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A CHARACTERIZATION OF GENERALIZED QUASI-EINSTEIN MANIFOLDS

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Abstract. The aim of this paper is to give a characterisation of generalized quasi-Einstein manifolds in terms of scalar curvatures of subspaces of the tangent space.

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1. Introduction

According to ([2]) we have the following definition.

Definition 1.1. A non-flat Riemannian manifold (M, g), $n > 2$, is said to be a *quasi-Einstein* manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and satisfies the condition $Ric(X, Y) = ag(X, Y) + bA(X)A(Y)$ for every $X, Y \in \Gamma(TM)$, where a, b are real scalars, $b \neq 0$ and A is a non-zero 1-form on M, such that $A(X) = g(X, U)$ for all vector field $X \in \Gamma(TM)$, U being a unit vector field which is called the generator of the manifold.

According to ([4]) we have the following definition.

Definition 1.2. A non-flat Riemannian manifold (M, g), $n > 2$, is said to be a *generalized quasi-Einstein* manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and satisfies the condition $Ric_M(X, Y) = ag(X, Y) + bA(X)A(Y)+cB(X)B(Y)$ for every $X, Y \in \Gamma(TM)$, where a, b, c are real scalars, $b \neq 0, c \neq 0$ and A, B are non-zero 1-form on M such that $A(X) = g(X, U)$, $B(X) = g(X, V)$, $g(U, V) = 0$ for all vector field $X \in \Gamma(TM)$, U, V being unit vector fields which are called the generators of the manifold.

Let M be a Riemannian n-manifold. Let $p \in M$ and $L \subset T_pM$ a subspace of dimension $r \leq n$. Let $\{e_1, ..., e_r\}$ be a basis for L. We will denote by $\tau(L) = \sum_{1\leq i<j\leq r} K(e_i \wedge e_j)$, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane determinated by $\{e_i, e_j\}$. $\tau(L)$ is called the scalar curvature of L. In these conditions, the orthoganal complement of L is the plane spanned by $\{e_{r+1}, ..., e_n\}$ and is denoted by $L^{\perp}$.

We give now some characterisations of Einstein-type manifolds.

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Theorem 1.1. *([6]) Let M be a Riemannian 4-manifold. Then M is an Einstein space if and only if $K(\pi) = K(\pi^\perp)$ for any plane section $\pi \subset T_pM$, where $\pi^\perp$ denotes the orthogonal complement of π in T_pM for every $p \in M$.*

Theorem 1.2. *([3]) Let M be a Riemannian $(2n)$-manifold. Then M is an Einstein space if and only if $\tau(L) = \tau(L^\perp)$ for any n-plane section $L \subset T_pM$, where $L^\perp$ denotes the orthogonal complement of L in T_pM for every $p \in M$.*

Theorem 1.3. *([5]) Let M be a Riemannian $(2n+1)$-manifold. Then M is an Einstein space of constant λ if and only if $\tau(L) + \frac{\lambda}{2} = \tau(L^\perp)$ for any n-plane section $L \subset T_pM$, where $L^\perp$ denotes the orthogonal complement of L in T_pM for every $p \in M$.*

Theorem 1.4. *([1]) Let (M, g) be a Riemannian $(2n+1)$-manifold with $n \geq 2$. Then M is quasi-Einstein if and only if the Ricci operator Ric has an eigenvector ξ such that at any $p \in M$, there exist two real numbers a, b satisfying $\tau(P) + a = \tau(P^\perp)$ and $\tau(N) + b = \tau(N^\perp)$ for any n-plane section P and $(n+1)$-plane section N, both orthogonal to ξ in T_pM, where $P^\perp$ and $N^\perp$ denote respectively the orthogonal complement of P and N in T_pM.*

Theorem 1.5. *([1]) Let (M, g) be a Riemannian $(2n)$-manifold with $n \geq 2$. Then M is quasi-Einstein if and only if the Ricci operator Ric has an eigenvector ξ such that at any $p \in M$, there exists the real number c satisfying $\tau(P) + c = \tau(P^\perp)$ for any n-plane section P orthogonal to ξ in T_pM, where $P^\perp$ denotes the orthogonal complement of P in T_pM.*

2. Main results

The aim of this paper is to extend in some sense the results from ([1, 3, 5, 6]) to the case when the ambient space is a generalized quasi-Einstein space. Thus, we will give a characterization of generalized quasi-Einstein manifolds in terms of scalar curvatures of n-planes included in the tangent space.

Theorem 2.1. *Let M be a Riemannian $(2n+1)$-manifold, $n \geq 2$. Then the following conditions are equivalent:*

1) M is a generalized quasi-Einstein manifold with $Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$ for every $X, Y \in \Gamma(TM)$, where a, b, c are real scalars and A, B are non-zero 1-forms on M such that $A(X) = g(X, U)$, $B(X) = g(X, V)$, $g(U, V) = 0$ for all vector field $X \in \Gamma(TM)$, U, V being unit vector fields.

2) a) $\tau(L_1^\perp) = \tau(L_1) + \frac{1}{2}(a + b + c)$ for any n-plane section $L_1 \subset T_pM$ such that $U, V \notin L_1$,

b) $\tau(L_2^\perp) = \tau(L_2) + \frac{1}{2}(a - b - c)$ for any n-plane section $L_2 \subset T_pM$ such that $U, V \in L_2$,

c) $\tau(L_3^\perp) = \tau(L_3) + \frac{1}{2}(a + b - c)$ for any n-plane section $L_3 \subset T_pM$ such that $U \notin L_1, V \in L_3$,

d) $\tau(L_4^\perp) = \tau(L_4) + \frac{1}{2}(a - b + c)$ for any n-plane section $L_4 \subset T_pM$ such that $U \in L_4, V \notin L_4$,

where $L^\perp$ denotes the orthogonal complement of L in T_pM for every $p \in M$.

Proof. "1) $\Rightarrow$ 2)". Let $p \in M$ and $\{e_1, .., e_n, ..., e_{2n+1}\}$ an orthonormal frame of T_pM such that $U = e_1$ and $V = e_2$. We know that

$$Ric(X,Y) = \sum_{i=1}^{2n+1} R(X, e_i, Y, e_i) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$$

for every $X, Y \in \Gamma(TM)$. Let $X = Y = e_i$. This implies that $Ric(e_i) = Ric(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_i, e_j, e_i, e_j) = a$ for every $i \in \{3, ..., 2n+1\}$. In the same way we obtain that $Ric(U) = Ric(e_1) = a + b$ and $Ric(V) = Ric(e_2) = a + c$.

We will write now all the equations and by the formula of Ricci cuvature, we will have the following system of $2n+1$ equations:

$$\begin{array}{rl}
1) & Ric(e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \ldots + K(e_1 \wedge e_{2n+1}) = a + b \\
2) & Ric(e_2) = K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + \ldots + K(e_2 \wedge e_{2n+1}) = a + c \\
3) & Ric(e_3) = K(e_3 \wedge e_1) + K(e_3 \wedge e_2) + \ldots + K(e_3 \wedge e_{2n+1}) = a \\
 & \vdots \\
2n{+}1) & Ric(e_{2n+1}) = K(e_{2n+1} \wedge e_1) + K(e_{2n+1} \wedge e_2) + \ldots + K(e_{2n+1} \wedge e_{2n}) = a
\end{array}$$

Without any loss of generality we will consider the following n-planes from T_pM :

$$L_1 = Sp(\{e_3, e_4, ..., e_{n+2}\}), L_2 = Sp(\{e_1, e_2, ..., e_n\}),$$
$$L_3 = Sp(\{e_2, e_3, ..., e_{n+1}\}), L_4 = Sp(\{e_1, e_3, e_4, ..., e_{n+1}\}).$$

Now, by summing the first n equations we have the following relation:

$$(i) \qquad 2\tau(L_2) + \sum_{1 \le i \le n < j \le 2n+1} K(e_i \wedge e_j) = na + b + c$$

By summing the last $n+1$ equations we have another relation:

$$(ii) \qquad 2\tau(L_2^{\perp}) + \sum_{1 \le j \le n < i \le 2n+1} K(e_i \wedge e_j) = (n+1)a$$

Then $(ii) - (i)$ implies:

$$\begin{aligned}
a - b - c =& 2\tau(L_2^{\perp}) - 2\tau(L_2) + \sum_{1 \le j \le n < i \le 2n+1} K(e_i \wedge e_j) - \sum_{1 \le i \le n < j \le 2n+1} K(e_i \wedge e_j) \\
\Rightarrow& \tau(L_2^{\perp}) = \tau(L_2) + \frac{1}{2}(a - b - c).
\end{aligned}$$

In a similar way, by summing the equations from 3) to n+2) we have:

$$(iii) \qquad 2\tau(L_1) + \sum_{3\le i\le n+2<j\le 2n+1} K(e_i\wedge e_j) + \sum_{3\le i\le n+2,\ j\in\{1,2\}} K(e_i\wedge e_j) = na$$

Also, by summing the remaining equations we have:

$$(iv) \qquad 2\tau(L_1^{\perp}) + \sum_{3\le i\le n+2<j\le 2n+1} K(e_i\wedge e_j) + \sum_{3\le i\le n+2,\ j\in\{1,2\}} K(e_i\wedge e_j) = (n+1)a + b + c$$

Then $(iv) - (iii)$ implies:

$$a+b+c = 2\tau(L_1^{\perp}) - 2\tau(L_1) \Rightarrow \tau(L_1^{\perp}) = \tau(L_1) + \frac{1}{2}(a+b+c).$$

In a similar way, by summing the equations from 2) to n+1) we have:

$$(v) \qquad 2\tau(L_3) + \sum_{2\le i\le n+1<j\le 2n+1} K(e_i\wedge e_j) + \sum_{2\le i\le n+1} K(e_i\wedge e_1) = na + c$$

Also, by summing the remaining equations we have:

$$(vi) \qquad 2\tau(L_3^{\perp}) + \sum_{2\le i\le n+1<j\le 2n+1} K(e_i\wedge e_j) + \sum_{2\le i\le n+1} K(e_i\wedge e_1) = (n+1)a + b$$

Then $(vi) - (v)$ implies:

$$a+b-c = 2\tau(L_3^{\perp}) - 2\tau(L_3) \Rightarrow \tau(L_3^{\perp}) = \tau(L_3) + \frac{1}{2}(a+b-c).$$

In a similar way, by summing the equation 1) with all the equations from 3) to n+1) we have:

$$(vii) \qquad 2\tau(L_4) + \sum_{1\le i\le n+1<j\le 2n+1} K(e_i\wedge e_j) - \sum_{1\le i\le n+1} K(e_i\wedge e_2) = na + b$$

Also, by summing the remaining equations we have:

$$(viii) \qquad 2\tau(L_4^{\perp}) + \sum_{1\le i\le n+1<j\le 2n+1} K(e_i\wedge e_j) - \sum_{1\le i\le n+1} K(e_i\wedge e_2) = (n+1)a + c$$

Then $(viii) - (vii)$ implies:

$$a-b+c = 2\tau(L_4^{\perp}) - 2\tau(L_4) \Rightarrow \tau(L_4^{\perp}) = \tau(L_4) + \tfrac{1}{2}(a-b+c).$$

"2) $\Leftarrow$ 1)". Let $p \in M$ and $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n+1}\}$ be an orthonormal frame of T_pM such that $U = e_1$ and $V = e_2$.

Let $L = Sp(\{e_{n+2}, ..., e_{2n+1}\})$ and $L_0 = Sp(\{e_2, ..., e_{n+1}\})$.

Then $L^{\perp} = Sp(\{e_1, ..., e_{n+1}\})$ and $L_0^{\perp} = Sp(\{e_1, e_{n+2}, ..., e_{2n+1}\})$. Thus:

$$\begin{aligned} Ric(e_1) &= [K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \ldots + K(e_1 \wedge e_{n+1})] + \\ &\quad +[K(e_1 \wedge e_{n+2}) + \ldots + K(e_1 \wedge e_{2n+1})] \\ &= [\tau(L^{\perp}) - \sum_{2\leq i<j\leq n+1} K(e_i \wedge e_j)] + [\tau(L_0^{\perp}) - \sum_{n+2\leq i<j\leq 2n+1} K(e_i \wedge e_j)] \\ &= [\tau(L) + \frac{1}{2}(a+b+c) - \tau(L_0)] + [\tau(L_0) + \frac{1}{2}(a+b-c) - \tau(L)] \\ &= a + b. \end{aligned}$$

In a similar way one can prove that $Ric(e_2)= a + c$ and $Ric(e_i) = a$ for all $i \in \{3, ..., 2n+1\}$. We define now the 1-forms $A(X) = g(X, U)$, $B(X) = g(X, V)$ such that $A(U) = B(V) = 1$ and we consider the following $(0, 2)$-tensor $P(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$. Then $Ric(X, X) = P(X, X)$ for every $X \in \Gamma(TM)$. Because the tensors Ric and P are symmetric, it follows that $Ric(X, Y) = P(X, Y)$ for every $X, Y \in \Gamma(TM)$ and then M is a generalized quasi-Einstein manifold. □

We give the particular version for dimension three.

Theorem 2.2. *Let M be a Riemannian 3-manifold. Then the following conditions are equivalent:*

1) M is a generalized quasi-Einstein manifold with $Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$ for every $X, Y \in \Gamma(TM)$, where a, b, c are real scalars and A, B are non-zero 1-forms on M such that $A(X) = g(X, U)$, $B(X) = g(X, V)$, $g(U, V) = 0$ for all vector field $X \in \Gamma(TM)$, U, V being unit vector fields.

2) a) $\tau(L_1) = \frac{1}{2}(a + b + c)$, where $L_1 = Sp(\{U, V\})$,

b) $\tau(L_2) = \frac{1}{2}(a + b - c)$, where L_2 is a 2-plane section orthogonal to V,

c) $\tau(L_3) = \frac{1}{2}(a - b + c)$, where L_3 is a 2-plane section orthogonal to U.

Proof. Similar to that of Theorem 2.1. □

We can state now the even dimension version of Theorem 2.1. from above.

Theorem 2.3. *Let M be a Riemannian $(2n)$-manifold, $n \geq 2$. Then the following conditions are equivalent:*

1) M is a generalized quasi-Einstein manifold with $Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$ for every $X, Y \in \Gamma(TM)$ where a, b, c are real scalars and A, B are non-zero 1-forms on M such that $A(X) = g(X, U)$, $B(X) = g(X, V)$, $g(U, V) = 0$ for all vector field $X \in \Gamma(TM)$, U, V being unit vector fields.

2) a) $\tau(L_1^\perp) = \tau(L_1) + \frac{1}{2}(b+c)$ *for any n-plane section* $L_1 \subset T_pM$ *such that* $U, V \notin L_1$,

b) $\tau(L_2^\perp) = \tau(L_2) - \frac{1}{2}(b+c)$ *for any n-plane section* $L_2 \subset T_pM$ *such that* $U, V \in L_2$,

c) $\tau(L_3^\perp) = \tau(L_3) + \frac{1}{2}(b-c)$ *for any n-plane section* $L_3 \subset T_pM$ *such that* $U \notin L_1, V \in L_3$,

d) $\tau(L_4^\perp) = \tau(L_4) + \frac{1}{2}(-b+c)$ *for any n-plane section* $L_4 \subset T_pM$ *such that* $U \in L_4, V \notin L_4$,

where $L^\perp$ *denotes the orthogonal complement of* L *in* T_pM *for every* $p \in M$.

Proof. Similar to that of Theorem 2.1. □

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