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ON CONCIRCULARLY $\phi-$RECURRENT PARA-SASAKIAN MANIFOLDS

Amit Prakash[1]

Abstract. The present paper deals with the study of concircularly $\phi-$recurrent para-Sasakian manifolds.

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1. Introduction

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus, the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curavture tensor.

The notion of local symmetry of a Riemannian manifold has been studied by many authors ([6], [7]) in several ways and to a different extent. As a weaker version of local symmetry, in 1977, Takahashi [9] introduced the notion of locally $\phi-$symmetric Sasakian manifold and obtained their several interesting results. Later in 2009, De, Yildiz and Yaliniz [5] studied $\phi-$recurrent Kenmotsu manifold and obtained some interesting results too. In this paper we study a concircularly $\phi-$recurrent para-Sasakian manifold which generalizes the notion of locally concircular $\phi-$symmetric para-Sasakian manifold. Again, it is proved that a concircularly $\phi-$recurrent para-Sasakian manifold is an Einstein manifold and in a concircularly $\phi-$ recurrent para-Sasakian manifold, the characteristic vector field ξ and the vector field ρ associated to the $1-$form A are co-directional. Finally, we proved that a three-dimensional locally concircularly $\phi-$recurrent para-Sasakian manifold is of constant curvature.

2. Preliminaries

Let $M^n(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold, where ϕ is a $(1, 1)$ tensor field, ξ is the structure vector field, η is a $1-$form and g is the

[1]Department of Mathematics, Allenhouse Institute of Technology, Rooma, Kanpur, U.P. India, e-mail: apoct0185@rediffmail.com

Riemannian metric. It is well known that the structure (ϕ, ξ, η, g) satisfy

$$\phi^2 X = X - \eta(X)\xi, \tag{2.1}$$

$$(a)\ \eta(\xi) = 1, (b)\ g(X, \xi) = \eta(X),\ \ (c)\ \eta(\phi X) = 0, (d)\ \phi\xi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(D_X\phi)(Y) = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

$$(a)\ D_X\xi = \phi(X), (b)\ (D_X\eta)(Y) = g(\phi X, Y), (c)\ d\eta = 0. \tag{2.5}$$

for all vector fields X, Y, Z, where D denotes the operator of covariant differentiation with respect to g, then $M^n(\phi, \xi, \eta, g)$ is called a para-Sasakian manifold or briefly a P-Sasakian manifold [1],[8]. In particular a para-Sasakian manifold M is called a Special para-Sasakian manifold or briefly SP-Sasakian manifold if M admitts a 1-form η satisfying

$$(D_X\eta)(Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.6}$$

In a para-Sasakian manifold, the following relations hold:([1], [2], [3])

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \tag{2.9}$$

$$S(X, \xi) = -(n - 1)\eta(X), \tag{2.10}$$

for all vector fields X, Y, Z, where S is the Ricci tensor of type $(0, 2)$ and R is the Riemannian curvature tensor of the manifold.

A para-Sasakian manifold is said to be Einstein manifold if the Ricci tensor S is of the form

$$S(X, Y) = \lambda g(X, Y),$$

where λ is a constant.

Definition 2.1. *A para-Sasakian manifold is said to be a locally $\phi-$symmetric manifold if [9]*

$$\phi^2((D_W R)(X, Y)Z) = 0, \tag{2.11}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2. *A para-Sasakian manifold is said to be a locally concircularly $\phi-$symmetric manifold if*

$$\phi^2((D_W C)(X,Y)Z) = 0, \tag{2.12}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.3. *A para-Sasakian manifold is said to be concircularly $\phi-$recurrent para-Sasakian manifold if there exists a non-zero 1-form A such that*

$$\phi^2((D_W C)(X,Y)Z) = A(W)C(X,Y)Z, \tag{2.13}$$

for arbitrary vector fields X, Y, Z, W, where C is a concircular curvature tensor given by [10]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.14}$$

where R is the Riemann curvature tensor and r is the scalar curvature.

If the 1-form A vanishes, then the manifold reduces to a locally concircularly $\phi-$symmetric manifold.

3. Concircularly $\phi-$recurrent para-Sasakian manifold

Let us consider a concircularly $\phi-$recurrent para-Sasakian manifold. Then, by virtue of (2.1) and (2.13), we get

$$(D_W C)(X,Y)Z - \eta((D_W C)(X,Y)Z)\xi = A(W)C(X,Y)Z, \tag{3.1}$$

from which it follows that

$$g((D_W C)(X,Y)Z,U) - \eta((D_W C)(X,Y)Z)\eta(U) = A(W)g(C(X,Y)Z,U). \tag{3.2}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then, putting $X = U = e_i$ in (3.2) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}(D_W S)(Y,Z) &= \frac{dr(W)}{n}g(Y,Z) + \frac{dr(W)}{n(n-1)}[g(Y,Z) - \eta(Y)\eta(Z)] \\ &+ A(W)[S(Y,Z) - \frac{r}{n}g(Y,Z)].\end{aligned} \tag{3.3}$$

Replacing Z by ξ in (3.3) and using (2.5) and (2.10), we get

$$(D_W S)(Y,\xi) = \frac{dr(W)}{n}\eta(Y) - A(W)[\frac{r}{n} + (n-1)]\eta(Y). \tag{3.4}$$

Now we have

$$(D_W S)(Y,\xi) = D_W S(Y,\xi) - S(D_W Y,\xi) - S(Y, D_W \xi).$$

Using (2.5), (2.6) and (2.10) in the above relation, it follows that

$$(D_W S)(Y,\xi) = -(n-1)g(Y,\phi W) - S(Y,\phi W). \tag{3.5}$$

In view of (3.4) and (3.5), we get

$$S(Y,\phi W) = -(n-1)g(Y,\phi W) - \frac{dr(W)}{n}\eta(Y) + A(W)[\frac{r}{n} + (n-1)]\eta(Y). \tag{3.6}$$

Replacing Y by ϕY in (3.6), we get

$$S(\phi Y,\phi W) = -(n-1)g(\phi Y,\phi W). \tag{3.7}$$

Using (2.3) and (2.9) in (3.7), we get

$$S(Y,W) = -(n-1)g(Y,W),$$

for all Y, W.

Hence, we can state the following theorem:

Theorem 3.1. *A concircularly $\phi-$recurrent para-Sasakian manifold (M^n, g) is an Einstein manifold.*

Using (2.14) in (3.1), we get

$$\begin{aligned}(D_W R)(X,Y)Z &= \eta((D_W R)(X,Y)Z)\xi + A(W)R(X,Y)Z \\ &-\frac{dr(W)}{n(n-1)}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \\ &+\frac{dr(W)}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \\ &-\frac{r}{n(n-1)}A(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.8}$$

From (3.8) and the Bianchi identity, we get

$$\begin{aligned}&A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \\ &= \frac{r}{n(n-1)}A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+\frac{r}{n(n-1)}A(X)[g(Z,W)\eta(Y) - g(Y,Z)\eta(W)] \\ &+\frac{r}{n(n-1)}A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)].\end{aligned} \tag{3.9}$$

Putting $Y = Z = e_i$ in (3.9) and taking summation over i, $1 \leq i \leq n$, we get

$$A(W)\eta(X) = A(X)\eta(W), \tag{3.10}$$

for all vector fields X, W. Replacing X by ξ in (3.10), we get

$$A(W) = \eta(W)\eta(\rho), \tag{3.11}$$

for any vector field W, where $A(\xi) = g(\xi,\rho) = \eta(\rho), \rho$ being the vector field associated to the $1-$form A i.e., $A(X) = g(X,\rho)$. From (3.10) and (3.11), we can state the following theorem:

Theorem 3.2. *In a concircularly $\phi-$ recurrent para-Sasakian manifold (M^n, g) $(n \geq 3)$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional, and the 1-form A is given by (3.11).*

4. On 3-dimensional locally concircularly ϕ-recurrent para-Sasakian manifolds

It is known that in a three-dimensional para-Sasakian manifold the curvature tensor has the following form [4]

$$\begin{aligned} R(X,Y)Z &= (\frac{r+4}{2})[g(Y,Z)X - g(X,Z)Y] \\ &\quad -\frac{(r+6)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.1}$$

Taking covariant differentiation of (4.1), we get

$$\begin{aligned} (D_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &\quad -\frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &\quad -\frac{r+6}{2}[g(Y,Z)(D_W\eta)(X)\xi + g(Y,Z)\eta(X)(D_W\xi) \\ &\quad -g(X,Z)(D_W\eta)(Y)\xi - g(X,Z)\eta(Y)(D_W\xi) \\ &\quad +(D_W\eta)(Y)\eta(Z)X + (D_W\eta)(Z)\eta(Y)X \\ &\quad -(D_W\eta)(X)\eta(Z)Y - (D_W\eta)(Z)\eta(X)Y]. \end{aligned} \tag{4.2}$$

Taking X, Y, Z, W orthogonal to ξ and using (2.5) and (2.6), we get

$$\begin{aligned} (D_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &\quad -\frac{r+6}{2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\xi. \end{aligned} \tag{4.3}$$

From (4.3) it follows that

$$\phi^2(D_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y]. \tag{4.4}$$

Now, taking X, Y, Z, W orthogonal to ξ and using (2.1) and (2.2) in (4.4), we get

$$\phi^2(D_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.5}$$

Differentiating covariantly (2.14) with respect to W (for n=3), we get

$$(D_W C)(X,Y)Z = (D_W R)(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{4.6}$$

Now, applying ϕ^2 to the both sides of (4.6), we get

$$\begin{aligned} &\phi^2(D_W C)(X,Y)Z = \\ &\quad \phi^2(D_W R)(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y]. \end{aligned} \tag{4.7}$$

Using (2.13), (4.5), (2.1) in (4.7), we obtain

$$\begin{aligned} A(W)C(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &\quad -\frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y \\ &\quad +g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]. \end{aligned} \tag{4.8}$$

Taking X, Y, Z, W orthogonal to ξ, we get

$$C(X,Y)Z = \frac{dr(W)}{3A(W)}[g(Y,Z)X - g(X,Z)Y]. \tag{4.9}$$

Putting W=$\{e_i\}$ in (4.9), where $\{e_i\}$, i=1, 2, 3 is an orthonormal basis of the tangent space at any point of the manifold, and taking summation over i, $1 \le i \le 3$, we obtain

$$C(X,Y)Z = \frac{dr(e_i)}{3A(e_i)}[g(Y,Z)X - g(X,Z)Y]. \tag{4.10}$$

Using (2.14) in (4.10), we get

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y], \tag{4.11}$$

where $\lambda = [\frac{r}{6} + \frac{dr(e_i)}{3A(e_i)}]$ is a scalar, since A is a non-zero 1-form. Then, by Schur's theorem λ will be a constant on the manifold. Therefore, M^3 is of constant curvatare λ.

Hence, we can state the following theorem:

Theorem 4.1. *A 3-dimensional locally concircularly ϕ-recurrent para-Sasakian manifold is of constant curvature.*

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