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OBTAINING SHOCK SOLUTIONS VIA MASLOV'S THEORY AND COLOMBEAU'S ALGEBRA FOR CONSERVATION LAWS WITH ANALYTICAL COEFFICIENTS

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Abstract

In this paper, via the algebra of generalized functions, we investigate the generalized Riemann's problem associated to conservation laws with analytical coefficients. This allows us to transform the problem into a system of ordinary differential equations. In some particular cases, such as Burgers' and conservative Richard's equation, approximated solutions are obtained by the truncation of the so-called Hugoniot-Maslov's chain and numerical simulations are also presented in the case of equations with polynomial coefficients.

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1 Introduction

Physical phenomena, such as fluid flows in electromagnetic fields or particles infiltration and transport in porous media, can be modelled by equations of Burgers' or Richard's type. But these last ones are particular cases of more general conservation laws with analytical coefficients as follows:

$$u_t(x,t) + f(u(x,t))u_x(x,t) = 0 \ , \ x \in \mathbb{R} \ , \ t > 0, \tag{1.1}$$

where f is an analytical function.

On the one hand, when $f(u) = u$, we recognize Burgers' equation for which existence and uniqueness results are well known. But even in this simple case, we notice the appearance of shock type solutions (even in the case of smooth initial conditions). Indeed, the function u may be discontinuous, then the term

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u_x has the appearance of a Dirac delta function. Thus, the use of the algebra of generalized functions of Colombeau [3, 4, 5, 6, 7, 11] seems to be necessary in order to make sense of product of distributions $u \cdot u_x$.

In particular, if $f(u)$ does not depend on u but only on the variables x and t, and if the initial condition is a generalized function, the existence and uniqueness of a solution have been proved by M. Oberguggenberger in this algebra [11]. In addition, a generalized solution has been built via numerical schemes of Godunov type in [2, 3].

On the other hand, the results obtained by V. P. Maslov [10] in the seventies, about the structure of singular solutions of hyperbolic non-linear equations, have been successfully applied in order to obtain approximate singular solutions of many physical problems (see, for instance, S. P. Dobrokhotov [8]). From the numerical point of view, one of the main possibilities of this theory is the reduction of the problem of finding solutions of partial differential equations to the search of an infinite chain of ordinary differential equations, the so-called Hugoniot-Maslov's chain.

Rodríguez-Bermúdez and Valiño-Alonso [13] obtained the Hugoniot-Maslov's chain for the case of conservation laws with polynomial flow in the context of Colombeau algebra of simplified generalized functions. Here, via this algebra of generalized functions, we study the generalized Riemann problem associated to the equation (1.1). By using properties of Heaviside and Dirac generalized functions, we transform (1.1) into a system of partial differential equations. Then, this system is investigated both from the theoretical and numerical point of view, in some particular cases when $f(u)$ is a polynomial function. Numerical simulations give an idea of the solutions and show the possibilities of Maslov's theory from the numerical point of view.

2 Some remarks on Colombeau's algebra

In 1954, L. Schwartz proved the nonexistence of a differential algebra A (of any kind of generalized functions on $\mathbb{R}$) containing the algebra $\mathcal{C}(\mathbb{R})$ of continuous functions on $\mathbb{R}$ and preserving the classical differentiation of functions in $\mathcal{C}^1(\mathbb{R})$, in such a way that also be preserved another natural properties like the Leibnitz rule for the derivation of a product, or the fact that the unity, that is the constant function 1, and the Dirac distribution belong to the given algebra A.

This theorem, known as the impossibility of multiplication of distributions, makes the theory of Schwartz distributions unable to be applied in nonlinear differential equations. Many people worked by trying to solve this inconvenience by different ways. One of them was the French mathematician Jean François Colombeau, who developed a theory about a set of new generalized functions that form a differential algebra containing canonically the topological vector space of Schwartz distributions. The idea of Colombeau in the construction of this algebra of functions is the usual procedure of construction of an algebra as the quotient of the given algebra of functions by an ideal of the same algebra. If we denote by Ω any region in the Euclidien n-dimensional space $\mathbb{R}^n$, then the set of Colombeau generalized functions is a differential algebra (that is to say that it has the same operations and rules that characterize the differential algebra of $\mathcal{C}^\infty$-functions on Ω), which contains the topological vector space $\mathcal{D}'(\mathbb{R}^n)$ of Schwartz distributions. Of course, the product of two distributions in this algebra, considered as an element of it, may be not an element of the vector space of distributions.

The Colombeau differential algebra is quite complicated for practical uses: it is fruitfully applied in theoretical problems, like questions of existence of unique solutions, a lot of literature has been produced about this algebra and applications derived from it. Our objective is less theoretical, we intend only to apply the Maslov idea to get approximations of singular solutions (in particular shock waves) by simpler methods than other specific numerical methods previously employed. For that purpose it will be enough to use another algebra of Colombeau, the so-called simplified algebra of generalized functions.

Let us show the process of construction of the Colombeau simplified algebra. First we introduce the following set of families of real (or complex) valued functions, defined on $\mathbb{R}^n$:

$$\mathcal{E}_S(\mathbb{R}^n) = \{R: \ (\epsilon, x) \in (0,\infty) \times \mathbb{R}^n \to R(\epsilon, x) \in \mathbb{R} \ / \ R \text{ is } \mathcal{C}^\infty \text{ in the variable } x \text{ for all } \epsilon > 0\}.$$

This is clearly an algebra with the usual pointwise operations of addition, inner multiplication and exterior multiplication by scalars. Next we consider a minor subalgebra, containing moderate families of functions of $\mathcal{E}_S(\mathbb{R}^n)$, and a linear subspace $\mathcal{N}_S(\mathbb{R}^n)$, containing the negligible elements of the precedent algebra:

$$\mathcal{E}_{M,S}(\mathbb{R}^n) = \left\{ \begin{array}{c} R \in \mathcal{E}_S(\mathbb{R}^n) \ / \ \forall K \subset \mathbb{R}^n \text{ compact}, \forall D \text{ derivative operator}, \ \exists q \in \mathbb{N}, \\ \exists\ c > 0, \ \exists\ \eta > 0, \forall x \in K, \forall\ 0 < \epsilon < \eta, \ |DR(\epsilon, x)| \le c\ \epsilon^{-q} \end{array} \right\},$$

$$\mathcal{N}_S(\mathbb{R}^n) = \left\{ \begin{array}{c} R \in \mathcal{E}_{M,S}(\mathbb{R}^n) \ / \ \forall K \subset \mathbb{R}^n \text{ compact}, \forall D \text{ derivative operator}, \ \exists q \in \mathbb{N}, \\ \forall p \ge q, \ \exists\ c > 0, \ \exists\ \eta > 0, \forall x \in K, \forall\ 0 < \epsilon < \eta, \ |DR(\epsilon, x)| \le c\ \epsilon^{p-q} \end{array} \right\}.$$

$\mathcal{N}_S(\mathbb{R}^n)$ is an ideal of $\mathcal{E}_{M,S}(\mathbb{R}^n)$.

Definition 2.1. The algebra of simplified generalized functions is defined as the quotient of $\mathcal{E}_{M,S}(\mathbb{R}^n)$ by $\mathcal{N}_S(\mathbb{R}^n)$, and is denoted by $\mathcal{G}_S(\mathbb{R}^n)$.

In this algebra, there are two equalities, one strong and one weak. The strong one is the classical algebraic equality, in other words, two simplified generalized functions are equal if the difference of two of their representatives is in the ideal $\mathcal{N}_S(\mathbb{R}^n)$. The weak one is called association and is denoted by the symbol $\approx$.

Definition 2.2. We say that $G_1, G_2 \in \mathcal{G}_S(\mathbb{R}^n)$ are associated, if and only if, for any $\Psi \in \mathcal{D}(\mathbb{R}^n)$, one has

$$\int_{\mathbb{R}^n} [R_1(\epsilon, x) - R_2(\epsilon, x)]\, \Psi(x)\ dx \to 0 \text{ when } \epsilon \to 0,$$

with $R_i \in \mathcal{E}_{M,S}(\mathbb{R}^n)$ as a representative of G_i, $i = 1, 2$.

The association is stable by differentiation but is not by multiplication. Let H be a Heaviside function, then H^n is also one, for any $n \in \mathbb{N}$. But $H^n \neq H$ in $\mathcal{G}_S(\mathbb{R}^n)$, for $n > 1$. This leads to the notions of generalized functions of Heaviside and Dirac types in the algebra of simplified generalized functions $\mathcal{G}_S(\mathbb{R}^n)$.

Definition 2.3. A generalized function H in $\mathcal{G}_S(\mathbb{R})$ is called a Heaviside generalized function if it has a representative $R \in \mathcal{E}_{M,S}(\mathbb{R})$ satisfying the following condition: there is a function $A(\epsilon) > 0$, $A(\epsilon) \to 0$ as $\epsilon \to 0$, such that

a) $\forall \epsilon > 0,\ \forall x < -A(\epsilon),\ R(\epsilon, x) = 0,$

b) $\forall \epsilon > 0,\ \forall x > A(\epsilon),\ R(\epsilon, x) = 1,$

c) $\sup\limits_{\epsilon>0, x\in\mathbb{R}} |R(\epsilon, x)| < +\infty.$

Accordingly, any power of a Heaviside generalized function is a Heaviside generalized function, and the two Heaviside generalized functions are associated.

Definition 2.4. A Dirac generalized function on $\mathbb{R}$ is an element δ of $\mathcal{G}_S(\mathbb{R})$ with a representative $R \in \mathcal{E}_{M,S}(\mathbb{R})$ satisfying the following condition: there is a function $A(\epsilon) > 0$, $A(\epsilon) \to 0$ as $\epsilon \to 0$, such that

a) $\forall \epsilon > 0,\ \forall |x| > A(\epsilon),\ R(\epsilon, x) = 0,$

b) $\forall \epsilon > 0,\ \int_{\mathbb{R}} R(\epsilon, x)\, dx = 1,$

c) $\sup\limits_{\epsilon>0} \displaystyle\int_{\mathbb{R}} |R(\epsilon, x)|\, dx < +\infty.$

3 Obtaining the Hugoniot-Maslov's Chain for the Generalized Riemann's problem

In this section, our aim is to obtain the so-called Hugoniot-Maslov's chain from the following generalized Riemann's problem:

$$\begin{cases} u_t(x,t) + f(u(x,t))u_x(x,t) \approx 0\ ,\ t > 0, \\ u(\cdot, 0) = u_l(\cdot) + (u_r(\cdot) - u_l(\cdot))L(\cdot), \end{cases} \tag{3.2}$$

where L is a Heaviside generalized function and u_l and u_r are real smooth functions such that $u_l(0) \neq u_r(0)$.

As f is an analytical function, it can be written as

$$f(u) = \sum_{n=0}^{+\infty} a_n u^n \tag{3.3}$$

in some neighborhood of zero, where a_n are real coefficients.

We look for shock type solutions of the form

$$u(x,t) = A(x,t) + B(x,t)H(x - X(t)), \tag{3.4}$$

where A, B and X are smooth functions and H is an Heaviside generalized function. Consequently,

$$f(u(x,t)) = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t)H^k(x - X(t)), \tag{3.5}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. By replacing (3.4) and (3.5) in the first equation of (3.2), we obtain that

$$\begin{aligned}
&\frac{\partial A}{\partial t}(x,t) + \frac{\partial B}{\partial t}(x,t)H(x - X(t)) - X'(t)B(x,t)H'(x - X(t))\\
&\quad + \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t)\frac{\partial A}{\partial x}(x,t)H^k(x - X(t))\\
&\quad + \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t)\frac{\partial B}{\partial x}(x,t)H^{k+1}(x - X(t))\\
&\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k}(x,t)B^{k+1}(x,t)H^k(x - X(t))H'(x - X(t)) \approx 0.
\end{aligned}$$

Then, we use the fact that $H^k \approx H$, for all $k \neq 0$ and $H^k H' \approx \frac{1}{k+1}H'$, for all $k \in \mathbb{N}$, in order to have

$$\begin{aligned}
&\frac{\partial A}{\partial t}(x,t) + \sum_{n=0}^{+\infty} a_n A^n(x,t)\frac{\partial A}{\partial x}(x,t)\\
&+ \Big[\tfrac{\partial B}{\partial t}(x,t) + \textstyle\sum_{n=0}^{+\infty}\sum_{k=1}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t)\tfrac{\partial A}{\partial x}(x,t)\\
&+ \sum_{n=0}^{+\infty}\sum_{k=1}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t)\frac{\partial B}{\partial x}(x,t)\Big] H(x - X(t))\\
&+ \left[\sum_{n=0}^{+\infty}\sum_{k=1}^{n} \frac{a_n}{k+1} C_n^k A^{n-k}(x,t)B^{k+1}(x,t) - X'(t)B(x,t)\right] H'(x - X(t)) \approx 0.
\end{aligned}$$

As $(1, H, H')$ are linearly independent, we get the following system:

$$\begin{cases}
\dfrac{\partial A}{\partial t}(x,t) + \displaystyle\sum_{n=0}^{+\infty} a_n A^n(x,t)\frac{\partial A}{\partial x}(x,t) = 0,\\
\dfrac{\partial B}{\partial t}(x,t) + \displaystyle\sum_{n=0}^{+\infty}\sum_{k=1}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t)\frac{\partial A}{\partial x}(x,t)\\
\quad + \textstyle\sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t)\tfrac{\partial B}{\partial x}(x,t) = 0,\\
X'(t)B(x,t) = \displaystyle\sum_{n=0}^{+\infty}\sum_{k=1}^{n} \frac{a_n}{k+1} C_n^k A^{n-k}(x,t)B^{k+1}(x,t).
\end{cases}$$

Some simple computations allow us to write that

$$\begin{cases} \displaystyle\sum_{n=0}^{+\infty} a_n A^n(x,t) = f(A(x,t)), \\ \displaystyle\sum_{n=0}^{+\infty}\sum_{k=1}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t) = f((A+B)(x,t)) - f(A(x,t)), \\ \displaystyle\sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k}(x,t)B^k(x,t) = f((A+B)(x,t)), \\ \displaystyle\sum_{n=0}^{+\infty}\sum_{k=0}^{n} \frac{a_n}{k+1} C_n^k A^{n-k}(x,t)B^{k+1}(x,t) = g((A+B)(x,t)) - g(B(x,t)), \end{cases}$$

where g is a primitive of f. Hence, we have just to find A, B and X, smooth functions satisfying the following system of three partial differential equations:

$$\begin{cases} \dfrac{\partial A}{\partial t}(x,t) + f(A(x,t))\dfrac{\partial A}{\partial x}(x,t) = 0, \\ \dfrac{\partial B}{\partial t}(x,t) + [f((A+B)(x,t)) - f(A(x,t))]\,\dfrac{\partial A}{\partial x}(x,t) \\ \quad + f((A+B)(x,t))\frac{\partial B}{\partial x}(x,t) = 0, \\ X'(t)B(x,t) = g((A+B)(x,t)) - g(A(x,t)). \end{cases} \tag{3.6}$$

In addition, A and B are smooth functions, so we can write them as formal Taylor's series in the neighborhood of the position of singularity $x = X(t)$, that is

$$A(x,t) = \sum_{k=0}^{+\infty} A_k(t)(x - X(t))^k \text{ and } B(x,t) = \sum_{k=0}^{+\infty} B_k(t)(x - X(t))^k,$$

and compute the different terms of system (3.6). Now, we are going to use this system in order to obtain the Hugoniot-Maslov's chain in the following particular cases: $f(u) = u$ and $f(u) = au^2 + bu + c$, with a, b and c in $\mathbb{R}$.

3.1 Burgers' equation

When $f(u) = u$, the first equation of (3.6) is

$$\frac{\partial A}{\partial t}(x,t) + A(x,t)\frac{\partial A}{\partial x}(x,t) = 0,$$

which becomes

$$\sum_{k=0}^{+\infty} A_k'(t)(x - X(t))^k - X'(t)\sum_{k=1}^{+\infty} kA_k(t)(x - X(t))^{k-1} \\ + \sum_{k=0}^{+\infty} A_k(t)(x - X(t))^k \sum_{k=0}^{+\infty} kA_k(t)(x - X(t))^{k-1} = 0.$$

By appropriate changes of variables and applying Cauchy formula to multiply series, we obtain that

$$\sum_{k=0}^{+\infty}\left[A'(t)-(k+1)X'(t)A_{k+1}(t)+\sum_{p=0}^{k}(k-p+1)A_p(t)A_{k-p+1}(t)\right](x-X(t))^k=0.$$

Consequently, the term between brackets is equal to zero for all non-negative k. But, since $g(u)=u^2/2$, the last equation of (3.6) gives

$$X'(t)=\frac{1}{2}B(x,t)+A(x,t)=\frac{1}{2}\sum_{k=0}^{+\infty}B_k(t)(x-X(t))^k+\sum_{k=0}^{+\infty}A_k(t)(x-X(t))^k,$$

for all x in $\mathbb{R}$, so in particular for $x=X(t)$, which allows us to deduce that

$$X'(t)=\frac{1}{2}B_0(t)+A_0(t).$$

Then, for all $k\geq 0$,

$$(k+1)\left(\frac{1}{2}B_0(t)+A_0(t)\right)A_{k+1}(t)=A_k'(t)+\sum_{p=0}^{k}(k-p+1)A_p(t)A_{k-p+1}(t),$$

so

$$A_k'(t)=(k+1)\left(\frac{1}{2}B_0(t)+A_0(t)\right)A_{k+1}(t)-\sum_{p=0}^{k}(k-p+1)A_p(t)A_{k-p+1}(t). \tag{3.7}$$

Now, let us use the second equation of (3.6), that is

$$\frac{\partial B}{\partial t}(x,t)+B\frac{\partial A}{\partial x}(x,t)+(A+B)\frac{\partial B}{\partial x}(x,t)=0,$$

and apply similar arguments to obtain for all $k\geq 0$,

$$\begin{aligned}&(k+1)(\frac{1}{2}B_0(t)+A_0(t))B_{k+1}(t)\\ &=B_k'(t)+\sum_{p=0}^{k}(k-p+1)\left[B_p(t)A_{k-p+1}(t)+\left(A_p(t)+B_p(t)\right)B_{k-p+1}(t)\right].\end{aligned}$$

The equation that must be satisfied by the functions $B_k(t)$, for all $k\geq 0$, is

$$B_k'(t)=(k+1)(\frac{1}{2}B_0(t)+A_0(t))B_{k+1}(t)-\sum_{p=0}^{k}(k-p+1)B_p(t)A_{k-p+1}(t). \tag{3.8}$$

But

$$X'(t)=\frac{1}{2}B_0(t)+A_0(t)=\frac{1}{2}\sum_{k=0}^{+\infty}B_k(t)(x-X(t))^k+\sum_{k=0}^{+\infty}A_k(t)(x-X(t))^k,$$

so $\frac{1}{2}B_k(t) + A_k(t) = 0$, for $k \geq 1$. When k greater than 1, after replacing $B_k(t)$ by $-2A_k(t)$ in (3.8), we deduce that $A'_k(t) = 0$ for $k \geq 1$ and (3.7) gives

$$\forall k \geq 1 \ , \ \frac{1}{2}(k+1)B_0 A_{k+1} = \sum_{p=1}^{k}(k-p+1)A_p A_{k-p+1}.$$

For example, $A_2 = \dfrac{A_1^2}{B_0}, A_3 = 2\dfrac{A_1^3}{B_0^2}$, etc..., for B_0 non equal to zero. An inductive argument allows us to write that, for all $j \geq 1$,

$$A_j = C_j \frac{A_1^j}{B_0^{j-1}},$$

with $C_1 = 1$ and $C_{j+1} = \dfrac{2}{j+1}\displaystyle\sum_{p=1}^{j}(j-p+1)C_p C_{j-p+1}$. Let us assume that $\displaystyle\sum_{l=0}^{+\infty} C_l z^l$ converges for all z in $\mathbb{C}$ and $|z| \geq \rho$, where ρ is the radius of convergence of this series, and set $b_k = kC_k$, for $k \in \mathbb{N}$. We can write, for $l > 2$, that

$$b_l = 2\sum_{p=1}^{l-1} C_p b_{l-p} = 2\left[\sum_{p+k=l} C_p b_k - C_0 b_l - C_l b_0\right],$$

that is

$$(1+2C_0)b_l = 2\sum_{p+k=l} C_p b_k.$$

Thus, $(1+2C_0)\displaystyle\sum_{l=2}^{+\infty} b_l z^l = 2\sum_{l=2}^{+\infty}\sum_{p+k=l} C_p b_k z^l$, for all z in $\mathbb{C}$. Let us use the Z-transform by setting $F(z) = \displaystyle\sum_{l=2}^{+\infty} b_l z^l$ and $G(z) = \displaystyle\sum_{l=2}^{\infty} C_l z^l$. We have $(1 + 2C_0)F(z) = 2F(z)G(z)$. But $F \neq 0$, so $G(z) = \frac{1}{2} + C_0$, that is $G(z)$ does not depend on z. Consequently,

$$\sum_{l=2}^{+\infty} C_l z^l - \frac{1}{2} - C_0 = 0,$$

which implies that $C_0 = \frac{1}{2}$ and $C_l = 0$, for all $l \geq 2$, which is not the case. Thus, $\displaystyle\sum_{l=0}^{+\infty} C_l z^l$ does not converge for $z \neq 0$, as well as $\displaystyle\sum_{k=0}^{+\infty} B_k(t)(x - X(t))^k$, when $x \neq X(t)$. This study leads us to follow Prasad-Ravindrah's method [12], by making $A_{k+1} = B_{k+1} = 0$, which will allow us to have an approximation of A and B, i.e. of the shock solution.

3.2 Equation with $f(u) = au + b$, where $a, b \in \mathbb{R}$, $a \neq 0$

In this case the system becomes

$$\begin{cases} \dfrac{\partial A}{\partial t}(x,t) + [aA(x,t) + B(x,t)]\,\dfrac{\partial A}{\partial x}(x,t) = 0, \\ \dfrac{\partial B}{\partial t}(x,t) + aB(x,t)\dfrac{\partial A}{\partial x}(x,t) + [aA(x,t) + aB(x,t) + b]\,\dfrac{\partial B}{\partial x}(x,t) = 0, \\ X'(t)B(x,t) = aA(x,t)B(x,t) + \dfrac{a}{2}B^2(x,t) + bB(x,t). \end{cases}$$

From the last equation and writing $A(x,t) = \sum_{k=0}^{+\infty} A_k(t)(x - X(t))^k$ and $B(x,t) = \sum_{k=0}^{+\infty} B_k(t)(x - X(t))^k$, the expression for the equation of the trajectory of the singularity is

$$X'(t) = a\left[A_0(t) + \frac{B_0(t)}{2}\right] + b.$$

Now, the substitution of these expressions in the equation

$$\frac{\partial A}{\partial t}(x,t) + [aA(x,t) + B(x,t)]\,\frac{\partial A}{\partial x}(x,t) = 0$$

gives

$$\sum_{k=0}^{+\infty}[A'_k - (k+1)A_{k+1}X' + a\sum_{l=0}^{k}(k-l+1)A_l A_{k-l+1} + b(k+1)A_{k+1}](x - X(t))^k = 0,$$

which allows to have, for all $k = 0, 1, 2, ...,$

$$A'_k - (k+1)A_{k+1}X' + a\sum_{l=0}^{k}(k-l+1)A_l A_{k-l+1} + b(k+1)A_{k+1} = 0.$$

But $X'(t) = a\left[A_0(t) + \frac{B_0(t)}{2}\right] + b$, so we finally get

$$A'_k = a\left[(k+1)\left(A_0 + \frac{B_0}{2}\right)A_{k+1} - \sum_{l=0}^{k}(k-l+1)A_l Ak - l + 1\right].$$

To obtain the equations for B_k, we consider the second equation

$$\frac{\partial B}{\partial t}(x,t) + aB(x,t)\frac{\partial A}{\partial x}(x,t) + [aA(x,t) + aB(x,t) + b]\,\frac{\partial B}{\partial x}(x,t) = 0.$$

The substitution and use of the formal developments of the functions A and B gives

$$\sum_{k=0}^{+\infty}\Bigg[B_k' - (k+1)B_{k+1}X' + a\sum_{l=0}^{k}(k-l+1)B_lA_{k-l+1}$$
$$+a\sum_{l=0}^{k}(k-l+1)A_lB_{k-l+1}\sum_{l=0}^{k}(k-l+1)B_lB_{k-l+1} + b(k+1)B_{k+1}\Bigg]$$
$$\cdot (x-X(t))^k = 0,$$

so that

$$B_k' - (k+1)B_{k+1}\left[a\left(A_0(t)+\frac{B_0(t)}{2}\right)+b\right] + a\sum_{l=0}^{k}(k-l+1)B_lA_{k-l+1}$$
$$+ a\sum_{l=0}^{k}(k-l+1)A_lB_{k-l+1}\sum_{l=0}^{k}(k-l+1)B_lB_{k-l+1} + b(k+1)B_{k+1} = 0.$$

Thus the equations for the functions B_k are

$$B_k' = a(k+1)(A_0+\frac{B_0}{2})B_{k+1} - a\sum_{l=0}^{k}(k-l+1)\left[B_lA_{k-l+1} + A_lB_{k-l+1} + B_lB_{k-l+1}\right].$$

In the case $f(u) = au + b$, we obtain the following Hugoniot-Maslov's chain:

$$\begin{cases} X'(t) = a\left[A_0(t)+\dfrac{B_0(t)}{2}\right] + b, \\ A_k' = a\left[(k+1)(A_0+\dfrac{B_0}{2})A_{k+1} - \displaystyle\sum_{l=0}^{k}(k-l+1)A_lA_{k-l+1}\right], \\ B_k' = a(k+1)\left(A_0+\dfrac{B_0}{2}\right)B_{k+1} \\ \quad -a\displaystyle\sum_{l=0}^{k}(k-l+1)\left[B_lA_{k-l+1} + A_lB_{k-l+1} + B_lB_{k-l+1}\right]. \end{cases}$$

3.3 The quadratic polynomial case

A cubic polynomial flow have been investigated by L. Alvarez and B. Valiño in [1], considering a conservative version of the Richard's equation of particles infiltration and transport in porous medias. Let us consider in this paragraph the quadratic polynomial $f(u) = au^2 + bu + c$, with a, b, c in $\mathbb{R}$. The system (3.6) becomes

$$\begin{cases} \frac{\partial A}{\partial t}(x,t) + \left[aA^2(x,t) + bA(x,t) + c\right]\frac{\partial A}{\partial x}(x,t) = 0, \\ \frac{\partial B}{\partial t}(x,t) + \left[aB^2 + 2aBA + bB\right](x,t)\frac{\partial A}{\partial x}(x,t) \\ \quad + \left[a(A+B)^2 + b(A+B) + c\right](x,t)\frac{\partial B}{\partial x}(x,t) = 0, \\ X'(t) = aA(x,t)B(x,t) + aA^2(x,t) + \frac{1}{3}aB^2(x,t) + bA(x,t) + \frac{b}{2}B(x,t) + c. \end{cases}$$

Using a similar method as in previous sections, the Hugoniot Maslov chain is

$$\begin{cases} X' &= a\left(\dfrac{B_0^2}{3}+A_0^2+A_0B_0\right)+b\left(\dfrac{B_0}{2}+A_0\right)+c, \\ A_k' &= (k+1)A_{k+1}\left[a\left(\dfrac{B_0^2}{3}+A_0^2+A_0B_0\right)+b\left(\dfrac{B_0}{2}+A_0\right)\right] \\ & \quad -a\displaystyle\sum_{p=0}^{k}\sum_{s=0}^{p}(k-p+1)A_sA_{p-s}A_{k-p+1} \\ & \quad -b\displaystyle\sum_{p=0}^{k}(k-p+1)A_pA_{k-p+1}, \\ B_k' &= (k+1)B_{k+1}\left[a\left(\dfrac{B_0^2}{3}+A_0^2+A_0B_0\right)+b\left(\dfrac{B_0}{2}+A_0\right)\right] \\ & \quad -b\displaystyle\sum_{p=0}^{k}(k-p+1)\left[A_{k-p+1}B_p+(A_p+B_p)B_{k-p+1}\right] \\ & \quad -a\displaystyle\sum_{r=0}^{k}\sum_{p=0}^{r}(k-r+1)B_p(B_{r-p}+2A_{r-p})A_{k-r+1} \\ & \quad -a\displaystyle\sum_{r=0}^{k}\sum_{p=0}^{r}(k-r+1)(A_p+B_p)(A_{r-p}+B_{r-p})B_{k-r+1}, \end{cases}$$

for all $k \geq 0$, where we set $X' = X'(t), A_k = A_k(t)$ and $B_k = B_k(t)$ in order to simplify the notations.

3.4 Hugoniot-Maslov's chain for the polynomial general case

P. Rodríguez Bermúdez and B. Valiño Alonso [13] calculated the Hugoniot-Maslov's chain of shock waves in scalar conservation laws with polynomial flow. They considered an equation of type (1.1), where $f(u) = a_{n+1}u^n + a_n u^{n-1} + ... + a_2 u + a_1$, $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$. They showed that if there exists a weak solution of the type (3.4), in the algebra of simplified generalized functions of Colombeau, then the functions $X(t), A_l(t), B_l(t), l = 0, 1, 2, \ldots$, must satisfy an infinite system of ordinary differential equations.

4 Numerical simulations

Next we show some numerical simulations of the solutions corresponding to several examples of conservation laws with polynomial flow with initial conditions of the generalized Riemann's problem. By such a problem for the equation (1.1), we consider the following: find a generalized function u , belonging to Colombeau's algebra of simplified generalized functions, such that

$$\frac{\partial u}{\partial t} + \left(a_{n+1}u^n + a_n u^{n-1} + ... + a_2 u + a_1\right)\frac{\partial u}{\partial x} \approx 0,$$

with the initial condition

$$u(x,0) = \begin{cases} u_l(x), & x < 0, \\ u_r(x), & x \geq 0, \end{cases}$$

where $u_l(0) \neq u_r(0)$. This initial condition can be written in the form $u(x,0) = u_l(x) + [u_r(x) - u_l(x)]\, H(x)$, where H denotes a Heaviside generalized function.

A necessary condition for the existence of such a solution is giving by the corresponding Hugoniot-Maslov's chain, see [13]. If the solution exists, we can calculate an approximation with some grade of accuracy by interrupting the chain in a certain step, that is "truncate" the chain, for instance, in the $k-th$ index. Consequently, we obtain a system of ordinary differential equations with $2n+3$ equations and $2n+5$ unknowns $X, A_0, A_1, ..., A_k, A_{k+1}, B_0, B_1, ..., B_k, B_{k+1}$. That is, the truncated ODE' system is overdeterminated. So we need to close this system in order to solve it. To do this, we follow the procedure applied by Prasad and Ravindrah in [12] vanishing the variables A_{k+1}, B_{k+1}. This is an easy way to get an approximated solution of the infinite ODE' system corresponding to the particular solution of the Generalized Riemann's Problem, and consequently, to obtain a graph of this solution. According to Dobrokhotov [8], this procedure gives a reasonable error, and for this reason is commonly used.

All the simulations made in this work are obtained from a truncated chain for $k=2$, so we have always a closed truncated ODE' system with 7 equations and 7 unknowns $X, A_0, A_1, A_2, B_0, B_1, B_2$. For instance, in the case of $f(u) = au + b$, we obtain the closed truncated ODEs' system:

$$\begin{cases} X'(t) = a\left(A_0(t) + \frac{B_0(t)}{2}\right) + b, \\ A_0' = a\frac{A_1 B_0}{2}, \\ A_1' = a\left(A_2 B_0 - A_1^2\right), \\ A_2' = -3aA_1A_2, \\ B_0' = aB_1\left(\frac{B_0}{2} - A_1 - B_1\right), \\ B_1' = -a\left(2A_1B_1 + 2A_2B_0 + B_0B_2 + B_1^2\right), \\ B_2' = -3a(A_1B_2 + A_2B_1 + B_1B_2). \end{cases}$$

For the values $a=1$ and $b=0$, we obtain the corresponding closed truncated chain for the Burgers's equation:

$$\begin{cases} X'(t) = A_0(t) + \frac{B_0(t)}{2}, \\ A_0' = \frac{A_1 B_0}{2}, \\ A_1' = \ aA_2 B_0 - A_1^2, \\ A_2' = -3A_1A_2, \\ B_0' = aB_1\left(\frac{B_0}{2} - A_1 - B_1\right), \\ B_1' = -\left(2A_1B_1 + 2A_2B_0 + B_0B_2 + B_1^2\right), \\ B_2' = -3(A_1B_2 + A_2B_1 + B_1B_2). \end{cases}$$

In the same way, we can obtain the Hugoniot-Maslov's chain for the conservative Richard's equation:

$$\begin{cases} X' &= a\left(\frac{B_0^2}{3}+A_0^2+A_0B_0\right)+b\left(\frac{B_0}{2}+A_0\right)+c,\\ A_0' &= A_1\left[aB_0\left(A_0+\frac{B_0}{3}\right)+b\frac{B_0}{2}\right],\\ A_1' &= A_2B_0\left[2a\left(A_0+\frac{B_0}{2}\right)+b\right]-A_1^2\left(2aA_0+b\right),\\ A_2' &= -A_1A_2\left(6aA_0+3b\right)-aA_1^3,\\ B_0' &= -aB_0^2\left(\frac{2B_1}{3}+A_1\right)-A_0B_0(B_1+2A_1)-bB_0\left(\frac{B_1}{2}+A_1\right),\\ B_1' &= -\frac{4}{3}aB_2B_0^2-2a\left[2\left(A_0A_1B_1+A_0A_2B_0+A_0B_0B_2+A_1B_0B_1\right)\right.\\ & \quad \left.+A_2B_0^2+A_1^2B_0+B_0B_1^2+A_0B_1^2-A_0B_0B_2\right]\\ & \quad -b\left[2\left(A_2B_0+A_1B_1\right)+B_1^2+B_0B_2\right],\\ B_2' &= -3a[2(A_1A_2B_0+A_0A_1B_2+A_0B_1B_2+A_0A_2B_1\\ & \quad +B_0B_1B_2+A_2B_0B_1+A_1B_0B_2\\ & \quad +A_1B_1^2+A_1^2B_1+B_1^3]-3b\left[A_1B_2+A_2B_1+B_1B_2\right]. \end{cases}$$

In the polynomial general case, the "closed truncated" ODE system is:

$$X'(t) = \sum_{k=0}^{n}\frac{1}{k+1}\sum_{i=k}^{n}a_{i+1}f_i(k)A_0^{i-k}B_0^k,$$

$$A_0'(t) = \sum_{k=1}^{n}\frac{1}{k+1}\sum_{i=k}^{n}a_{i+1}f_i(k)A_0^{i-k}B_0^kA_1,$$

$$A_1'(t) = 2A_2\sum_{k=1}^{n}\frac{1}{k+1}\sum_{i=k}^{n}a_{i+1}f_i(k)A_0^{1-k}B_0^k-A_1^2\sum_{i=1}^{n}ia_{i+1}A_0^{i-1},$$

$$A_2'(t) = -3A_1A_2\sum_{i=1}^{n}ia_{1+1}A_0^{i-1}-A_1^3\sum_{i=2}^{n}\frac{i(i-1)}{2}a_{i+1}A_0^{i-2},$$

$$B_0'(t) = B_1\sum_{k=1}^{n}\frac{1}{k+1}\sum_{i=k}^{n}a_{i+1}f_i(k)A_0^{i-k}B_0^k-(A_1+B_1)\sum_{k=1}^{n}\sum_{i=k}^{n}a_{i+1}f_i(k)A_0^{i-k}B_0^k,$$

$$\begin{aligned} B_1'(t) = &2B_2\sum_{k=1}^{n}\frac{1}{k+1}\sum_{i=k}^{n}a_{i+1}f_i(k)A_0^{i-k}B_0^k\\ &-B_1(A_1+B_1)\sum_{k=1}^{n}\sum_{i=k}^{n}ka_{i+1}f_i(k)A_0^{i-k}B_0^{k-1}\\ &-2(A_2+B_2)\sum_{k=1}^{n}\sum_{i=k}^{n}a_{i+1}f_i(k)A_0^{i-k}B_0^k\\ &-A_1B_1\sum_{k=1}^{n}\sum_{i=k}^{n}(i-k+1)a_{i+1}f_i(k-1)A_0^{i-k}B_0^k, \end{aligned}$$

$$B_2'(t) = -3A_1A_2\sum_{k=1}^{n}\sum_{i=k}^{n}(i-k)a_{i+1}f_i(k)A_0^{i-k-1}B_0^k$$

$$
\begin{aligned}
&- A_2B_1\sum_{k=1}^{n}\sum_{i=k}^{n}(i-k+1)a_{i+1}f_i(k-1)A_0^{i-k}B_0^{k-1}\\
&- A_1^3\sum_{k=1}^{n}\sum_{i=k}^{n}\frac{(i-k)\,(i-k-1)}{2}a_{i+1}f_i(k-1)A_0^{i-k-2}B_0^{k}\\
&- A_1^2B_1\sum_{k=1}^{n}\sum_{i=k}^{n}\frac{(i-k)\,(i-k+1)}{2}a_{i+1}f_i(k-1)A_0^{i-k-1}B_0^{k-1}\\
&- A_1^2B_1\sum_{k=1}^{n}\sum_{i=k}^{n}k\,(i-k)\,a_{i+1}f_i(k)A_0^{i-k-1}B_0^{k-1}\\
&- A_1B_1^2\sum_{k=1}^{n}\sum_{i=k}^{n}k\,(i-k)\,a_{i+1}f_i(k)A_0^{i-k-1}B_0^{k-1}\\
&- 2A_1B_2\sum_{k=1}^{n}\sum_{i=k}^{n}(i-k+1)\,a_{i+1}f_i(k-1)A_0^{i-k}B_0^{k-1}\\
&- A_1B_2\sum_{k=1}^{n}\sum_{i=k}^{n}ka_{i+1}f_i(k)A_0^{i-k}B_0^{k-1}\\
&- B_1B_2\sum_{k=1}^{n}\sum_{i=k}^{n}ka_{i+1}f_i(k)A_0^{i-k}B_0^{k-1}\\
&- A_1B_1^2\sum_{k=1}^{n}\sum_{i=k}^{n}\frac{i(i-1)}{2}a_{i+1}f_i(k)A_0^{i-k}B_0^{k-2}\\
&- 2B_1B_2\sum_{k=1}^{n}\sum_{i=k}^{n}ka_{i+1}f_i(k)A_0^{i-k-1}B_0^{k-1}\\
&- B_1^3\sum_{k=1}^{n}\sum_{i=k}^{n}\frac{i\,(i-1)}{2}a_{i+1}f_i(k)A_0^{i-k}B_0^{k-2}\\
&- 2A_2B_1\sum_{k=1}^{n}\sum_{i=k}^{n}ka_{i+1}f_i(k)A_0^{i-k}B_0^{k-1}.
\end{aligned}
$$

where $f_i(k)$, for $k \in \mathbb{N}$ is given by the following expression

$$
f_i(k) = \begin{cases} \frac{(i-k+1)(i-k+2)\ldots(i-1)i}{k!} & \text{if } k > 0; \\ 1 & \text{if } k = 0. \end{cases} \tag{4.9}
$$

To solve this ODEs' system, it is necessary to give the initial values $X(0)$, $A_0(0)$, $A_1(0)$, $A_2(0)$, $B_0(0)$, $B_1(0)$ and $B_2(0)$, then a solution, uniquely determined, will be obtained. The initial values can be calculated from the initial condition of the problem. In fact, the smooth functions $u_l(x), u_r(x)$ can be formally expressed as Taylor's series in the form

$$
u_l(x) = \sum_{k=0}^{+\infty}\frac{u_l^{(k)}(0)}{k!}x^k, \qquad u_r(x) = \sum_{k=0}^{+\infty}\frac{u_r^{(k)}(0)}{k!}x^k;
$$

then putting these developments in the expression of the initial function $u(x,0)$ we obtain

$$u(x,0) = \left[\sum_{k=0}^{+\infty} \frac{u_l^{(k)}(0)}{k!} x^k\right] + \left[\sum_{k=0}^{+\infty} \frac{u_r^{(k)}(0) - u_l^{(k)}(x)}{k!} x^k\right] H(x).$$

Now, remember that we also have the series development:

$$u(x,t) = \left[\sum_{k=0}^{+\infty} A_k(x)(x - X(t))^k\right] + \left[\sum_{k=0}^{+\infty} B_k(x)(x - X(t))^k\right] H(x - X(t)),$$

which, for $t = 0$, gives:

$$u(x,0) = \left[\sum_{k=0}^{+\infty} A_k(0)(x - X(0))^k\right] + \left[\sum_{k=0}^{+\infty} B_k(0)(x - X(0))^k\right] H(x - X(0)).$$

Applying the identity principle of power series, we finally obtain that

$$X(0) = 0, \qquad A_k(0) = \frac{u_l^{(k)}(0)}{k!}, \qquad B_k(0) = \frac{u_r^{(k)}(0) - u_l^{(k)}(0)}{k!}, k = 0, 1, 2, 3, ...$$

Then, the initial conditions for the closed-truncated ODEs' system, obtained for $k = 2$, is:

$$X(0) = 0, \qquad A_k(0) = \frac{u_l^{(k)}(0)}{k!}, \qquad B_k(0) = \frac{u_r^{(k)}(0) - u_l^{(k)}(0)}{k!}, k = 0, 1, 2.$$

The numerical calculations can be made now by using MATLAB. The approximate shock wave can also be obtained this way, making substitutions of the numerical approximations obtained for $X(t)$, $A_0(t)$, $A_1(t)$, $A_2(t)$, $B_0(t)$, $B_1(t)$, $B_2(t)$, in the expression

$$u_{approx}(x,t) = \left[\sum_{k=0}^{2} A_k(x)(x - X(t))^k\right] + \left[\sum_{k=0}^{2} B_k(x)(x - X(t))^k\right] H(x - X(t)).$$

This can be done by means of the corresponding program in MATLAB. In this case, the macroscopic aspect of the Heaviside generalized function employed in the formulation of the initial condition (i.e. the Heaviside distribution) must be used to obtain the graph. Next, we show some numerical simulations obtained by this method.

Example 4.1. (Riemann problem for the Burgers' equation)
Consider the equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, with the initial condition $u(x,0) = \begin{cases} 5 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$. Note that in this case we can obtain the exact solution $u(x,t) = 5 - 6H(x - 2t)$, where H(x) is the Heaviside distribution. Fig. 1 and Fig. 2 show, respectively, the graphs of the singularity trajectory and of the approximated shock wave.

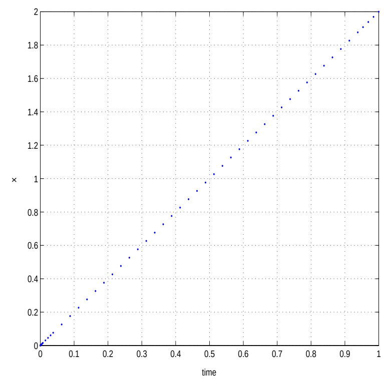

Figure 1: Graph of the singularity trajectory for Ex. 4.1

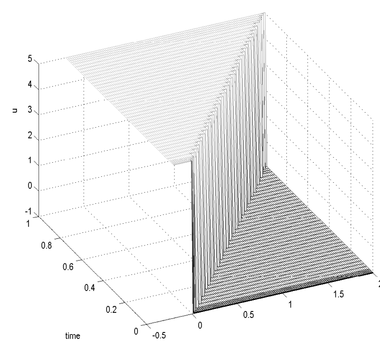

Figure 2: Graph of the approximated shock wave for Ex. 4.1

Example 4.2. (Generalized Riemann Problem for the Burgers' equation)
Consider again Burgers' equation, with the initial condition

$$u(x,0) = \begin{cases} x-1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} .$$

The corresponding graphs for this problem are shown in Fig. 3 and Fig. 4.

Example 4.3. (Generalized Riemann Problem for a conservation law with polynomial cubic flow)
Consider the equation

$$\frac{\partial u}{\partial t} + \left(-\frac{1}{2}u^2 + u + 2\right)\frac{\partial u}{\partial x} = 0,$$

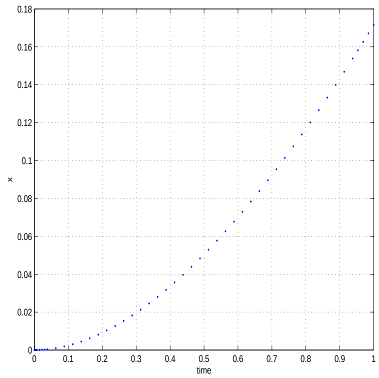

Figure 3: Graph of the singularity trajectory for Ex. 4.2

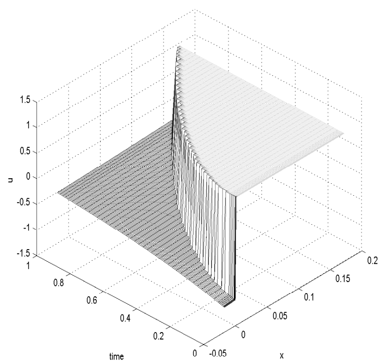

Figure 4: Graph of the approximated shock wave for Ex. 4.2

with the initial condition

$$u(x,0) = \begin{cases} 5x-3 & \text{if } x<0 \\ -5x+10 & \text{if } x \geq 0 \end{cases}$$

The corresponding graphs are shown in Fig. 5 and Fig. 6.

Example 4.4. (Generalized Riemann Problem for a conservation law with polynomial flow of fourth degree)
Consider the conservation law

$$\frac{\partial u}{\partial t} + \left(2u^3 - u^2 - 3u\right)\frac{\partial u}{\partial x} = 0,$$

with the initial condition

$$u(x,0) = \begin{cases} -x-1 & \text{if } x<0 \\ x+2 & \text{if } x \geq 0 \end{cases}$$

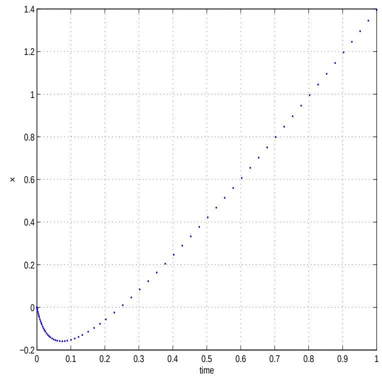

Figure 5: Graph of the singularity trajectory for Ex. 4.3

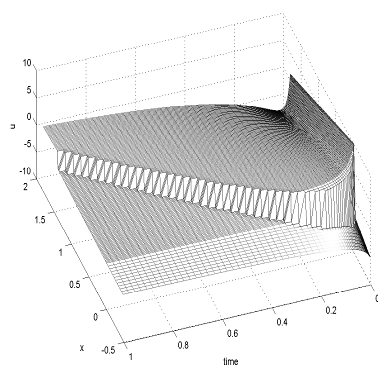

Figure 6: Graph of the approximated shock wave for Ex. 4.3

Fig. 7 shows the singularity trajectory. Fig. 8 and Fig. 9 show the graphs of the solution from two different angles. These graphs show perfectly (as the previous ones) that the wave profile and the magnitude of the jump may have smooth variations with time. This corresponds to the self-similarity of these solutions.

Example 4.5. Let us introduce little variations in the coefficients of the previous conservation law, and also in the initial condition. These assumptions do not have a significant influence on the corresponding graphics, because of the structural stability of the proposed shock wave solution. Consider the equation

$$\frac{\partial u}{\partial t} + \left(2.006u^3 - 0.998u^2 - 3.003u + 0.004\right)\frac{\partial u}{\partial x} = 0,$$

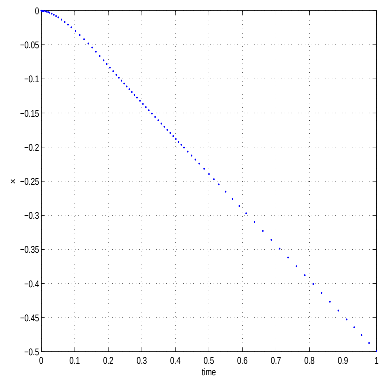

Figure 7: Graph of the singularity trajectory for Ex. 4.4

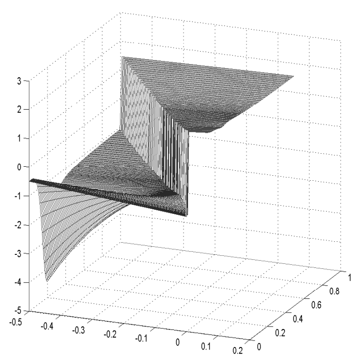

Figure 8: Graph of the approximated shock wave for Ex. 4.4-View 1

with the initial condition

$$u(x,0) = \begin{cases} -0.008x^2 - 1.007x - 1.003 & \text{if } x < 0 \\ 0.005x^2 + x + 1.998 & \text{if } x \geq 0 \end{cases}$$

The corresponding graphs are shown in Fig. 10, Fig. 11 and Fig. 12.

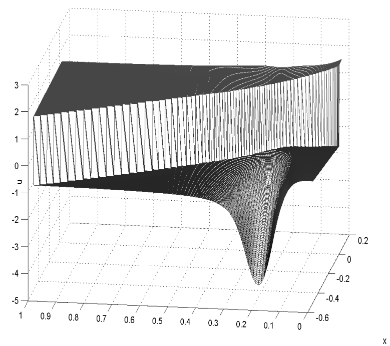

Figure 9: Graphic of the approximated shock wave for Ex. 4.4-View 2

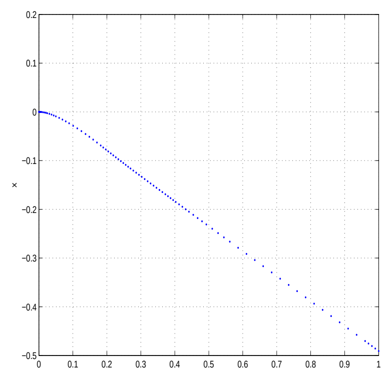

Figure 10: Graph of the singularity trajectory for Ex. 4.5

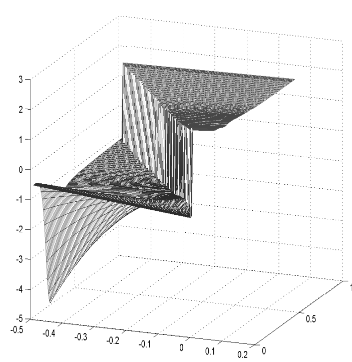

Figure 11: Graph of the approximated shock wave for Ex. 4.5-View 1

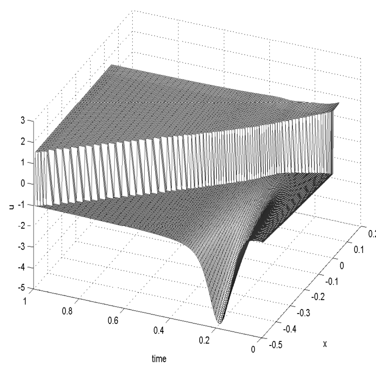

Figure 12: Graph of the approximated shock wave for Ex. 4.5-View 2

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