)) P X + g(PX, PY) \phi(\nabla \mu), \tag{63}$$

for any $X, Y \in \Gamma(TM)$ and for some function μ on M satisfying $Z\mu = 0, Z \in \Gamma(\bar{D})$.

Proof. Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$ satisfying (63). Let $X, Y \in \Gamma(D \oplus \{V\})$, then (63) implies that $(\nabla_X f)Y = 0$ then (32) gives $Bh(X, Y) = 0$. Thus $D \oplus \{V\}$ defines a totally geodesic foliation in M and consequently it is totally geodesic in M and integrable using Theorem 4.12.

Let $X, Y \in \Gamma(\bar{D})$, then (63) gives

$$(\nabla_X f)Y = g(PX, PY)\phi(\nabla\mu). \tag{64}$$

Let $U \in \Gamma(D_0)$, then (64) implies that

$$g((\nabla_X f)Y, U) = g(PX, PY)g(\phi(\nabla\mu), U). \tag{65}$$

Also, using (21) with (32), we have

$$g((\nabla_X f)Y, U) = g(A_{wY}X, U) = \bar{g}(\bar{\nabla}_X Y, \phi U) = g(\nabla_X Y, \phi U), \tag{66}$$

therefore, from (65) and (66) we get

$$g(\nabla_X Y, \phi U) = -g(\nabla\mu, \phi U)g(X, Y). \tag{67}$$

Let h' and ∇' be the second fundamental form and the metric connection of $\bar{D}$, respectively in M then

$$g(h'(X, Y), \phi U) = g(\nabla_X Y - \nabla'_X Y, \phi U) = g(\nabla_X Y, \phi U), \tag{68}$$

therefore, from (67) and (68) we get $g(h'(X, Y), \phi U) = -g(\nabla\mu, \phi U)g(X, Y)$ then the non-degeneracy of the distribution D_0 implies that

$$h'(X, Y) = -\nabla\mu g(X, Y), \tag{69}$$

this gives that the distribution $\bar{D}$ is totally umbilical in M, and using Theorem 4.5, the distribution $\bar{D}$ is integrable. Also, $Z\mu = 0$ for $Z \in \Gamma(\bar{D})$, hence as in Theorem 5.3 each leaf of $\bar{D}$ is an extrinsic sphere in M. Thus M is locally a GCR-lightlike warped product $N_T \times_\lambda N_\perp$ of $\bar{M}$, where $\lambda = e^\mu$. □

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