Novi Sad J. Math.
Vol. 42, No. 1, 2012, 59-66

HOMOMORPHISM-HOMOGENEOUS GRAPHS WITH LOOPS

Andreja Ilić[1], Dragan Mašulović[2][3] and Uroš Rajković[4]

Abstract. In 2006, P. J. Cameron and J. Nešetřil introduced the following variant of homogeneity: we say that a structure is homomorphism-homogeneous if every homomorphism between finite substructures of the structure extends to an endomorphism of the structure. In this paper we classify finite homomorphism-homogeneous graphs where some vertices may have loops, but only up to a certain point. We focus on disconnected graphs, and on connected graphs whose subgraph induced by loops is disconnected. In a way, this is the best one can hope for, since it was shown in a recent paper by M. Rusinov and P. Schweitzer that there is no polynomially computable characterization of finite homomorphism-homogeneous graphs whose subgraph induced by loops is connected (unless P = coNP).

AMS Mathematics Subject Classification (2010): 05C75

Key words and phrases: finite graphs, homomorphism-homogeneity

1. Introduction

A structure is *ultrahomogeneous* if every isomorphism between finite substructures of the structure extends to an automorphism of the structure. The theory of (countable) ultrahomogeneous structures gained its momentum in 1953 with the famous theorem of Fraïssé [3] which states that countable ultrahomogeneous structures can be recognized by the fact that their collections of finitely induced substructures have the amalgamation property. Nowadays it is a well-established theory with deep consequences in many areas of mathematics.

Ultrahomogeneous objects have been determined for many important classes of structures. For example, countably infinite ultrahomogeneous posets were characterized in [10]; countably infinite ultrahomogeneous graphs were described in [7], while the finite ones were determined in [4]; countably infinite ultrahomogeneous digraphs were described in [2], while finite and countably infinite ultrahomogeneous tournaments were described in [6].

In their paper [1], the authors introduce the following variant of homogeneity: a structure is called *homomorphism-homogeneous* if every homomorphism

[1]Faculty of Science and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia, e-mail: andrejko@bankerinter.net
[2]Supported by the Ministry of Science and Environmental Protection of the Republic of Serbia, Grant No. 174019
[3]Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia, e-mail: dragan.masulovic@dmi.uns.ac.rs
[4]Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia, e-mail: urajkovic@gmail.com

between finite substructures of the structure extends to an endomorphism of the structure. Not much is known about homomorphism-homogeneous structures. For example, homomorphism-homogeneous posets were characterized in [8], and finite homomorphism-homogeneous tournaments (with loops) were characterized in [5]. Finite homomorphism-homogeneous graphs without loops were characterized in [1]: a finite graph G with no loops is homomorphism-homogeneous if and only if G is a disjoint union of complete graphs of the same size.

In this paper we classify finite homomorphism-homogeneous graphs where some vertices may have loops, but only up to a certain point. We focus on disconnected graphs, and on connected graphs whose subgraph induced by loops is disconnected. In a way, this is the best one can hope for, since it was shown in [9] that deciding homomorphism-homogeneity for finite graphs whose subgraph induced by loops is connected is coNP-complete. Therefore, unless P = coNP, there is no polynomially computable characterization of the latter class of graphs.

All graphs in this paper are finite.

2. Preliminaries

Throughtout the paper G is a finite graph with at least one loop. Let $x \sim_G y$ denote that x and y are adjacent in G. For $S \subseteq V(G)$ let $N_S(v) = \{x \in S : x \sim v\}$, and $\delta_S(v) = |N_S(v)|$. We shorthand $N_{V(G)}(v)$ and $\delta_{V(G)}(v)$ to $N_G(v)$ and $\delta_G(v)$. Moreover, if G is obvious, we simply write $\sim$, $N(v)$ and $\delta(v)$. For $S \subseteq V(G)$, by $G[S]$ we denote the subgraph of G induced by S. Let $P(G)$ denote the set of all vertices of G that have a loop and let $Q(G) = V(G) \setminus P(G)$.

Recall that $G \cup H$ denotes the disjoint union of G and H, and $G \nabla H$ denotes the graph obtained from the disjoint union $G \cup H$ by adding all edges $\{u, v\}$, where $u \in V(G)$ and $v \in V(H)$. By $n \cdot G$ we denote the disjoint union of n copies of G. A complete graph (or a clique) on S is the graph K_S with the set of vertices S where each pair of distinct vertices is adjacent. We often write K_n instead of K_S where $n = |S|$. Let $K(m, n)$ denote the complete graph with $m+n$ vertices where m vertices have a loop and the remaining n do not. Hence, $K(0, n) \cong K_n$, and $K(m, 0)$ is the complete graph on m vertices where each vertex has a loop. In particular, $K(1, 0)$ is just a vertex with a loop.

An *isomorphism* between graphs G and H is a bijective mapping $f : V(G) \to V(H)$ such that $x \sim_G y$ if and only if $f(x) \sim_H f(y)$, for all $x, y \in V(G)$. By $G \cong H$ we denote that G and H are isomorphic. A *homomorphism* between graphs G and H is a mapping $f : V(G) \to V(H)$ such that $x \sim_G y$ implies $f(x) \sim_H f(y)$, for all $x, y \in V(G)$. An *endomorphism* of G is a homomorphism from G into itself. If f is anendomorphism of G and $S \subseteq V(G)$, then:

- $f(P(G)) \subseteq P(G)$;
- if $G[S]$ is connected, then so is $G[f(S)]$; in particular, f maps a connected component into a connected component;

- if $G[S] = K_S$ and $f(S) \cap P(G) = \varnothing$ then $f|_S$ is injective.

Definition 1. Following [1], we say that a graph G with loops allowed is *homomorphism-homogeneous* if every homomorphism $f : S \to T$ between finite induced subgraphs $G[S]$ and $G[T]$ of G extends to an endomorphism of G.

Lemma 2. *Let G be a homomorphsim-homogeneous graph.*

(1) if K_r and K_s are connected components of G then $r = s$;

(2) if S is a connected component of G such that $S \cap P(G) = \varnothing$ then $G[S] = K_S$.

(3) $G[P(G)]$ is a homomorphism-homogeneous graph.

Proof. (1) Take any $x \in V(K_r)$ and $y \in V(K_s)$. Then $f : x \mapsto y$ extends to an endomorphism f^*. Now, f^* is injective on $V(K_r)$ and $f^*(V(K_r)) \subseteq V(K_s)$. Hence $r \leqslant s$. The same argument yields $s \leqslant r$.

(2) Assume that $G[S] \neq K_S$. Take $x, y_1 \in S$ such that $d(x, y_1) = 2$ and let z be a common neighbour of x and y_1. Let $N(z) = \{x, y_1, \dots, y_k\}$ and consider $f = \begin{pmatrix} x & y_1 & \dots & y_k \\ z & y_1 & \dots & y_k \end{pmatrix}$. This is easily seen to be a homomorphism from $G[x, y_1, \dots, y_k]$ to $G[z, y_1, \dots, y_k]$ (since $z \sim y_i$ for all i), so it extends to an endomorphsim f^* of G. From $z \sim x$ it follows that $f^*(z) \sim f^*(x) = z$ i.e. $f^*(z) \in N(z)$. If $f^*(z) = y_i$ for some i then $y_i \sim z$ implies $y_i = f^*(y_i) \sim f^*(z) = y_i$, which contradicts $S \cap P(G) = \varnothing$. If, however, $f^*(z) = x$ then $z \sim y_1$ implies $x \sim y_1$, which contradicts $d(x, y_1) = 2$.

(3) Obvious, since $f(P(G)) \subseteq P(G)$ for every endomorphism f of G. □

We say that u and v are of the same type and write $u \equiv v$ if either both u and v have a loop, or both u and v do not have a loop. More precisely, u and v are of the same type if $\{u, v\} \subseteq P(G)$ or $\{u, v\} \subseteq Q(G)$.

Lemma 3. *Let G be a connected homomorphism-homogeneous graph.*

(1) If $d(x, y) = 3$ and if $x \sim a \sim b \sim y$ then $x \not\equiv a$ and $y \not\equiv b$.

(2) If $d(x, y) = 3$ and if $x \sim a \sim b \sim y$ then $x, y \in Q(G)$ and $a, b \in P(G)$.

(3) $\operatorname{diam}(G) \leqslant 3$.

(4) If $\{x, y\} \nsubseteq Q(G)$ then $d(x, y) \leqslant 2$.

Proof. (1): Suppose that $x \equiv a$ and consider $f : a \mapsto x, y \mapsto y$. This is a homomorphism between $G[a, y]$ and $G[x, y]$ so it extends to an endomorphism f^* of G. Now, $a \sim b \sim y$ implies $f^*(a) \sim f^*(b) \sim f^*(y)$ i.e. $x \sim f^*(b) \sim y$, which contradicts $d(x, y) = 3$. By the same argument, $y \not\equiv b$.

(2): According to (1), there are only three possibilities to consider:

- $x, b \in P(G)$ and $a, y \in Q(G)$;

- $x, y \in P(G)$ and $a, b \in Q(G)$; and
- $x, y \in Q(G)$ and $a, b \in P(G)$.

In the first two cases $f : a \mapsto x, y \mapsto y$ is a homomorphism between the two induced substructures and extends to an endomorphism f^* of G. Now $a \sim b \sim y$ implies $x \sim f^*(b) \sim y$, which contradicts $d(x, y) = 3$.

(3): Suppose $\operatorname{diam}(G) \geqslant 4$. Take x, y such that $d(x, y) = 4$ and let $x \sim a \sim b \sim c \sim y$. Applying (2) to $x \sim a \sim b \sim c$ yields $c \in Q(G)$, while applying (2) to $a \sim b \sim c \sim y$ yields $c \in P(G)$. Contradiction.

(4) is a direct consequence of (2) and (3). □

Note that it is not possible to improve (2) and (3) in the previous lemma since the graph on four vertices x, a, b, y and with edges $x \sim a \sim b \sim y$, $a \sim a$, $b \sim b$ is homomorphism-homogeneous and its diameter is 3.

3. The main result

In this section we first characterize finite homomorphism-homogeneous disconnected graphs. Using this characterization, we then obtain the characterization of all finite homomorphism-homogeneous connected graphs G whose subgraph induced by loops is not connected.

Lemma 4. *Let G be a disconnected homomorphism-homogeneous graph.*

(1) If x and y are distinct vertices which belong to the same connected component of G and at least one of them is in $Q(G)$ then $x \sim y$.

(2) If there exists a vertex with a loop adjacent to a vertex with no loop, then every connected component of G has a vertex with a loop.

(3) Assume that at least two connected components of G have vertices with loops. Then in each connected component of G each pair of vertices with loops is adjacent.

Proof. (1) Let S be a connected component of G such that $x, y \in S$, let $y \in Q(G)$ and assume that $x \not\sim y$. Let S' be another connected component of G and let $z \in S'$. Then $f : x \mapsto x, y \mapsto z$ is a homomorphism between the induced subgraphs and hence extends to an endomorphism f^* of G. Now, x and y belong to the same connected component, so there is a path $x \sim v_1 \sim \ldots \sim y$. Applying f^* to this path yields $x \sim f^*(v_1) \sim \ldots \sim z$ whence follows $S \cap S' \neq \varnothing$. Contradiction.

(2) Let p be a vertex with a loop and suppose that $p \sim x$, where x is a vertex with no loop. Let S be a connected component that does not contain p and let $y \in S$ be arbitrary. Then $f : x \mapsto y$ is a homomorphism, and hence extends to an endomorphism f^*. Now, $p \sim x$ implies $f^*(p) \sim y$. Since $f^*(p)$ is a vertex with a loop, the statement follows.

(3) Let S be a connected component of G and assume there exist $p_1, p_2 \in S \cap P(G)$ such that $p_1 \not\sim p_2$. By the assumption, there exist another connected component S' and a vertex $p' \in S' \cap P(G)$. The mapping $f : p_1 \mapsto p_1, p_2 \mapsto p'$ is a homomorphism, and hence extends to an endomorphism f^*. Now, p_1 and p_2 belong to the same connected component, so there is a path $p_1 \sim v_1 \sim \ldots \sim p_2$. Applying f^* to this path yields $p_1 \sim f^*(v_1) \sim \ldots \sim p'$ whence follows $S \cap S' \neq \varnothing$. Contradiction. □

Proposition 5. *Let G be a disconnected graph. Then G is homomorphism-homogeneous if and only if G is one of the following graphs:*

(1) $K(m_1, n_1) \cup \ldots \cup K(m_t, n_t)$ for some integers $t \geqslant 2$, $m_i \geqslant 1$ and $n_i \geqslant 0$ such that $n_1 + \ldots + n_t \geqslant 1$;

(2) $K(n_1, 0) \cup \ldots \cup K(n_t, 0) \cup s \cdot K_m$ for some integers $t \geqslant 1$, $s \geqslant 0$, $n_i \geqslant 1$, $m \geqslant 1$ such that $s + t \geqslant 2$;

(3) $H \cup s \cdot K_n$ for some integers $s \geqslant 1$, $n \geqslant 1$, and some connected homomorphism-homogeneous graph H whose every vertex has a loop.

Proof. (⇐) is easy.

(⇒): If G has a vertex with a loop that has a neighbour without a loop, then Lemma 4 implies that every connected component of G is isomorphic to $K(m, n)$ for some m and n, and we have case (1).

If no vertex with a loop is adjacent to a vertex without a loop, and there exist at least two connected components containing a vertex with a loop, then Lemmas 2 and 4 imply that $G \cong K(n_1, 0) \cup \ldots \cup K(n_t, 0) \cup s \cdot K_m$ for some integers $t \geqslant 1$, $s \geqslant 0$, $n_i \geqslant 1$, $m \geqslant 1$ such that $s + t \geqslant 2$, and we have case (2).

Finally, if no vertex with a loop is adjacent to a vertex without a loop, and there exists only one connected component containing a vertex with a loop, then the other connected components have to be complete graphs of the same size according to Lemma 2, while the connected component that contains vertices with loops has to be a connected homomorphism-homogeneous graph where every vertex has a loop. So, we have case (3). □

Proposition 6. *Let G be a connected homomorphism-homogeneous graph such that $G[P(G)]$ is not connected. Then $G \cong K_s \nabla (K(t_1, 0) \cup \ldots \cup K(t_m, 0))$ for some $m \geqslant 2$ and some $s, t_1, \ldots, t_m \geqslant 1$.*

Proof. Clearly, $Q(G) \neq \varnothing$ since G is connected and $G[P(G)]$ is not. Let T_1, $\ldots$, T_m, $m \geqslant 2$, be connected components of $G[P(G)]$, and for for each $i \in \{1, \ldots, m\}$ let $S_i \subseteq Q(G)$ be the set of all the vertices in $Q(G)$ that have at least one neighbour in T_i.

One can easily adapt the idea of the proof of Lemma 4 (3) to show that $G[T_i] \cong K(|T_i|, 0)$, for every $i \in \{1, \ldots, m\}$.

Let us show that $G[S_i] \cong K(0, |S_i|)$, for every $i \in \{1, \ldots, m\}$. Suppose, to the contrary, that there exists an $i \in \{1, \ldots, m\}$ and some $a, b \in S_i$ such that $a \neq b$ and $a \nsim b$. By the definition of S_i, there exist $u, v \in T_i$ such that $a \sim u$ and $b \sim v$. Moreover, $u \sim v$ since $G[T_i] \cong K(|T_i|, 0)$. Take any $j \neq i$ and let w be an arbitrary vertex of T_j. The mapping $f = \begin{pmatrix} a & b \\ u & w \end{pmatrix}$ is a homomorphism from $G[a, b]$ to $G[u, w]$, so it extends to an endomorphism f^* of G. Now, $a \sim u \sim v \sim b$ implies $u \sim f^*(u) \sim f^*(v) \sim w$, which is a path from u to w which consists of vertices from $P(G)$ – contradiction with the fact that T_i and T_j are distinct connected components of $G[P(G)]$.

Next, let us show that for every $i \in \{1, \ldots, m\}$, every vertex in T_i is adjacent to every vertex in S_i. Suppose, to the contrary, that there exists an $i \in \{1, \ldots, m\}$ and some $a \in S_i$ and $u \in T_i$ such that $a \nsim u$. By the definition of S_i, there exists a $v \in T_i$ such that $a \sim v$. Moreover, $u \sim v$ since $G[T_i] \cong K(|T_i|, 0)$. Take any $j \neq i$ and let w be an arbitrary vertex of T_j. The mapping $f = \begin{pmatrix} u & a \\ u & w \end{pmatrix}$ is a homomorphism from $G[a, u]$ to $G[u, w]$, so it extends to an endomorphism f^* of G. Now, $u \sim v \sim a$ implies $u \sim f^*(v) \sim w$, which is a path from u to w which consists of vertices from $P(G)$ – contradiction.

As the next step, let us show that $S_i \cap S_j = \varnothing$ or $S_i = S_j$, whenever $i \neq j$. Suppose, to the contrary, that there exist $i \neq j$ such that $S_i \cap S_j \neq \varnothing$ and $S_i \setminus S_j \neq \varnothing$. Take any $a \in S_i \setminus S_j$, $b \in S_i \cap S_j$, $u \in T_i$ and $v \in T_j$ and let $X = S_i \cup S_j \cup \{u, v\}$. Because b is adjacent to every $x \in X$, the mapping $f : X \setminus \{b\} \to X \setminus \{a\}$ given by

$$f(x) = \begin{cases} b, & x = a \\ x, & x \in X \setminus \{a, b\} \end{cases}$$

is a homomorphism from $G[X \setminus \{b\}]$ to $G[X \setminus \{a\}]$, so by the homogeneity assumption, it extends to an endomorphism f^* of G. From $u \sim b \sim v$ it follows that $f^*(u) \sim f^*(b) \sim f^*(v)$, that is, $u \sim f^*(b) \sim v$. Since u and v belong to distinct connected components of $G[P(G)]$, it follows that $f^*(b)$ cannot be an element of $P(G)$. Therefore, $f^*(b) \in Q(G)$, and consequently, $f^*(b) \in S_i \cap S_j$. If $f^*(b) = b$ then $a \sim b$ implies $f^*(a) \sim f^*(b)$, that is $b \sim b$ – contradiction with $b \in S_i \cap S_j \subseteq Q(G)$. On the other hand, if $f^*(b) = c$ for some $c \neq b$ then $b \sim c$ implies $f^*(b) \sim f^*(c)$. From $c \neq b$ it follows that $f^*(c) = f(c) = c$, so $f^*(b) \sim f^*(c)$ implies $c \sim c$ – contradiction with $c \in S_i \cap S_j \subseteq Q(G)$.

Finally, let us show that $S_i = S_j$ for all i and j. Take any i and j such that $i \neq j$, and any $u \in T_i$ and $v \in T_j$. Since $u \nsim v$, Lemma 3 (4) yields $d(u, v) = 2$, so there is an $x \in V(G)$ such that $u \sim x \sim v$. But $x \in Q(G)$ since T_i and T_j are distinct connected components of $G[P(G)]$. Therefore, $x \in S_i \cap S_j$, whence $S_i = S_j$ by the previous paragraph.

Putting it all together, we obtain $G \cong K_{|S_1|} \nabla (K(|T_1|, 0) \cup \ldots \cup K(|T_m|, 0))$. □

Theorem 7. *Let G be a graph such that $G[P(G)]$ is not connected. Then G is homomorphism-homogeneous if and only if G is one of the following graphs:*

(1) $K(m_1, n_1) \cup \ldots \cup K(m_t, n_t)$ *for some integers* $t \geqslant 2$, $m_i \geqslant 1$ *and* $n_i \geqslant 0$ *such that* $n_1 + \ldots + n_t \geqslant 1$*;*

(2) $K(n_1, 0) \cup \ldots \cup K(n_t, 0) \cup s \cdot K_m$ *for some integers* $t \geqslant 1$, $s \geqslant 0$, $n_i \geqslant 1$, $m \geqslant 1$ *such that* $s + t \geqslant 2$*;*

(3) $K_s \nabla (K(t_1, 0) \cup \ldots \cup K(t_m, 0))$ *for some* $m \geqslant 2$ *and some* $s, t_1, \ldots, t_m \geqslant 1$*.*

Proof. (⇐) It is easy to see that all the graphs in (1)–(3) are homomorphism-homogeneous.

(⇒) Let G be a homomorphism-homogeneous graph such that $G[P(G)]$ is not connected. If G is not connected then, according to Proposition 5, we have that G is isomorphic to a graph in (1) or (2). However, if G is connected, Proposition 6 yields that G is isomorphic to a graph in (3). □

4. Acknowledgements

This paper was brought to its final form during the second author's stay at the Johannes Kepler University in Linz, Austria, in the winter semester of 2010/11, where he was holding a position of a Research Fellow. The second author would like to thank the staff of the Institute for Algebra of the Johannes Kepler University in Linz for a stimulating research environment.

References

[1] Cameron, P. J., Nešetřil, J., Homomorphism-homogeneous relational structures. Combinatorics, Probability and Computing 15 (2006), 91–103.

[2] Cherlin G. L., The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments. Memoirs of the American Mathematocal Society Vol. 621, 1998.

[3] Fraïssé, R., Sur certains relations qui généralisant l'ordre des nombres rationnels. C. R. Acad. Sci. Paris 237 (1953), 540–542

[4] Gardiner, A. D., Homogeneous graphs. J. Combinatorial Theory (B) 20 (1976), 94–102.

[5] Ilić A., Mašulović D., Rajković U., Finite homomorphism-homogeneous tournaments with loops. Journal of Graph Theory, Vol. 59 No. 1 (2008), 45–58

[6] Lachlan, A. H., Countable Homogeneous Tournaments. Transactions of the American Mathematical Society, Vol. 284, No. 2 (1984), 431–461.

[7] Lachlan, A. H., Woodrow, R. E., Countable ultrahomogeneous undirected graphs. Transactions of the American Mathematical Society, Vol. 262 (1980), 51–94.

[8] Mašulović D., Homomorphism-homogeneous partially ordered sets. Order Vol. 24 No. 4 (2007), 215–226.

[9] Rusinov, M., Schweitzer, P., Homomorphism-homogeneous graphs. (to appear in Journal of Graph Theory)

[10] Schmerl, J. H.: Countable homogeneous partially ordered sets. Algebra Universalis 9 (1979), 317–321

Received by the editors January 13, 2011