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ON WEAKLY $m-$PROJECTIVELY SYMMETRIC MANIFOLDS

S. K. Chaubey[1]

Abstract. The present paper deals with the study of weakly $m-$projectively symmetric and $m-$projectively flat weakly Ricci-symmetric manifolds. In the end, examples of $(WMPS)_n$ and $(WRS)_n$ are given.

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1. Introduction

The notion of weakly symmetric Riemannian manifolds have been introduced by Tamassy and Binh [13] in 1989. A non-flat Riemannian manifold (M_n, g) $(n > 2)$ is called weakly-symmetric manifold if its curvature tensor $'K$ of type $(0, 4)$ satisfies the relation

$$(D_X{}'K)(Y,Z,U,V) = A(X)'K(Y,Z,U,V) + B(Y)'K(X,Z,U,V)$$
$$+C(Z)'K(Y,X,U,V) + D(U)'K(Y,Z,X,V) + E(V)'K(Y,Z,U,X), \tag{1.1}$$

for arbitrary vector fields $X, Y, Z, U, V \in \chi(M_n)$, where D denotes the operator of covariant differentiation with respect to the Riemannian metric g and A, B, C, D and E are $1-$forms (not simultaneously zero). The $1-$forms are called the associated $1-$forms of the manifold and an $n-$dimensional manifold of this kind is denoted by $(WS)_n$. Tamassy and Binh [14] further studied weakly symmetric Sasakian manifolds and proved that such manifold does not always exits. In [3] the authors established the existence of $(WS)_n$ by an example and proved that in $(WS)_n$, the associated $1-$forms $B = C$ and $D = E$. So (1.1) reduces to the following form

$$(D_X{}'K)(Y,Z,U,V) = A(X)'K(Y,Z,U,V) + B(Y)'K(X,Z,U,V)$$
$$+B(Z)'K(Y,X,U,V) + D(U)'K(Y,Z,X,V) + D(V)'K(Y,Z,U,X). \tag{1.2}$$

Some authors, like De and Bandyopadhyay [4], Shaikh and Baishya [12] extended this notion for conformal curvature tensor, quasi-conformal curvature tensor respectively. Recently, Malek and Samavaki [8] have also studied weakly symmetric Riemannian manifolds.

[1]Department of Mathematics, GLA University, Mathura, India, e-mail: sk22_math@yahoo.co.in

In 1971, Pokhariyal and Mishra [11] defined a tensor field W^* on a Riemannian manifold as

$$W^*(X,Y)Z = K(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{1.3}$$

so that

$$'W^*(X,Y,Z,U) \overset{\text{def}}{=} g(W^*(X,Y)Z,U) =' W^*(Z,U,X,Y)$$

and

$$'W^*_{ijkl}w^{ij}w^{kl} =' W_{ijkl}w^{ij}w^{kl},$$

where $'W^*_{ijkl}$ and $'W_{ijkl}$ are components of $'W^*$ and $'W$, w^{kl} is a skew-symmetric tensor [15], [10], Q is the Ricci operator, defined by

$$S(X,Y) \overset{\text{def}}{=} g(QX,Y) \tag{1.4}$$

and S is the Ricci tensor for arbitrary vector fields X, Y, Z. Such a tensor field W^* is known as $m-$projective curvature tensor. Ojha [9], [10] defined and studied the properties of $m-$projective curvature tensor in Sasakian and Kähler manifolds. He has also shown that it bridges the gap between conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one side and $H-$projective curvature tensor on the other.

A non $m-$projectively flat Riemannian manifold (M_n, g) $(n > 2)$ is said to be weakly $m-$projectively symmetric manifold if the $m-$projective curvature tensor $'W^*$ of type $(0,4)$ satisfies the relation

$$(D_X{'W^*})(Y,Z,U,V) = A(X){'W^*}(Y,Z,U,V) + B(Y){'W^*}(X,Z,U,V) + C(Z){'W^*}(Y,X,U,V) + D(U){'W^*}(Y,Z,X,V) + E(V){'W^*}(Y,Z,U,X), \tag{1.5}$$

for all vector fields X, Y, Z, U, $V \in \chi(M_n)$, where A, B, C, D are E are defined as above. Such an $n-$dimensional manifold is denoted by $(WMPS)_n$. If $B = C$, $D = E$ and hence (1.5) reduces to the form

$$(D_X{'W^*})(Y,Z,U,V) = A(X){'W^*}(Y,Z,U,V) + B(Y){'W^*}(X,Z,U,V) + B(Z){'W^*}(Y,X,U,V) + D(U){'W^*}(Y,Z,X,V) + D(V){'W^*}(Y,Z,U,X), \tag{1.6}$$

for arbitrary vector fields X, Y, Z, U, $V \in \chi(M_n)$ and non-vanishing $1-$forms A, B and D.

This paper is organized as follows. Section 2 is equipped with some prerequisites about $m-$projective curvature tensor and $(WMPS)_n$. In section 3, we study the nature of scalar curvature of $(WMPS)_n$ and prove that the Ricci tensor S in $(WMPS)_n$ has an eigen value $\frac{r}{n}$ corresponding to the eigen vector $\tilde{\rho}$. The next section is devoted to the study of $m-$projectively flat Ricci-symmetric manifolds and it proves that an $m-$projectively flat $(WMPS)_n$ $(n > 3)$ has a proper concircular vector field. In the last section, we construct some examples of $(WMPS)_4$ and $(WRS)_4$ which support the existence of $(WMPS)_4$ and $(WRS)_4$.

2. Preliminaries

In this section, we obtain some formulas which will be useful to the study of a $(WMPS)_n$. Let us suppose that $\{e_i\}$, $i = 1, 2, \ldots\ldots, n$, be an orthonormal basis of the tangent space at any point of the manifold, then equation (1.3) gives

$$\sum_{i=1}^{n} {}'W^*(e_i, Y, Z, e_i) = \frac{n}{2(n-1)} W(Y, Z), \tag{2.1}$$

where

$$W(Y, Z) \stackrel{\text{def}}{=} S(Y, Z) - \frac{r}{n} g(Y, Z,), \tag{2.2}$$

$$\sum_{i=1}^{n} {}'W^*(X, Y, e_i, e_i) = 0, \tag{2.3}$$

and

$$\sum_{i=1}^{n} W(e_i, e_i) = 0. \tag{2.4}$$

Proposition 1. *In a Riemannian manifold (M_n, g) $(n > 2)$, the m−projective curvature tensor satisfies the following relations*

$$\begin{aligned} &(i) \quad {}'W^*(X, Y, Z, U) + {}'W^*(Y, Z, X, U) + {}'W^*(Z, X, Y, U) = 0, \\ &(ii) \quad {}'W^*(X, Y, U, Z) + {}'W^*(Y, Z, U, X) + {}'W^*(Z, X, U, Y) = 0. \end{aligned} \tag{2.5}$$

Proposition 2. *The defining condition of $(WMPS)_n$ can always be expressed in the form (1.6).*

Proof. Interchanging Y and Z in (1.5), we get

$$\begin{aligned} (D_X{}'W^*)(Z, Y, U, V) &= A(X){}'W^*(Z, Y, U, V) + B(Z){}'W^*(X, Y, U, V) \\ &+ C(Y){}'W^*(Z, X, U, V) + D(U){}'W^*(Z, Y, X, V) + E(V){}'W^*(Z, Y, U, X). \end{aligned} \tag{2.6}$$

Adding (1.5) and (2.6) and then using skew-symmetric properties of ${}'W^*$, we get

$$\mu(Y){}'W^*(X, Z, U, V) + \mu(Z){}'W^*(X, Y, U, V) = 0, \tag{2.7}$$

where $\mu(Y) = B(Y) - C(Y)$, for all $Y \in \chi(M_n)$.

Now we choose a particular vector field ρ such that $\mu(\rho) \neq 0$. Substituting $Y = Z = \rho$ in (2.7), we get ${}'W^*(X, \rho, U, V) = 0$. Again, replacing ρ for Z in (2.7), we obtain ${}'W^*(X, Y, U, V) = 0$, for all vector fields X, Y, U, V $\in \chi(M_n)$, which is inadmissible because in our assumption the manifold is not $m-$projectively flat. Hence we must have $\mu(X) = 0$, for all $X \in \chi(M_n)$ and thus $B = C$. In the same fashion, by interchanging U and V in (1.5) and proceeding as above, we can easily see that $D = E$. Thus, all the associated $1-$forms A, B, C, D and E coincide because $B = C$, $D = E$. Therefore (1.5) can be written as (1.6). □

3. The nature of the scalar curvature of $(WMPS)_n$

Let Q be the symmetric endomorphism of the tangent bundle of the manifold corresponding to the Ricci tensor S, *i.e.*, $S(X,Y) = g(QX,Y)$ for all vector fields X, $Y \in \chi(M_n)$.

Theorem 3.1. *The Ricci tensor S in an $n-$dimensional Riemannian manifold (M_n, g) $(n > 2)$ is Codazzi type if and only if the relation (3.3) holds.*

Proof. Covariant differentiation of (1.3) along X with Bianchi identity gives

$$\begin{aligned}(D_X{'W^*})(Y,Z,U,V) &+ (D_Y{'W^*})(Z,X,U,V) + (D_Z{'W^*})(X,Y,U,V)\\ &= -\frac{1}{2(n-1)}[\{(D_XS)(Z,U) - (D_ZS)(X,U)\}\,g(Y,V)\\ &+ \{(D_YS)(X,U) - (D_XS)(Y,U)\}\,g(Z,V)\\ &+ \{(D_ZS)(Y,U) - (D_YS)(Z,U)\}\,g(X,V)\\ &+ \{(D_XS)(Y,V) - (D_YS)(X,V)\}\,g(Z,U)\\ &+ \{(D_ZS)(X,V) - (D_XS)(Z,V)\}\,g(Y,U)\\ &+ \{(D_YS)(Z,V) - (D_ZS)(Y,V)\}\,g(X,U)]. \qquad (3.1)\end{aligned}$$

If the Ricci tensor is of Codazzi type [5], i. e.

$$(D_XS)(Y,Z) = (D_YS)(X,Z), \qquad (3.2)$$

then, in consequence of (3.1) and (3.2), we get

$$(D_X{'W^*})(Y,Z,U,V)+(D_Y{'W^*})(Z,X,U,V)+(D_Z{'W^*})(X,Y,U,V) = 0. \qquad (3.3)$$

The converse part is obvious from (3.2) and (3.3). □

Now we suppose that the Ricci tensor is of Codazzi type, so by virtue of (1.6), (2.5) and (3.1), we have

$$\lambda(X){'W^*}(Y,Z,U,V) + \lambda(Y){'W^*}(Z,X,U,V) + \lambda(Z){'W^*}(X,Y,U,V) = 0, \qquad (3.4)$$

where $\lambda(X) = A(X) - 2B(X)$, for all $X \in \chi(M_n)$. Putting $Y = V = e_i$ in (17) and then taking summation over i, $1 \le i \le n$, we obtain by virtue of (7) that

$$\frac{n}{2(n-1)}\{\lambda(X)W(Z,U) - \lambda(Z)W(X,U)\} + \lambda(W^*(Z,X)U) = 0. \qquad (3.5)$$

Again, putting $X = U = e_i$ in (18) and then taking summation over i, $1 \le i \le n$ and then using (2.2) and (2.4), we obtain

$$\lambda(QZ) = \frac{r}{n}\lambda(Z), \qquad (3.6)$$

which gives

$$S(Z,P) = \frac{r}{n}g(Z,P). \qquad (3.7)$$

Hence we state the following theorem.

Theorem 3.2. *If in a* $(WMPS)_n$ *the Ricci tensor is of Codazzi type,* $\frac{r}{n}$ *is an eigen value of the Ricci tensor* S *corresponding to the eigen vector* P *defined by* $g(X,P) = \lambda(X)$*, for all* X*.*

Next, substituting e_i for Y and V in (1.6) and then taking summation over i, $1 \le i \le n$, we get by virtue of (2.1) that

$$(3.8)\quad \begin{aligned}(D_X W)(Z,U) &= \{A(X)W(Z,U) + B(Z)W(X,U) + D(U)W(Z,X)\}\\ &+ \frac{2(n-1)}{n}\{B(W^*(X,Z)U) + D(W^*(Z,U)X)\}.\end{aligned}$$

Let ρ_1, ρ_2, ρ_3 be the unit vector fields associated to the 1−forms A, B and D respectively, i. e.

$$A(X) = g(X,\rho_1), \quad B(X) = g(X,\rho_2) \quad \textit{and} \quad D(X) = g(X,\rho_3).$$

Putting $Z = U = e_i$ in (3.8) and then taking summation over i, $1 \le i \le n$, we get by virtue of (2.1) and (2.4)

$$(3.9)\qquad W(X,\rho_2) + W(X,\rho_3) = 0,$$

which gives by (2.2) that

$$(3.10)\qquad S(X,\rho_2) + S(X,\rho_3) = \frac{r}{n}\{g(X,\rho_2) + g(X,\rho_3)\}.$$

In view of (3.10), we have

$$(3.11)\qquad S(X,\tilde{\rho}) = \frac{r}{n}g(X,\tilde{\rho}),$$

where $g(X,\tilde{\rho}) = T(X) = B(X) + D(X)$, $\tilde{\rho} = \rho_2 + \rho_3$. From (3.11), it is clear that $\frac{r}{n}$ is an eigen value of S corresponding to the eigen vector $\tilde{\rho}$. Thus we state the following theorem.

Theorem 3.3. *The Ricci tensor* S *in* $(WMPS)_n$ *has eigen value* $\frac{r}{n}$ *corresponding to the eigen vector* $\tilde{\rho}$*.*

If the scalar curvature r of $(WMPS)_n$ is zero, then equation (3.11) gives $S(X,\tilde{\rho}) = 0$ and hence by virtue of (1.3), we obtain

$$(3.12)\qquad 'W^*(X,Y,\tilde{\rho},U) =' K(X,Y,\tilde{\rho},U) - \frac{1}{2(n-1)}\{S(X,U)g(Y,\tilde{\rho}) - S(Y,U)g(X,\tilde{\rho})\}.$$

Also, if (3.12) holds in $(WMPS)_n$, then by virtue of (3.11) it follows from (1.3) that $r = 0$ for $T(X) \neq 0$ for all $X \in \chi(M_n)$. Thus we have the following corollary.

Corollary 3.1. *If the scalar curvature of a* $(WMPS)_n$ *vanishes, then the relation (3.12) holds.*

Again, if the Ricci tensor S of $(WMPS)_n$ is zero, then (1.3) gives

$$'W^*(X,Y,Z,U) =' K(X,Y,Z,U). \tag{3.13}$$

Thus, in consequence of (1.6) and (3.13), we obtain the relation (1.2). Hence we can state the following corollary.

Corollary 3.2. *A $(WMPS)_n$ with vanishing Ricci tensor is a $(WS)_n$.*

4. m-projectively flat weakly Ricci-symmetric manifolds

In [14], Tamassy and Binh introduced the notion of weakly Ricci-symmetric manifold and studied its properties.

A non-flat Riemannian manifold (M_n, g) $(n > 2)$ is called weakly Ricci-symmetric if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the relation

$$(D_X S)(Y,Z) = A(X)g(Y,Z) + B(Y)S(X,Z) + D(Z)S(X,Y), \tag{4.1}$$

where A, B and D are defined as before. Such an $n-$dimensional manifold is denoted by $(WRS)_n$. In [6], Jana and Shaikh have studied quasi-conformally flat weakly Ricci-symmetric manifolds although Pseudo-projectively flat weakly Ricci-symmetric manifolds was studied by Jaiswal and Ojha [7].

Corollary 4.1. *In a $(WRS)_n$ with $\sigma(X) \neq 0$ the scalar curvature can not be zero and the Ricci tensor will be of the form $S(X,Y) = rH(X)H(Y)$, where the vector field σ associated with the $1-$form H is a unit vector field.*

Proof. In consequence of (4.1) and symmetric properties of S, it follows that

$$\{B(Y) - D(Y)\} S(X,Z) = \{B(Z) - D(Z)\} S(X,Y). \tag{4.2}$$

Let $\sigma(X) = B(X) - D(X)$ for any vector field X, then (4.2) becomes

$$\sigma(Y)S(X,Z) = \sigma(Z)S(X,Y). \tag{4.3}$$

Let $\{e_i\}$, $i = 1, 2, \ldots\ldots, n$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = Z = e_i$ in (4.3) and then taking summation over i, $1 \leq i \leq n$, we get

$$r\sigma(Y) = \sigma(QY), \tag{4.4}$$

where $\sigma(X) = g(X, \delta)$ for any vector field X and r is the scalar curvature. From (4.3), we have

$$\sigma(\delta)S(X,Z) = \sigma(Z)S(X,\delta) = \sigma(Z)\sigma(QX). \tag{4.5}$$

From (4.4) and (4.5), we get

$$S(X,Z) = rH(X)H(Z), \tag{4.6}$$

where $H(X) = \frac{\sigma(X)}{\sqrt{\sigma(\delta)}}$ and $g(X,\rho) = H(X)$, ρ is a unit vector field. Now from (4.6) it follows that if $r = 0$, then $S(X,Z) = 0$, which is inadmissible by the definition of the $(WRS)_n$. So $r \neq 0$. □

Proposition 3. *In a $(WRS)_n$ with $\sigma(X) \neq 0$, r is an eigen value of the Ricci tensor corresponding to the eigen vector δ.*

Proof. From (4.4) it follows that $S(Y,\delta) = rg(Y,\delta)$, which shows that r is an eigen value of the Ricci tensor corresponding to the eigen vector δ. □

Theorem 4.1. *In an m−projectively flat $(WRS)_n$ $(n > 3)$ with $\sigma(X) \neq 0$, the vector field ρ defined by $g(X,\rho) = H(X)$ is a proper concircular vector field.*

Proof. Differentiating (1.3) covariantly along U, we get

$$\begin{aligned}(D_U W^*)(X,Y)Z = (D_U R)(X,Y)Z - \frac{1}{2(n-1)}[(D_U S)(Y,Z)X \\ -(D_U S)(X,Z)Y + g(Y,Z)(D_U Q)(X) - g(X,Z)(D_U Q)(Y)].\end{aligned} \tag{4.7}$$

Contracting (4.7) with respect to U, we get

$$\begin{aligned}(divW^*)(X,Y)Z = (divR)(X,Y)Z - \frac{1}{2(n-1)}[(D_X S)(Y,Z) \\ -(D_Y S)(X,Z) + g(Y,Z)(divQ)(X) - g(X,Z)(divQ)(Y)].\end{aligned} \tag{4.8}$$

We know that in a Riemannian manifold

$$(divR)(X,Y)Z = (D_X S)(Y,Z) - (D_Y S)(X,Z). \tag{4.9}$$

In view of (4.9), (4.8) becomes

$$\begin{aligned}&(divW^*)(X,Y)Z = \\ &(D_X S)(Y,Z) - (D_Y S)(X,Z) - \frac{1}{2(n-1)}[(D_X S)(Y,Z) \\ &-(D_Y S)(X,Z) + g(Y,Z)(divQ)(X) - g(X,Z)(divQ)(Y)].\end{aligned} \tag{4.10}$$

Since the manifold is m−projectively flat, therefore $div(W^*) = 0$ and hence (4.10) gives

$$\begin{aligned}(2n-3)\{(D_X S)(Y,Z) - (D_Y S)(X,Z)\} \\ = g(Y,Z)(divQ)(X) - g(X,Z)(divQ)(Y).\end{aligned} \tag{4.11}$$

From (4.6), we have

$$\begin{aligned}&(D_Y S)(X,Z) = \\ &dr(Y)H(X)H(Z) + r\{(D_Y H)(X)H(Z) + (D_Y H)(Z)H(X)\}.\end{aligned} \tag{4.12}$$

In consequence of (4.12) and $(divQ)(X) = \frac{1}{2}dr(X)$, (4.11) becomes

$$\begin{aligned}&2(2n-3)[r\{(D_X H)(Y)H(Z) + (D_X H)(Z)H(Y) - (D_Y H)(X)H(Z)\} \\ &-(D_Y H)(Z)H(X) + dr(X)H(Y)H(Z) - dr(Y)H(X)H(Z)] \\ &= \{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}.\end{aligned} \tag{4.13}$$

Putting $Y = Z = e_i$ in (4.13) and then taking summation over i, $1 \leq i \leq n$, we get

$$2(2n-3)\left[dr(\rho)H(X) + r\left\{(D_\rho H)(X) + H(X)\sum_{i=1}^{n}(D_{e_i}H)(e_i)\right\}\right] = (3n-5)dr(X). \tag{4.14}$$

Again, substituting $Y = Z = \rho$ in (4.13), we find

$$2(2n-3)r(D_\rho H)(X) = (4n-7)\left\{dr(X) - dr(\rho)H(X)\right\}. \tag{4.15}$$

By virtue of (4.15), (4.14) takes the form

$$2(2n-3)rH(X)\sum_{i=1}^{n}(D_{e_i}H)(e_i) = -(n-2)dr(X) + dr(\rho)H(X). \tag{4.16}$$

Putting $X = \rho$ in the equation (4.16), we get

$$2(2n-3)r\sum_{i=1}^{n}(D_{e_i}H)(e_i) = -(n-3)dr(\rho). \tag{4.17}$$

In consequence of (4.17), (4.16) becomes

$$dr(X) = dr(\rho)H(X). \tag{4.18}$$

Again, replacing ρ for Z in (4.13) and then using (4.18), we have

$$2(2n-3)r\left\{(D_X H)(Y) - (D_Y H)(X)\right\} = 0,$$

which shows that

$$(D_X H)(Y) = (D_Y H)(X), \tag{4.19}$$

for $(n > 3)$ (since $r \neq 0$). By virtue of (4.18), (4.15) gives

$$(D_\rho H)(X) = 0. \tag{4.20}$$

Again, replacing Y by ρ in (4.13) and then using (4.18) and (4.20), we find

$$(D_X H)(Z) = \frac{dr(\rho)}{2r(2n-3)}\left[H(X)H(Z) - g(X,Z)\right]. \tag{4.21}$$

Let us consider a scalar function $f = \frac{dr(\rho)}{2r(2n-3)}$, then we have

$$D_X f = \frac{1}{2r^2(2n-3)}\left[rd^2r(\rho,X) - dr(\rho)dr(X)\right]. \tag{4.22}$$

In consequence of (4.18), we get

$$d^2r(X,Y) = d^2r(\rho,Y)H(X) + dr(\rho)(D_Y H)(X). \tag{4.23}$$

Now, in a Riemannian manifold the second covariant differentiation of any function $h \in C^\infty(M_n)$ is defined by

$$d^2h(X,Y) = X(Yh) - (D_XY)h,$$

for all X, $Y \in \chi(M_n)$, which shows that

$$d^2h(X,Y) = d^2h(Y,X),$$

for all X, $Y \in \chi(M_n)$ and hence by virtue of (4.19), (4.23) becomes

$$d^2r(\rho,Y)H(X) = d^2r(\rho,X)H(Y). \tag{4.24}$$

Putting $Y = \rho$ in (4.24), we find

$$d^2r(\rho,X) = d^2r(\rho,\rho)H(X) = \phi H(X), \tag{4.25}$$

where $\phi = d^2r(\rho,\rho)$ is a scalar function. Now, in consequence of (4.18) and (4.25), (4.22) assumes the form

$$D_Xf = \nu H(X), \tag{4.26}$$

where $\nu = \frac{1}{2r^2(2n-3)}\left\{r\phi - (dr(\rho))^2\right\}$. If we consider a $1-$form α given by

$$\alpha(X) = \frac{dr(\rho)}{2r(2n-3)}H(X) = fH(X), \tag{4.27}$$

then, in consequence of (4.19), (4.26) and (4.27), we obtain

$$d\alpha(X,Y) = 0, \tag{4.28}$$

i.e. the $1-$form α is closed. So (4.21) can be written as

$$(D_XH)(Y) = \alpha(X)H(Y) - fg(X,Y), \tag{4.29}$$

which implies that the vector field ρ corresponding to the $1-$form H defined by $g(X,\rho) = H(X)$ is a proper concircular vector field [16]. □

Theorem 4.2. *An $m-$projectively flat $(WRS)_n$ $(n > 2)$ is a quasi-Einstein manifold.*

Proof. Let us consider an $m-$projectively flat $(WRS)_n$ manifold, then (1.3) gives

$$\begin{aligned} 'K(X,Y,Z,U) &= \frac{1}{2(n-1)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &\quad + g(Y,Z)S(X,U) - g(X,Z)S(Y,U)]. \end{aligned} \tag{4.30}$$

In view of (4.6), (4.30) becomes

$$\begin{aligned} 'K(X,Y,Z,U) &= \frac{r}{2(n-1)}[H(Y)H(Z)g(X,U) - H(X)H(Z)g(Y,U) \\ &\quad + g(Y,Z)H(X)H(U) - g(X,Z)H(Y)H(U)]. \end{aligned} \tag{4.31}$$

Substituting $X = U = e_i$ in (4.31), and then taking summation over i, $1 \leq i \leq n$, we get

$$S(Y, Z) = ag(Y, Z) + bH(Y)H(Z), \tag{4.32}$$

where $a = \frac{r}{2(n-1)}$ and $b = \frac{r(n-2)}{2(n-1)}$. Hence, the manifold is quasi-Einstein [2]. □

5. Examples of $(WMPS)_n$ and $(WRS)_n$

Let $(x^1, x^2, \ldots\ldots, x^n) \in R^n$, where R^n denotes $n-$dimensional real number space. In this section we give the suitable examples of $(WMPS)_n$ and $(WRS)_n$ by defining the Riemannian metric g on R^n.

Example 1. If R^4 be the 4−dimensional real number space, then we define a Riemannian metric as

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (kx^1)^2 v(x^4)(dx^4)^2, \tag{5.1}$$

where $(i, j = 1, 2, 3, 4)$, $f = a_0 + a_1x^3 + a_2(x^3)^2$, a_0, a_1, a_2 are non-constant functions of x^1 only, v is a function of x^4 and let k be a non-zero arbitrary constant. It can be seen from (5.1) that the non-vanishing components of the Riemannian metric are

$$g_{11} = f, \;\; g_{12} = g_{21} = 1, \;\; g_{33} = 1, \;\; g_{44} = (kx^1)^2 v(x^4). \tag{5.2}$$

Also, its associated components of the Riemannian metric are

$$g^{11} = 0, \;\; g^{12} = g^{21} = -1, \;\; g^{33} = 1, \;\; g^{44} = 1. \tag{5.3}$$

From equations (5.2) and (5.3), it can be easily calculated that the only non-vanishing components of the Christoffel symbols, curvature tensor and the Ricci tensor are given by the following relations

$$\begin{aligned} &\Gamma^2_{11} = \frac{1}{2}f_{.1}, \;\; \Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2}f_{.3}, \;\; \Gamma^4_{14} = \frac{1}{x^1}, \\ &\Gamma^2_{44} = -kx^1 v, \;\; \Gamma^4_{44} = \frac{(v)_{.4}}{2v}, \;\; K_{1331} = \frac{1}{2}f_{.33}, \;\; S_{11} = \frac{1}{2}f_{.33} \end{aligned} \tag{5.4}$$

and the components which can be obtained from these by the symmetric properties. Here, S_{ij} represent the components of the Ricci tensor, whereas $'.'$ denotes the partial differentiation with respect to the coordinates. From equation (5.4), it is clear that the 4−dimensional space R^4 with the metric defined in (5.1) is a Riemannian manifold. In consequence of (5.4), (1.3) yields that the only non-zero components of the $m-$projective curvature tensor are

$$'W^*_{1331} = \frac{5}{12}f_{.33} = \frac{5}{6}a_2 \neq 0 \tag{5.5}$$

and the components which can be obtained from (5.5) by symmetric properties. The only non-zero covariant derivative of $'W^*$ are

$$'W^*_{1331,1} = \frac{5}{12}f_{.331} = \frac{5}{6}(a_2)_{.1} \neq 0 \tag{5.6}$$

and the components which can be obtained from (5.6) by the symmetric properties, where $','$ denotes the covariant differentiation with respect to the Riemannian metric g. Thus (R^4, g) is neither $m-$projectively flat nor $m-$projectively symmetric.

We shall also define the $1-$forms as

$$A_i(x) = \begin{cases} d(log a_2) & \text{for } \ i = 1, \\ 0 & \text{for } \ \text{otherwise}; \end{cases}$$

$$B_i(x) = \begin{cases} d(x^2x^3) & \text{for } \ i = 1, \\ 0 & \text{for } \ \text{otherwise}; \end{cases}$$

$$D_i(x) = \begin{cases} -d(x^2x^3) & \text{for } \ i = 1, \\ 0 & \text{for } \ \text{otherwise}, \end{cases}$$

for all $x \in R^4$. In view of (5.4), (5.5), (5.6) and above $1-$forms, we have

$$A_1'W^*_{1331} + B_1'W^*_{1331} + B_3'W^*_{1311} + D_3'W^*_{1311} + D_1'W^*_{1331}$$
$$= (A_1 + B_1 + D_1)'W^*_{1331} = \frac{5}{6}(a_2)_{.1} =' W^*_{1331,1}.$$

Thus,

$$(5.7) \qquad 'W^*_{1331,1} = A_1'W^*_{1331} + B_1'W^*_{1331} + B_3'W^*_{1311} + D_3'W^*_{1311} + D_1'W^*_{1331}.$$

Next,

$$A_3'W^*_{1131} + B_1'W^*_{1331} + B_1'W^*_{1331} + D_3'W^*_{1131} + D_1'W^*_{1133}$$
$$= d(x^2x^3)('W^*_{3131} +' W^*_{1331}) = 0,$$

by the skew-symmetric properties of $'W^*$. Hence,

$$(5.8) \qquad 'W^*_{1131,3} = A_3'W^*_{1131} + B_1'W^*_{3131} + B_1'W^*_{1331} + D_3'W^*_{1131} + D_1'W^*_{1133}.$$

In a similar fashion, we can also find that

$$(5.9) \qquad 'W^*_{1311,3} = A_3'W^*_{1311} + B_1'W^*_{3311} + B_3'W^*_{1311} + D_1'W^*_{1331} + D_1'W^*_{1313}.$$

It is obvious that the rest components of each term of (1.6) vanishes identically and hence the relation (1.6) holds trivially. Therefore, (R^4, g) will be a $(WMPS)_4$. In the same way it can be easily shown that (R^4, g) is a $(WRS)_4$.

Corollary 5.1. *Let (R^4, g) be a Riemannian manifold endowed with the metric*

$$(5.10) \qquad ds^2 = g_{ij}dx^idx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (kx^1)^2v(x^4)(dx^4)^2,$$

where $(i, j = 1, 2, 3, 4)$, $f = a_0 + a_1x^3 + e^{x^1}(x^3)^2$, a_0, a_1 are non-constant functions of x^1 only, v is a function of x^4 and let k be a non-zero arbitrary constant. Then, (R^4, g) is a weakly $m-$projectively symmetric manifold which is neither $m-$projectively flat nor $m-$projectively symmetric.

Proof. If we replace a_2 by e^{x^1} in (5.1), we get (5.10) and hence the equations (5.4) and (5.5) take the form

$$'W^*_{1331} = \frac{5}{12}e^{x^1} \neq 0. \tag{5.11}$$

and

$$'W^*_{1331,1} = \frac{5}{12}e^{x^1} =' W^*_{1331} \neq 0. \tag{5.12}$$

Thus, in consequence of the (5.11) and (5.12), it is clear that the manifold R^4 with metric (5.10) is neither $m-$projectively flat nor $m-$projectively symmetric.

If we consider the $1-$forms as

$$A_i(x) = \begin{cases} \frac{1}{3} & \text{for } \ i = 1, \\ 0 & \text{for } \ \text{otherwise}; \end{cases}$$

$$B_i(x) = \begin{cases} -\frac{1}{6} & \text{for } \ i = 1, \\ 0 & \text{for } \ \text{otherwise}; \end{cases}$$

$$D_i(x) = \begin{cases} \frac{5}{6} & \text{for } \ i = 1, \\ 0 & \text{for } \ \text{otherwise}, \end{cases}$$

then it can be easily shown that the manifold R^4 endowed with metric (5.10) satisfies the relations (5.7), (5.8) and (5.9) and hence it is a $(WMPS)_4$. □

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