Novi Sad J. Math.
Vol. 42, No. 2, 2012, 103-115

A REMARK ON BAUM-KATZ TYPE THEOREM FOR ϕ-MIXING SEQUENCES OF RANDOM VARIABLES

Marcin Przystalski[1]

Abstract. We present a generalization of the Baum-Katz theorem for ϕ-mixing sequences of random variables with different distributions satisfying some cover condition.

AMS Mathematics Subject Classification (2010): 60F15

Key words and phrases: Complete convergence, ϕ-mixing sequence of random variables, regular cover

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Define

$$S_n = \sum_{k=1}^{n} X_k.$$

A sequence $\{X_n, n \geq 1\}$ is said to converge completely to a constant A if

$$\sum_{n=1}^{\infty} P\left(|X_n - A| > \epsilon\right) < \infty, \qquad \text{for all } \epsilon > 0.$$

This notion was first introduced and discussed by Hsu and Robbins in [3]. They proved that the sequence of arithmetic means of independent, identically distributed (i.i.d.) random variables converges completely to the expected value of summands, provided the variance is finite.

The result proved by Hsu and Robbins [3] was further generalized and extended by many authors (see e.g. [1, 2, 4]). Katz [4], Baum and Katz [1] formed the following generalization, with a normalization of Marcinkiewicz-Zygmund type (see [2]):

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Let $rp \geq 1$, $r > \frac{1}{2}$. The following statements are equivalent:*

(i) $E\,|X_1|^p < \infty$, *and, if* $p \geq 1$, $EX_1 = 0$,

(ii) $\sum_{n=1}^{\infty} n^{rp-2} P\left(|S_n| > n^r \epsilon\right) < \infty$ *for all* $\epsilon > 0$,

(iii) $\sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1 \leq k \leq n} |S_k| > n^r \epsilon\right) < \infty$ *for all* $\epsilon > 0$.

If $rp > 1$ and $r > \frac{1}{2}$ the above statements are also equivalent

[1]The Research Center for Cultivar Testing, 63 − 022 Słupia Wielka, Poland, e-mail: marprzyst@gmail.com

(iv) $\sum_{n=1}^{\infty} n^{rp-2} P\left(\sup_{k\geq n} k^{-r}\left|S_k\right| > \epsilon\right) < \infty$ *for all* $\epsilon > 0$.

Sometimes, in practical applications, it is difficult to verify the assumption that the samples are independent observations. For this reason, in the recent years the limit theorems for sequences of dependent random variables have been considered.

Peligrad and Gut [11] extended Theorem 1.1 to the case of a ρ^*-mixing sequence, i.e., the sequence of random variables $\{X_n, n \geq 1\}$ satisfying the condition

$$\rho^*(n) \to 0, \text{ as } n \to \infty,$$

where

$$\rho^*(n) = \sup_{S,T}\left\{\sup_{X\in L^2(\mathcal{F}_S), Y\in L^2(\mathcal{F}_T)} \frac{Cov\,(X,Y)}{\sqrt{VarXVarY}}\right\},$$

$S, T \subset \mathbb{N}$ such that $dist\,(S,T) \geq k$ and $\mathcal{F}_W$ is the σ-algebra generated by random variables X_i, $i \in W \subset \mathbb{N}$.

Peligrad [9, 10] and Kiesel [5, 6] extended Theorem 1.1 to the case of a ϕ-mixing sequence, i.e., the sequence of random variables $\{X_n, n \geq 1\}$ satisfying the condition

$$\phi(n) \to 0, \text{ as } n \to \infty,$$

where

$$\phi(n) = \sup\left\{\left|\frac{P(AB)}{P(A)} - P(B)\right| : k \geq 1, A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^{\infty}, P(A) \neq 0\right\},$$

and $\mathcal{F}_j^k$ is the σ-algebra generated by random variables X_l, $l = j, \ldots, k$.

The results in [5, 6, 9, 10, 11] were proved for the sequences of identically distributed (i.d.) random variables. In [12], Pruss introduced the notion of regular cover which allowed him to consider non-identically distributed sequences of random variables.

Definition 1. *Let* $X_1, X_2, \ldots, X_n$ *be random variables, and let* X *be a random variable possibly defined on a different probability space. Then,* $X_1, X_2, \ldots, X_n$ *are said to be a regular cover of* X*, provided we have*

$$E(G(X)) = \frac{1}{n}\sum_{k=1}^{n} E(G(X_k)), \tag{1}$$

for any measurable function G *for which both sides make sense.*

Using this concept, Pruss [12] obtained a generalization of the Hsu-Ribbins theorem for a sequence of non-identically distributed random variables. Recently, Kuczmaszewska [7], using a weaker cover condition

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_k| > x) = cP(|X| > x), \tag{2}$$

obtained a generalization of the Baum-Katz theorem for negatively associated random variables.

Definition 2. *A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated if for every pair of disjoint nonempty subset $A, B \in \{1, \dots, n\}$ and any real coordinatewise nondecreasing functions f and g*

$$Cov\left(f\left(X_i \in A\right), g\left(X_j, j \in B\right)\right) \le 0,$$

whenever f and g are such that the covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

The aim of this paper is to prove a generalization of the Baum-Katz theorem for ϕ-mixing sequences of random variables with different distributions satisfying condition (2). Using the inequalities proved by Nagaev [8], we improve the results obtained by Peligrad [9, 10] and Kiesel [5, 6].

2. Some technical lemmas

Let $\{X_n,\ n \ge 1\}$ be a sequence of ϕ-mixing random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let us define the partial sums $S_n = \sum_{k=1}^{n} X_k$, $M_n = \max_{1 \le k \le n} |S_k|$, and let $\mathcal{F}_j^k$ denote σ-algebra generated by random variables X_l, $l = j, \dots, k$. Define

$$\phi^{+}\left(m\right)=\sup\left\{\frac{P\left(AB\right)}{P\left(A\right)} - P\left(B\right): 1 \le k \le n-m, A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+m}^n, P\left(A\right) \neq 0\right\}$$

and

$$\phi^{-}\left(m\right)=sup\left\{P\left(B\right) - \frac{P\left(AB\right)}{P\left(A\right)}: 1 \le k \le n-m, A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+m}^n, P\left(A\right) \neq 0\right\}.$$

Let $\phi^{+}(1) < 1$ and let $\delta > 0$ satisfy the condition $\phi^{+}(1) + \delta < 1$. Define $\rho = \phi^{+}(1) + \delta$. Let α be a number such that the condition

$$P\left(2M_n > \alpha\right) < \delta$$

is satisfied. Define

$$Q\left(r\right) = \sum_{i=1}^{n} P\left(|X_i| > r\right).$$

Using the above notation, Nagaev [8] proved the following maximal inequalities.

Lemma 1. *For any $r > \alpha$ and $0 < \varepsilon < \frac{1}{6}$,*

$$P\left(M_n > r\right) < \frac{2}{\alpha\rho} \int_0^r Q\left(\frac{r\alpha\varepsilon^2}{2u}\right) \frac{du}{\left(1 + \varepsilon u/\alpha\right)^{s(\varepsilon)+1}} + \rho^{-1}\left(1 + \frac{\varepsilon r}{\alpha}\right)^{-s(\varepsilon)},$$

where $s(\varepsilon) = -\log\rho / \log(1+\varepsilon)$.

Lemma 2. *For any $p > 0$ and $0 < \varepsilon < \frac{1}{6}$ such that $s(\varepsilon) > p$,*

$$EM_n^p < c_1\left(p\right) \sum_{i=1}^{n} E\left|X_i\right|^p + c_2\left(p\right) \alpha^p,$$

where

$$c_1(p) < \tfrac{2^{p+1}}{\varepsilon^{3p+1}\rho} B(p+1, s(\varepsilon)-p+1), \qquad (3)$$
$$c_2(p) < \rho^{-1}\varepsilon^{-p} B(p+1, s(\varepsilon)-p)\, p+1,$$

and $B(\cdot,\cdot)$ is the Euler function.

Lemma 3. *Let the random variables X_j take real values and let $EX_j = 0$. Then for $p > 2$ and $0 < \varepsilon < \frac{1}{6}$, such that $s(\varepsilon) > p$,*

$$EM_n^p < c_1(p)\sum_{k=1}^{n} EX_k^p + c_2'(p)\left(\sum_{k=1}^{n} EX_k^2\right)^{p/2},$$

where $c_2'(p) = c_p(p)\left(32c(\phi)/\left((1-\phi^-(1))(1-\phi^+(1))\right)\right)^{p/2}$, $c(\phi) = \left(1 + 2\sum_{k=1}^{\infty}\phi^{1/2}(k)\right)$, $c_1(p)$ and $c_2(p)$ satisfying the conditions (3).

As it was pointed out by Nagaev [8], if $\sum_{k=1}^{\infty}\phi^{1/2}(k) < \infty$ and $EX_j = 0$ for $j = 1, \ldots, n$, then

$$E|S_n|^2 < c(\phi)\sum_{j=1}^{n} EX_j^2. \qquad (4)$$

In our considerations we will also need the following lemma.

Lemma 4. (Gut [2]) *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables satisfying a weak mean dominating condition with mean dominating random variable X, i.e. for some $c > 0$*

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_k| > x) \leq cP(|X| > x).$$

Let $r > 0$ and for some $A > 0$

$$X_k' = X_k I(|X_k| \leq A), \qquad X_k'' = X_k I(|X_k| > A),$$
$$X_k^* = X_k I(|X_k| \leq A) - AI(X_k < -A) + AI(X_k \leq A)$$

and

$$X' = XI(|X| \leq A), \qquad X'' = XI(|X| > A),$$
$$X^* = XI(|X| \leq A) - AI(X < -A) + AI(X \leq A).$$

Then

(*i*) *if $E|X|^r < \infty$, then $\frac{1}{n}\sum_{k=1}^{n} E|X_k|^r \leq CE|X|^r$,*

(*ii*) *$\frac{1}{n}\sum_{k=1}^{n} E|X_k'|^r \leq C\left(E|X'|^r + A^r P(|X| > A)\right)$ for any $A > 0$,*

(*iii*) *$\frac{1}{n}\sum_{k=1}^{n} E|X_k''|^r \leq CE|X''|^r$ for any $A > 0$,*

(*iv*) *$\frac{1}{n}\sum_{k=1}^{n} E|X_k^*|^r \leq CE|X^*|^r$ for any $A > 0$.*

Throughout this paper, C_1 and C_2 always stand for positive constants which differ from one place to another.

3. Main result

Theorem 3.1. *Let $rp > 1$ and $r > \frac{1}{2}$. Let $\{X_n,\ n \geq 1\}$ be a sequence of ϕ-mixing random variables with $\sum_{k=1}^{\infty} \phi^{1/2}(k) < \infty$ and let X be a random variable possibly defined on a different probability space satisfying the condition*

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_k| > x) = cP(|X| > x), \tag{5}$$

for all $n \geq 1$, all $x > 0$ and some $c > 0$. Additionally, assume that for all $p \geq 1$, $EX_n = 0$ for all $n \geq 1$. For $p > 0$ and any $0 < \varepsilon < \frac{1}{6}$ such that $s(\varepsilon) > p$, the following statements are equivalent:

(*i*) $E|X|^p < \infty$,

(*ii*) $\sum_{n=1}^{\infty} n^{rp-2} P(\max_{1\leq k\leq n} |S_k| > \beta n^r) < \infty, \qquad$ *for all* $\beta > 0$.

Corollary 1. *Let $rp > 1$ and $r > \frac{1}{2}$. Let $\{X_n,\ n \geq 1\}$ be a sequence of ϕ-mixing i.d. random variables with $\sum_{k=1}^{\infty} \phi^{1/2}(k) < \infty$. Moreover, assume that for all $p \geq 1$, $EX_1 = 0$. For any $p > 0$, $0 < \varepsilon < \frac{1}{6}$ such that $s(\varepsilon) > p$, the following statements are equivalent:*

(*i*) $E|X_1|^p < \infty$,

(*ii*) $\sum_{n=1}^{\infty} n^{rp-2} P(\max_{1\leq k\leq n} |S_k| > \beta n^r) < \infty, \qquad$ *for all* $\beta > 0$.

Proof of Theorem 3.1: First we prove that $(i) \Rightarrow (ii)$. For this purpose we distinguish two cases.

Case $0 < p < 1$. Note that

$$X_i = X_i I(|X_i| \leq n^r) + X_i I(|X_i| > n^r) = X_i' + X_i''$$

and

$$S_n = \sum_{i=1}^{n} X_i I(|X_i| \leq n^r) + \sum_{i=1}^{n} X_i I(|X_i| > n^r) = S_n' + S_n''.$$

By Lemma 1 and (5) we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\leq i\leq n} |S_n'| > \beta n^r\right) \\
&\leq C_1 \sum_{n=1}^{\infty} n^{rp-2} \int_0^{\beta n^r} \sum_{i=1}^{n} \frac{P\left(|X_i'| > \frac{\beta n^r \alpha \varepsilon^2}{2u}\right) du}{(1+\varepsilon u/\alpha)^{s(\varepsilon)+1}} \\
&+ C_2 \sum_{n=1}^{\infty} n^{rp-2} \left(1 + \frac{\varepsilon \beta n^r}{\alpha}\right)^{-s(\varepsilon)} \\
&\leq C_1 \sum_{n=1}^{\infty} n^{rp-2} \int_0^{\beta n^r} \sum_{i=1}^{n} \frac{P\left(|X_i'| > \frac{\beta n^r \alpha \varepsilon^2}{2u}\right) du}{(1+\varepsilon u/\alpha)^{s(\varepsilon)+1}} \\
&+ C_2 \sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)r} = I_1 + I_2.
\end{aligned}$$

Because for any $0<\varepsilon<\frac{1}{6}$, $s(\varepsilon)>p$, we have that $I_2<\infty$. Therefore, we have to prove that $I_1<\infty$. Indeed, let $x=\frac{\beta n^r\alpha\varepsilon^2}{2u}$, then

$$\begin{aligned}
I_1 &\leq C_1\sum_{n=1}^{\infty} n^{rp-2}\int_0^{\beta n^r}\sum_{i=1}^{n}\frac{P\left(|X_i'|>\frac{\beta n^r\alpha\varepsilon^2}{2u}\right)du}{(\varepsilon u/\alpha)^{s(\varepsilon)+1}}\\
&\leq C_1\sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)r}\sum_{i=1}^{n}\int_{2/\alpha\varepsilon^2}^{\infty} x^{s(\varepsilon)-1}P\left(|X_i'|>x\right)dx\\
&\leq C_1\sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)r}\sum_{i=1}^{n}\int_{0}^{\infty} x^{s(\varepsilon)-1}P\left(|X_i'|>x\right)dx.
\end{aligned}$$

By (5) we have

$$\begin{aligned}
I_1 &\leq C_1\sum_{n=1}^{\infty} n^{rp-1-s(\varepsilon)r}\int_0^{n^r} x^{s(\varepsilon)-1}P\left(|X|>x\right)dx\\
&\leq C_1\sum_{n=1}^{\infty} n^{rp-1-s(\varepsilon)r}\sum_{k=1}^{n}\int_{(k-1)^r}^{k^r} x^{s(\varepsilon)-1}P\left(|X|>x\right)dx\\
&= C_1\sum_{k=1}^{\infty}\int_{(k-1)^r}^{k^r} x^{s(\varepsilon)-1}P\left(|X|>x\right)dx\sum_{n=k}^{\infty} n^{rp-1-s(\varepsilon)r}\\
&\leq C_1\sum_{k=1}^{\infty} k^{rp-s(\varepsilon)r}\int_{(k-1)^r}^{k^r} x^{s(\varepsilon)-1}P\left(|X|>x\right)dx.
\end{aligned}$$

As $s(\varepsilon)>p$, we finally get

$$\begin{aligned}
I_1 &\leq C_1\sum_{k=1}^{\infty} k^{rp-s(\varepsilon)r}\cdot k^{s(\varepsilon)r-rp}\int_{(k-1)^r}^{k^r} x^{p-1}P\left(|X|>x\right)dx\\
&\leq C_1\sum_{k=1}^{\infty}\int_{(k-1)^r}^{k^r} x^{p-1}P\left(|X|>x\right)dx = E|X|^p<\infty.
\end{aligned}$$

Similarly, we show that

$$\sum_{n=1}^{\infty} n^{rp-2}P\left(\max_{1\leq i\leq n}|S_n''|>\beta n^r\right)<\infty.$$

Indeed,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\le i\le n} |S_n''| > \beta n^r\right) \\
&\le C_1 \sum_{n=1}^{\infty} n^{rp-2} \int_0^{\beta n^r} \sum_{i=1}^{n} \frac{P\left(|X_i''| > \frac{\beta n^r \alpha\varepsilon^2}{2u}\right) du}{(1+\varepsilon u/\alpha)^{s(\varepsilon)+1}} \\
&+ C_2 \sum_{n=1}^{\infty} n^{rp-2} \left(1+\frac{\varepsilon\beta n^r}{\alpha}\right)^{-s(\varepsilon)} \\
&\le C_1 \sum_{n=1}^{\infty} n^{rp-2} \int_0^{\beta n^r} \sum_{i=1}^{n} \frac{P\left(|X_i''| > \frac{\beta n^r \alpha\varepsilon^2}{2u}\right) du}{(1+\varepsilon u/\alpha)^{s(\varepsilon)+1}} \\
&+ C_2 \sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)r} = I_3 + I_4.
\end{aligned}
$$

Because $s(\varepsilon) > p$, we have that $I_4 < \infty$. It remains to prove that $I_3 < \infty$. Let $x = \frac{\beta n^r \alpha\varepsilon^2}{2u}$, then

$$
\begin{aligned}
I_3 &= C_1 \sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)r} \sum_{i=1}^{n} \int_{2/\alpha\varepsilon^2}^{\infty} x^{s(\varepsilon)-1} P(|X_i''| > x)\, dx \\
&\le C_1 \sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)r} \sum_{i=1}^{n} \int_{0}^{\infty} x^{s(\varepsilon)-1} P(|X_i''| > x)\, dx \\
&= C_1 \sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)r} \sum_{i=1}^{n} \int_{n^r}^{\infty} x^{s(\varepsilon)-1} P(|X_i| > x)\, dx.
\end{aligned}
$$

Because $s(\varepsilon) > p$, for any $0 < \varepsilon < \frac{1}{6}$, by (5) we have

$$
\begin{aligned}
I_3 &\le C_1 \sum_{n=1}^{\infty} n^{rp-1-s(\varepsilon)r} \int_{n^r}^{\infty} x^{s(\varepsilon)-1} P(|X| > x)\, dx \\
&= C_1 \sum_{n=1}^{\infty} n^{rp-2-s(\varepsilon)p} \sum_{k=n}^{\infty} \int_{k^r}^{(k+1)^r} x^{s(\varepsilon)-1} P(|X| > x)\, dx \\
&= C_1 \sum_{k=1}^{\infty} \int_{k^r}^{(k+1)^r} x^{s(\varepsilon)-1} P(|X| > x)\, dx \sum_{n=1}^{k} n^{rp-1-s(\varepsilon)p} \\
&\le C_1 \sum_{k=1}^{\infty} k^{rp-s(\varepsilon)p} \int_{k^r}^{(k+1)^r} x^{s(\varepsilon)-1} P(|X| > x)\, dx \\
&\le C_1 \sum_{k=1}^{\infty} k^{rp-s(\varepsilon)p} \cdot k^{s(\varepsilon)p-rp} \int_{k^r}^{(k+1)^r} x^{p-1} P(|X| > x)\, dx \\
&\le E\,|X|^p .
\end{aligned}
$$

By the condition (i),

$$I_3 < \infty,$$

which yields (ii) in the case of $0 < p < 1$.

Case $p \geq 1$.

Let us define $X'_{ni} = -n^r I(X_i < -n^r) + X_i I(|X_i| \leq n^r) + n^r I(X_i > n^r)$, for $1 \leq i \leq n$, $Y_{ni} = X'_{ni} - EX'_{ni}$ and $S'_{n,k} = \sum_{i=1}^{k} Y'_{ni}$, for $1 \leq k \leq n$.

Noting that $EXI(|X| \leq n^r) = -EXI(|X| > n^r)$, in view of the fact that $EX = 0$, by Lemma 4 (iii) we have

$$\begin{aligned}
\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} EX_i I(|X_i| \leq n^r) \right| &\leq \max_{1\leq k\leq n} \sum_{i=1}^{k} EX_i I(|X_i| > n^r) \\
&= \frac{1}{n^{r-1}} EXI(|X| > n^r) \\
&\leq C \frac{E|X|^p}{n^{rp-1}} \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

because $rp > 1$.

Additionally, by the Markov inequality and (5), for $1 \leq k \leq n$, we have

$$\begin{aligned}
\left| \frac{1}{n^r} \sum_{i=1}^{k} \left(n^r P(X_i > n^r) - n^r P(X_i < -n^r) \right) \right| &\leq \frac{1}{n^r} \sum_{i=1}^{k} n^r P(|X_i| > n^r) \\
&\leq CnP(|X| > n^r) \\
&\leq \frac{CE|X|^p}{n^{rp-1}} \to 0,
\end{aligned}$$

as $n \to \infty$ and $rp > 1$.

Hence, for a sufficiently large n we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\leq k\leq n} |S_k| > \beta n^r \right) &\leq \sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\leq i\leq n} \left|S'_{n,k}\right| > \beta n^r \right) \\
&\quad + \sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\leq i\leq n} |X_i| > n^r \right) \\
&\leq \sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\leq i\leq n} \left|S'_{n,k}\right| > \beta n^r \right) \\
&\quad + \sum_{n=1}^{\infty} n^{rp-2} \sum_{k=1}^{n} P(|X_k| > n^r).
\end{aligned}$$

Note that by (5) and (i), the second series on the right-hand side converges. Therefore, it remains to show that

$$\sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\leq i\leq n} \left|S'_{n,k}\right| > \beta n^r \right) < \infty. \tag{6}$$

By the Markov inequality and Lemma 2, for a sufficiently large $q > 2$ and any

$0 < \varepsilon < \frac{1}{6}$ such that $s(\varepsilon) > q$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\le i\le n}\left|S'_{n,k}\right| > \beta n^r\right) &\le \sum_{n=1}^{\infty} n^{rp-2-qr} E\left(\max_{1\le i\le n}\left|S'_{n,k}\right| > \beta n^r\right)^q \\ &\le C_1 \sum_{n=1}^{\infty} n^{rp-2-qr} \sum_{i=1}^{n} E\left|X'_{ni}\right|^q \\ &+ C_2 \sum_{n=1}^{\infty} n^{rp-2-qr} \\ &= I_5 + I_6.\end{aligned}$$

For $q > p$, we obtain that $I_6 < \infty$. By (5) we have that

$$\begin{aligned}I_5 &= C_1 \sum_{n=1}^{\infty} n^{rp-2-qr} \sum_{i=1}^{n} E\left|X'_{ni}\right|^q \\ &= C_1 \sum_{n=1}^{\infty} n^{rp-2-qr} \sum_{i=1}^{n} \int_0^{\infty} x^{q-1} P\left(\left|X'_{ni}\right| > x\right) dx \\ &\le C_1 \sum_{n=1}^{\infty} n^{rp-2-qr} \sum_{i=1}^{n} \int_0^{n^r} x^{q-1} P\left(\left|X_{ni}\right| > x\right) dx \\ &\le C_1 \sum_{n=1}^{\infty} n^{rp-1-qr} \int_0^{n^r} x^{q-1} P\left(\left|X\right| > x\right) dx.\end{aligned}$$

Hence, for $q > p$ we have

$$\begin{aligned}I_5 &\le C_1 \sum_{n=1}^{\infty} n^{rp-1-qr} \int_0^{n^r} x^{q-1} P\left(\left|X\right| > x\right) dx \\ &= C_1 \sum_{n=1}^{\infty} n^{rp-1-qr} \sum_{i=1}^{n} \int_{(i-1)^r}^{i^r} x^{q-1} P\left(\left|X\right| > x\right) dx \\ &= C_1 \sum_{i=1}^{\infty} \int_{(i-1)^r}^{i^r} x^{q-1} P\left(\left|X\right| > x\right) dx \sum_{n=i}^{\infty} n^{rp-1-qr} \\ &\le C_1 \sum_{i=1}^{\infty} \int_{(i-1)^r}^{i^r} x^{q-1} P\left(\left|X\right| > x\right) dx \sum_{n=i}^{\infty} n^{rp-1-qr} \\ &\le C_1 \sum_{i=1}^{\infty} i^{rp-qr} \int_{(i-1)^r}^{i^r} x^{q-1} P\left(\left|X\right| > x\right) dx \\ &\le C_1 \sum_{i=1}^{\infty} i^{rp-qr} \cdot i^{qr-rp} \int_{(i-1)^r}^{i^r} x^{p-1} P\left(\left|X\right| > x\right) dx \\ &= C_1 \sum_{i=1}^{\infty} \int_{(i-1)^r}^{i^r} x^{p-1} P\left(\left|X\right| > x\right) dx = E\left|X\right|^p.\end{aligned}$$

By (i) we obtain

$$I_5 < \infty,$$

which gives (ii) in the case of $p \geq 1$.

Now, we prove the converse. To prove that (ii) implies (i), it suffices to show that

$$\sum_{n=1}^{\infty} n^{rp-1} P\left(|X| > \beta n^r\right) < \infty.$$

From (ii), it follows that

$$\sum_{n=1}^{\infty} n^{rp-2} P\left(\max_{1\leq k\leq n} |X_k| > n^r\right) < \infty \tag{7}$$

and

$$P\left(\max_{1\leq k\leq n} |X_k| > \beta n^r\right) \to 0. \tag{8}$$

From the relation

$$\sum_{k=1}^{n} P\left(|X_k| > n^r, \max_{1\leq i\leq n} |X_i| \leq n^r\right) \leq P\left(\max_{1\leq i\leq n} |X_i| > n^r\right)$$

and (5), we obtain

$$\begin{aligned}
nP\left(|X| > n^r\right) &= \sum_{k=1}^{n} P\left(|X_k| > n^r\right) \\
&= \sum_{k=1}^{n} P\left(|X_k| > n^r, \max_{1\leq i\leq n} |X_i| > n^r\right) \\
&+ \sum_{k=1}^{n} P\left(|X_k| > n^r, \max_{1\leq i\leq n} |X_i| \leq n^r\right) \\
&\leq \sum_{k=1}^{n} P\left(|X_k| > n^r, \max_{1\leq i\leq n} |X_i| > n^r\right) \\
&+ P\left(\max_{1\leq i\leq n} |X_i| > n^r\right).
\end{aligned} \tag{9}$$

Let $J = \sum_{k=1}^{n} P\left(|X_k| > n^r, \max_{1\leq i\leq n} |X_i| > n^r\right)$. By centering we obtain that

$$\begin{aligned}
J &\leq E\left\{\sum_{k=1}^{n} \left[I\left(|X_k| > n^r\right) - P\left(|X_k| > n^r\right)\right] I\left(\max_{1\leq i\leq n} |X_i| > n^r\right)\right\} \\
&+ nP\left(\max_{1\leq i\leq n} |X_i| > n^r\right) P\left(|X| > n^r\right) = I_7 + I_8.
\end{aligned} \tag{10}$$

By the Cauchy-Schwarz inequality and (4), I_7 can be estimated as

$$\begin{aligned}
|I_7| &\leq \sqrt{E\left(\sum_{k=1}^{n}\left[I\left(|X_k|>n^r\right)-P\left(|X_k|>n^r\right)\right]\right)^2 P\left(\max_{1\leq i\leq n}|X_i|>n^r\right)} \\
&\leq \sqrt{\sum_{k=1}^{n}E\left[I\left(|X_k|>n^r\right)-P\left(|X_k|>n^r\right)\right]^2 P\left(\max_{1\leq i\leq n}|X_i|>n^r\right)} \\
&\leq \sqrt{c\left(\phi\right)\sum_{k=1}^{n}P\left(|X_k|>n^r\right)P\left(\max_{1\leq i\leq n}|X_i|>n^r\right)} \\
&\leq \sqrt{c\left(\phi\right)nP\left(|X|>n^r\right)P\left(\max_{1\leq i\leq n}|X_i|>n^r\right)} \\
&\leq \frac{1}{4}nP\left(|X|>n^r\right)+c\left(\phi\right)P\left(\max_{1\leq i\leq n}|X_i|>n^r\right). \qquad (11)
\end{aligned}$$

Combining (11) and (10), we get in (9)

$$\begin{aligned}
\frac{3}{4}nP\left(|X|>n^r\right) &\leq 2\left(1+c\left(\phi\right)\right)P\left(\max_{1\leq i\leq n}|X_i|>n^r\right) \\
&\quad + nP\left(|X|>n^r\right)P\left(\max_{1\leq i\leq n}|X_i|>n^r\right).
\end{aligned}$$

In the consequence, from (8), for sufficiently large n we have

$$nP\left(|X|>n^r\right)\leq 4\left(1+c\left(\phi\right)\right)P\left(\max_{1\leq i\leq n}|X_i|>n^r\right). \qquad (12)$$

Finally, (12) and (7) give

$$\sum_{n=1}^{\infty}n^{rp-1}P\left(|X|>\beta n^r\right)<\infty,$$

and (i) follows. This completes the proof of the theorem. □

Remark 1. *One can give an alternative proof of* (6)*. By the Markov inequality and Lemma 3, for a sufficiently large $q>2$ we have*

$$\begin{aligned}
\sum_{n=1}^{\infty}n^{rp-2}P\left(\max_{1\leq i\leq n}\left|S'_{n,k}\right|>\beta n^r\right) &\leq \sum_{n=1}^{\infty}n^{rp-2-qr}E\left(\max_{1\leq i\leq n}\left|S'_{n,k}\right|>\beta n^r\right)^q \\
&\leq C_1\sum_{n=1}^{\infty}n^{rp-2-qr}\sum_{i=1}^{n}E\left|X'_{ni}\right|^q \\
&\quad + C_2\sum_{n=1}^{\infty}n^{rp-2-qr}\left(\sum_{i=1}^{n}E\left|X'_{ni}\right|^2\right)^{q/2} \\
&= \tilde{I}_5+\tilde{I}_6.
\end{aligned}$$

Using similar arguments as in the estimation of I_5*, we obtain that* $\tilde{I}_5 < \infty$*. In order to estimate* $\tilde{I}_6$ *we distinguish two cases.*

Case $p > 2$.

$$\tilde{I}_6 \leq C_2 \sum_{n=1}^{\infty} n^{rp-2-qr} \cdot n^{q/2} \left(E\left|X\right|^2 \right)^{q/2} < \infty,$$

for $q > (rp-1) / (r-1/2)$.

Case $1 \leq p \leq 2$.

$$\tilde{I}_6 \leq C_2 \sum_{n=1}^{\infty} n^{rp-2-qr} \cdot n^{q/2} \cdot n^{(2-p)q/2} \left(E\left|X\right|^p \right)^{q/2} < \infty,$$

for $q > 2$.

Acknowledgements

The author would like to express his gratitude to Prof. Tadeusz Caliński for reading the draft of this paper and the referee for valuable comments and remarks.

References

[1] Baum I. E., Katz M. L., Convergence rates in the law of large numbers. Trans. Amer. Math. Soc. 120 (1965), 108-123.

[2] Gut A., Complete convergence for arrays. Period. Math. Hungar. 25 (1992), 51-75.

[3] Hsu P. L., Robbins H. Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. USA, 33 (1945), 153-162

[4] Katz M. L., The probability in the tail of distribution. Ann. Math. Statist. 31 (1963), 312-318.

[5] Kiesel R., Strong laws of large numbers and summability of ϕ-mixing random variables in Banach spaces. Electron. Commun. Probab. 2 (1997), 27-41.

[6] Kiesel R., Strong laws of large numbers and summability of ϕ-mixing random variables. J. Theoret. Probab. 11 (1998), 209–234.

[7] Kuczmaszewska A., On complete convergence in Marcinkiewicz-Zygmund type SLLN for negatively associated random variables. Acta Math. Hungar. 128 (2010), 116-130.

[8] Nagaev S. V., On probability and moment inequalities for dependent random variables. Theory Probab. Appl. 45 (2000), 152-160.

[9] Peligrad M., Convergence rates of the strong law of for stationary mixing sequences. Z. Wahrscheinkeitsth. 70 (1985), 307-314.

[10] Peligrad M. The r-quick version of the strong law for stationary ϕ-mixing sequences. In: Almost Everywhere Convergence (G. A. Edgar and L. Sucheston, eds.) pp. 335-348. Academic Press, Boston, MA.

[11] Peligrad M., Gut A., Almost sure results for the class of dependent random variables J. Theoret. Probab. 12 (1999), 87-112.

[12] Pruss A. R., Randomly sampled Riemann sums and complete convergence in the law of large numbers for a case without identical distirbution. Proc. Amer. Math. Soc. 124 (1996), 919-929.

Received by the editors January 19, 2012