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GENERALIZATIONS OF PRIMARY IDEALS IN COMMUTATIVE RINGS

Ahmad Yousefian Darani[1]

Abstract. Let R be a commutative ring with identity. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function where $\mathfrak{I}(R)$ denotes the set of all ideals of R. A proper ideal Q of R is called ϕ-primary if whenever $a, b \in R$, $ab \in Q - \phi(Q)$ implies that either $a \in Q$ or $b \in \sqrt{Q}$. So if we take $\phi_\emptyset(Q) = \emptyset$ (resp., $\phi_0(Q) = 0$), a ϕ-primary ideal is primary (resp., weakly primary). In this paper we study the properties of several generalizations of primary ideals of R.

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1. Introduction

Throughout this paper R will be a commutative ring with nonzero identity having total quotient ring $T(R)$. We will denote the set of ideals of R by $\mathfrak{I}(R)$. By a proper ideal I of R we mean an ideal I of R with $I \neq R$. Denote by $\mathfrak{I}^*(R)$ the set of proper ideals of R.

The concept of weakly prime ideals was introduced by Anderson and Smith (2003), where an ideal $P \in \mathfrak{I}^*(R)$ is called weakly prime if, for $a, b \in R$ with $0 \neq ab \in P$, either $a \in P$ or $b \in P$, [2]. In [4], Bhatwadekar and Sharma (2005) defined a proper ideal I of an integral domain R to be almost prime (resp., n-almost prime) if for $a, b \in R$ with $ab \in I - I^2$, (resp., $ab \in I - I^n (n \geq 2)$) either $a \in I$ or $b \in I$. This definition can obviously be made for any commutative ring R. Thus:

$$I \text{ prime} \Rightarrow I \text{ weakly prime} \Rightarrow I \text{ } n\text{-almost prime} \Rightarrow I \text{ almost prime.}$$

Later, Anderson and Batanieh (2008) gave a generalization of prime ideals which covers all the above mentioned definitions. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is said to be ϕ-prime if for $a, b \in R$ with $ab \in I - \phi(I)$, either $a \in I$ or $b \in I$, [1].

The radical of an ideal $I \in \mathfrak{I}(R)$ is defined to be the set of all $a \in R$ for which $a^n \in I$ for some positive integer n. Primary ideals have an important role in commutative ring theory. An ideal $Q \in \mathfrak{I}^*(R)$ of R is said to be primary provided that for $a, b \in R$, $ab \in Q$ implies that either $a \in Q$ or $b \in \sqrt{Q}$. We can generalize the concept of primary ideals by enlarging the set where a and

[1]Department of Mathematics, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran, e-mail: yousefian@uma.ac.ir, youseffian@gmail.com

b lie, or by restricting the set where ab lies. Let R be an integral domain with the quotient field K. Badawi and Houston [3] defined a proper ideal Q of R to be strongly primary if, whenever $ab \in Q$ with $a, b \in K$, we have $a \in Q$ or $b \in \sqrt{Q}$. The definition can obviously be made for any commutative ring R using $T(R)$ instead of K. A proper ideal Q of R is weakly primary if for $a, b \in R$ with $0 \neq ab \in Q$, either $a \in Q$ or $b \in \sqrt{Q}$. Weakly primary ideals were first introduced and studied by Ebrahimi Atani and Farzalipour in 2005, [5]. We say that a proper ideal Q of R is almost primary (resp., n-almost primary) provided that for $a, b \in R$, $ab \in Q - Q^2$ (resp., $ab \in Q - Q^n$ $(n \geq 2)$) implies that $a \in Q$ or $b \in \sqrt{Q}$.

In this paper we give some more generalizations of primary ideals and study the properties of these classes of ideals. Many of our results are analogous to the results in [1]. In fact, among the other results we prove the results mentioned below. It is shown in Lemma 2.7 that if Q is a ϕ-primary ideal of R with $\sqrt{\phi(Q)} = \phi(\sqrt{Q})$, then $\sqrt{Q}$ is a ϕ-prime ideal of R. Clearly, every primary ideal is ϕ-primary, but the converse does not necessarily hold. We prove in Theorem 2.11 that if Q is a ϕ-primary ideal of R with $Q^2 \nsubseteq \phi(Q)$, then Q is primary. Thus, if Q is not primary, then $\sqrt{Q} = \sqrt{\phi(Q)}$. Several characterizations of ϕ-primary ideals are given in Theorem 2.8

2. Results

Definition 2.1. Let R be a commutative ring and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal Q of R is called ϕ-primary (resp., strongly ϕ-primary) provided that for $a, b \in R$ (resp., $a, b \in T(R)$), $ab \in Q - \phi(Q)$ implies $a \in Q$ or $b \in \sqrt{Q}$.

Example 2.2. Let R be a commutative ring. Define the map $\phi_\alpha : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ as follows:

(1) $\phi_\emptyset : \phi(Q) = \emptyset$ defines primary ideals.

(2) $\phi_0 : \phi(Q) = 0$ defines weakly primary ideals.

(3) $\phi_2 : \phi(Q) = Q^2$ defines almost primary ideals.

(4) $\phi_n (n \geq 2) : \phi(Q) = Q^n$ defines n-almost primary ideals.

(5) $\phi_\omega : \phi(Q) = \cap_{n=1}^{\infty} Q^n$ defines ω-primary ideals.

(6) $\phi_1 : \phi(Q) = Q$ defines any ideals.

Let R be a commutative ring. Clearly, every strongly ϕ-primary ideal of R is ϕ-primary, but the converse does not necessarily hold. Now, consider the following result.

Proposition 2.3. *Let V ba a valuation ring with the quotient field K, and let $\phi : \mathfrak{I}(V) \to \mathfrak{I}(V) \cup \{\emptyset\}$ be a function. Then every ϕ-primary ideal of V is strongly ϕ-primary.*

Proof. Let Q be a ϕ-primary ideal of V. Let $a, b \in K$ be such that $ab \in Q - \phi(Q)$ but $a \notin Q$. Consider the two cases $a \notin V$ and $a \in V$. In the former case, $a^{-1} \in V$. So $b = a^{-1}ab \in Q$. So, assume that the latter case holds. Then, from $abb^{-1} = a \notin Q$ we get $b \in V$. Now $a, b \in V$, $ab \in Q - \phi(Q)$ and Q ϕ-primary imply that $b \in \sqrt{Q}$, that is, Q is strongly ϕ-primary. □

Remark 2.4.

1. Let R be a commutative ring, and let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be a function. Then, every ϕ-prime ideal of R is ϕ-primary.

2. Let R ba a commutative ring, Q an ideal of R and let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be a function with $\phi(Q) \in \Im(R)$. Then, if Q is a weakly primary ideal of R, it is ϕ-primary.

3. Given two functions $\psi_1, \psi_2 : \Im(R) \to \Im(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in \Im(R)$. Note in this case that

$$\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq ... \leq \phi_{n+1} \leq \phi_n \leq \phi_2 \leq \phi_1.$$

4. Since $I - \phi(I) = I - (I \cap \phi(I))$, without loss of generality we may assume that $\phi(I) \subseteq I$. We henceforth make this assumption.

5. For an ideal A of R we define the function $\phi_A : \Im(R) \to \Im(R) \cup \{\emptyset\}$ by $\phi_A(J) = AJ$.

Lemma 2.5. *Let R be a commutative ring, and let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be a function. An ideal Q of R is ϕ-primary if and only if $Q/\phi(Q)$ is a weakly primary ideal of $R/\phi(Q)$.*

Proof. First assume that Q is a ϕ-primary ideal of R. Let $a, b \in R$ be such that $0 \neq (a + \phi(Q))(b + \phi(Q)) \in Q/\phi(Q)$. Then, $ab \in Q - \phi(Q)$ implies that either $a \in Q$ or $b \in \sqrt{Q}$. Hence, either $a + \phi(Q) \in Q/\phi(Q)$ or $b + \phi(Q) \in \sqrt{Q}/\phi(Q) = \sqrt{Q/\phi(Q)}$. Consequently, $Q/\phi(Q)$ is weakly primary.

Conversely, assume that $Q/\phi(Q)$ is a weakly primary ideal of $R/\phi(Q)$. Let $a, b \in R$ be such that $ab \in Q - \phi(Q)$. Then $0 \neq (a + \phi(Q))(b + \phi(Q)) \in Q/\phi(Q)$. Since $Q/\phi(Q)$ is weakly primary, either $a + \phi(Q) \in Q/\phi(Q)$ or $b + \phi(Q) \in \sqrt{Q/\phi(Q)} = \sqrt{Q}/\phi(Q)$; so either $a \in Q$ or $b \in \sqrt{Q}$, as required □

Proposition 2.6. *Let R be a commutative ring and Q a proper ideal of R.*

(1) Let $\psi_1, \psi_2 : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be functions with $\psi_1 \leq \psi_2$. Then, if Q is ψ_1-primary, it is ψ_2-primary too.

(2) (a) Q primary $\Rightarrow$ Q weakly primary $\Rightarrow$ Q ω-primary $\Rightarrow$ Q $(n+1)$-almost primary $\Rightarrow$ Q n-almost primary $\Rightarrow$ Q almost primary.

(b) Q is ω-primary if and only if Q is n-almost primary for all $n \geq 2$.

Proof. (1) It is obvious.

(2) It follows from part (1) and the linear ordering in Remark 2.4.

□

Lemma 2.7. *Let R be a commutative ring, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function. If Q is a ϕ-primary ideal of R with $\sqrt{\phi(Q)} = \phi(\sqrt{Q})$, then $\sqrt{Q}$ is a ϕ-prime ideal of R.*

Proof. Set $P = \sqrt{Q}$ and let $a, b \in R$ be such that $ab \in P - \phi(P)$ but $a \notin P$. Then there exists a positive integer n with $a^n b^n \in Q$. If $(ab)^n \in \phi(Q)$, then $ab \in \sqrt{\phi(Q)} = \phi(\sqrt{Q}) = \phi(P)$ a contradiction. Since Q is ϕ-primary, it follows from $a^n b^n \in Q - \phi(Q)$ that $b \in \sqrt{Q} = P$, that is P is ϕ-prime. □

Theorem 2.8. *Let I be a proper ideal of the commutative ring R, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function. Then the following statements are equivalent:*

(i) I is ϕ-primary.

(ii) For every $a \in R - \sqrt{I}$, $(I :_R a) = I \cup (\phi(I) :_R a)$.

(iii) For every $a \in R - \sqrt{I}$, either $(I :_R a) = I$ or $(I :_R a) = (\phi(I) :_R a)$.

(iv) For the ideals A and B of R, $AB \subseteq I$ and $AB \nsubseteq \phi(I)$ imply $A \subseteq I$ or $B \subseteq \sqrt{I}$.

Proof. $(i) \Rightarrow (ii)$ Assume that I is ϕ-primary. Clearly, $I \cup (\phi(I) :_R a) \subseteq (I :_R a)$. On the other hand, for every $r \in (I :_R a)$, if $ra \in \phi(I)$, then $r \in (\phi(I) :_R a)$. Otherwise, from $ra \in I - \phi(I)$ and $a \notin \sqrt{I}$ we get $r \in I$. Hence, $(I :_R a) \subseteq I \cup (\phi(I) :_R a)$. Thus, $(I :_R a) = I \cup (\phi(I) :_R a)$.

$(ii) \Rightarrow (iii)$ Is clear because $(I :_R a)$ is an ideal of R.

$(iii) \Rightarrow (i)$ Assume that $ab \in I - \phi(I)$ for some $a, b \in R$. Obviously, $(I :_R a) \neq (\phi(I) :_R a)$. If $a \notin \sqrt{I}$, then by (iii), $(I :_R a) = I$. This implies that $b \in I$, that is I is ϕ-primary.

$(iii) \Rightarrow (iv)$ Let A and B be ideals of R with $AB \subseteq I$. Suppose that $A \nsubseteq I$ and $B \nsubseteq \sqrt{I}$. We will show that $AB \subseteq \phi(I)$. Let $b \in B$. We have two cases $b \notin \sqrt{I}$ and $b \in \sqrt{I}$. If the former case holds, then either $(I :_R b) = I$ or $(I :_R b) = (\phi(I) :_R b)$ by (iii). Now from $Ab \subseteq AB \subseteq I$ we have $A \subseteq (I :_R b)$. Choose $a \in A \setminus I$. Then from $a \in (I :_R b) \setminus I$ and (iii) we get $(I :_R b) = (\phi(I) :_R b)$. Therefore, $A \subseteq (I :_R b) = (\phi(I) :_R b)$, that is $Ab \subseteq \phi(I)$. Now suppose that the latter case holds. Then $b \in B \cap \sqrt{I}$. Choose $b' \in B \setminus \sqrt{I}$. Then $b + b' \in B \setminus \sqrt{I}$, and hence we have $Ab' \subseteq \phi(I)$ and $A(b + b') \subseteq \phi(I)$. Let $a \in A$. Then $ab = a(b + b') - ab' \in \phi(I)$. Hence, $Ab \subseteq \phi(I)$.

$(iv) \Rightarrow (i)$ Let $ab \in I \setminus \phi(I)$, where $a, b \in R$. Then $(a)(b) \subseteq I$, but $(a)(b) \nsubseteq \phi(I)$. The condition (iv) gives that $(a) \subseteq I$ or $(b) \subseteq \sqrt{I}$. Hence, I is ϕ-primary. □

Let J be an ideal of R and $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ a function. Define $\phi_J : \mathfrak{I}(R/J) \to \mathfrak{I}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $I \in \mathfrak{I}(R)$

with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$). In the following proposition we show that if I is a ϕ-primary ideal of R, then I/J is a ϕ_J-primary ideal of R/J.

Proposition 2.9. *Let Q be a proper ideal of the commutative ring R, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function. Assume that Q is a ϕ-primary ideal of R. Then*

(1) If J is an ideal of R with $J \subseteq Q$, then Q/J is a ϕ_J-primary ideal of R/J.

(2) If in addition $J \subseteq \phi(Q)$, and Q/J is ϕ_J-primary, then Q is ϕ-primary.

Proof.

(1) Assume that $a, b \in R$ are such that $(a + J)(b + J) \in Q/J - \phi_J(Q/J) = Q/J - (\phi(Q) + J)/J$. Then $ab \in Q - \phi(Q)$ and Q ϕ-primary gives $a \in Q$ or $b \in \sqrt{Q}$. Therefore, $a + J \in Q/J$ or $b + J \in \sqrt{Q}/J = \sqrt{Q/J}$. This shows that Q/J is ϕ_J-primary.

(2) Suppose that $ab \in Q - \phi(Q)$ for some $a, b \in R$. Then $(a + J)(b + J) \in Q/J - \phi(Q)/J = Q/J - \phi_J(Q/J)$. Since Q/J is assumed to be ϕ_J-primary, we get $a + J \in Q/J$ or $b + J \in \sqrt{Q/J} = \sqrt{Q}/J$. Consequently, either $a \in Q$ or $b \in \sqrt{Q}$, that is Q is ϕ-primary.

□

Let S be a multiplicatively closed subset of the commutative ring R. If Q is a P-primary ideal of R, it is easy to see that $P \cap S = \emptyset$ if and only if $Q \cap S = \emptyset$. It is also well known that if $Q \cap S = \emptyset$, then Q_S is a primary ideal of R_S and $Q_S \cap R = Q$. It is shown in [5, Proposition 2.8] that the second of these two results holds for weakly primary ideals. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function and define $\phi_S : \mathfrak{I}(R_S) \to \mathfrak{I}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = (\phi(J \cap R))_S$ (and $\phi_S(J) = \emptyset$ if $\phi(J \cap R) = \emptyset$) for every ideal J of R_S. Note that $\phi_S(J) \subseteq J$. In the following Proposition, we show that if Q is a ϕ-primary ideal of R with $Q \cap S = \emptyset$ and $\phi(Q)_S \subseteq \phi_S(Q_S)$, then Q_S is a ϕ_S-primary ideal of R_S.

Proposition 2.10. *Let R be a commutative ring, $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ a function and Q a ϕ-primary ideal of R. Suppose that S is a multiplicatively closed subset of R with $Q \cap S = \emptyset$ and $\phi(Q)_S \subseteq \phi_S(Q_S)$. Then Q_S is a ϕ_S-primary ideal of R_S. If $Q_S \neq \phi(Q)_S$, then $Q_S \cap R = Q$.*

Proof. Assume that $\frac{a}{s}, \frac{b}{t} \in R_S$ are such that $\frac{a}{s}\frac{b}{t} \in Q_S - \phi_S(Q_S)$. Then, there exists $c \in Q$ and $s' \in S$ such that $\frac{ab}{st} = \frac{c}{s'}$. Then $us'ab = ustc \in Q$ for some $u \in S$. Assume that $us'ab \in \phi(Q)$. Then $\frac{a}{s}\frac{b}{t} = \frac{us'ab}{us'st} \in \phi(Q)_S \subseteq \phi_S(Q_S)$, a contradiction. Hence $us'ab \in Q - \phi(Q)$. As Q is ϕ-primary, we get $us'a \in Q$ or $b \in \sqrt{Q}$. Therefore, either $\frac{a}{s} \in Q_S$ or $\frac{b}{t} \in (\sqrt{Q})_S = \sqrt{Q_S}$. This implies that Q_S is a ϕ_S-primary ideal of R_S.

Now assume that $Q_S \neq \phi(Q)_S$. Clearly, $Q \subseteq Q_S \cap R$. For the reverse containment, pick an element $a \in Q_S \cap R$. Then there exists $c \in Q$ and $s \in S$

with $\frac{a}{1} = \frac{c}{s}$. Therefore, $tsa = tc$ for some $t \in S$. If $(ts)a \notin \phi(Q)$, then $(ts)a \in Q - \phi(Q)$ and $\sqrt{Q} \cap S = \emptyset$ gives $a \in Q$. So assume that $(ts)a \in \phi(Q)$. In this case $a \in \phi(Q)_S \cap R$. Therefore, $Q_S \cap R \subseteq Q \cup (\phi(Q)_S \cap R)$. It follows that either $Q_S \cap R = \phi(Q)_S \cap R$ or $Q_S \cap R = Q$. If the former case holds, then $Q_S = \phi(Q)_S$ which is a contradiction. So the result follows. □

Let R be a commutative ring, and let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be a function. Clearly, every primary ideal of R is ϕ-primary. Theorems 2.11 and 2.13 provide some conditions under which a ϕ-primary ideal is primary.

Theorem 2.11. *Let R be a commutative ring, and let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be a function, and let Q be a ϕ-primary ideal of R.*

(1) If $Q^2 \nsubseteq \phi(Q)$, then Q is primary.

(2) If Q is not primary, then $\sqrt{Q} = \sqrt{\phi(Q)}$.

Proof.

(1) Assume that $a, b \in R$ are such that $ab \in Q$. If $ab \notin \phi(Q)$, since Q is ϕ-primary, either $a \in Q$ or $b \in \sqrt{Q}$. Hence we may assume that $ab \in \phi(Q)$. If $aQ \nsubseteq \phi(Q)$, then there exists an element $a_0 \in Q$ such that $aa_0 \notin \phi(Q)$. Now $a(a_0 + b) = aa_0 + ab \in Q - \phi(Q)$ and Q ϕ-primary imply that either $a \in Q$ or $a_0 + b \in \sqrt{Q}$. But $a_0 \in Q \subseteq \sqrt{Q}$. So, either $a \in Q$ or $b \in \sqrt{Q}$. Similarly, if $bQ \nsubseteq \phi(Q)$, we can show that either $a \in Q$ or $b \in \sqrt{Q}$. So we may assume that $aQ \subseteq \phi(Q)$ and $bQ \subseteq \phi(Q)$. Since $Q^2 \nsubseteq \phi(Q)$, there exist $c, d \in Q$ with $cd \notin \phi(Q)$. Now $(a+c)(b+d) = ab+ad+bc+cd \in Q-\phi(Q)$, imply that either $a + c \in Q$ or $b + d \in \sqrt{Q}$. Therefore, either $a \in Q$ or $b \in \sqrt{Q}$. Consequently, Q is primary.

(2) Since $\phi(Q) \subseteq Q$, we have $\sqrt{\phi(Q)} \subseteq \sqrt{Q}$. On the other hand, it follows from part (1) that $Q^2 \subseteq \phi(Q)$. Hence, $\sqrt{Q} = \sqrt{Q^2} \subseteq \sqrt{\phi(Q)}$; and hence $\sqrt{Q} = \sqrt{\phi(Q)}$.

□

Corollary 2.12. *Let Q be a ϕ-primary ideal where $\phi \leq \phi_3$. Then Q is ω-primary.*

Proof. If Q is primary, then it is ω-primary. So assume that Q is not primary. Then $Q^2 \subseteq \phi(Q) \subseteq Q^3$ by Theorem 2.11. Hence $\phi(Q) = Q^n$ for every $n \geq 2$. Consequently, Q is n-almost primary for every $n \geq 2$ and hence it is ω-primary by Proposition 2.6. □

Let R be a commutative ring. R is called decomposable if $R = R_1 \times R_2$ for some commutative rings R_1 and R_2. If I_1 is an ideal of R_1, then $Rad(I_1 \times R_2) = Rad(I_1) \times R_2$. Similarly, if I_2 is an ideal of R_2, then $Rad(R_1 \times I_2) = R_1 \times Rad(I_2)$.

Assume that both R_1 and R_2 are commutative rings with identity. Then, by [5, Theorem 2.6], the following hold:

(i) If P_1 is a primary ideal of R_1, then $P_1 \times R_2$ is a primary ideal of R.

(ii) If P_2 is a primary ideal of R_2, then $R_1 \times P_2$ is a primary ideal of R.

(iii) If P is a weakly primary ideal of R, then either $P = 0$ or P is primary.

Now consider the following results:

Theorem 2.13. *Let R_1 and R_2 be commutative rings and let $\psi_i : \Im(R_i) \to \Im(R_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. Set $R = R_1 \times R_2$, and $\phi = \psi_1 \times \psi_2$. Then, Q is a ϕ-primary ideal of R if and only if one of the following cases hold:*

(1) $Q = Q_1 \times Q_2$ where, for $i = 1, 2$, Q_i is a proper ideal of R_i with $\psi_i(Q_i) = Q_i$.

(2) $Q = Q_1 \times R_2$ where Q_1 is a ψ_1-primary ideal of R_1 which will be primary if $\psi_2(R_2) \neq R_2$.

(3) $Q = R_1 \times Q_2$ where Q_2 is a ψ_2-primary ideal of R_2 which will be primary if $\psi_1(R_1) \neq R_1$.

Proof. First assume that Q is a ϕ-primary ideal of R. Then $Q = Q_1 \times Q_2$ for some ideals Q_1 and Q_2 of R_1 and R_2, respectively. First we show that, for $i = 1, 2$, Q_i is a ψ_i-primary ideal of R_i. Let $a_1, b_1 \in R_1$ be such that $a_1 b_1 \in Q_1 - \psi_1(Q_1)$. Then, $(a_1, 0)(b_1, 0) = (a_1 b_1, 0) \in Q_1 \times Q_2 - \psi_1(Q_1) \times \psi_2(Q_2) = Q - \phi(Q)$. As Q is ϕ-primary, either $(a_1, 0) \in Q$ or $(b_1, 0) \in \sqrt{Q}$. So either $a_1 \in Q_1$ or $b_1 \in \sqrt{Q_1}$, that is Q_1 is a ψ_1-primary ideal of R_1. In a similar way, one can show that Q_2 is a ψ_2-primary ideal of R_2. Now we show that Q has one of the forms $(1) - (3)$. If $\phi(Q) = Q$, then $\psi_i(Q_i) = Q_i$ for $i = 1, 2$. So assume that $\phi(Q) \neq Q$. Then, either $Q_1 \neq \psi_1(Q_1)$ or $Q_2 \neq \psi_2(Q_2)$. If the former case holds, there exists $c \in Q_1 - \psi_1(Q_1)$. For every $d \in Q_2$, from $(c, 1)(1, d) = (c, d) \in Q_1 \times Q_2 - \psi_1(Q_1) \times \psi_2(Q_2) = Q - \phi(Q)$ we get $(c, 1) \in Q_1 \times Q_2$ or $(1, d) \in \sqrt{Q_1} \times \sqrt{Q_2}$. Hence, either $Q_2 = R_2$ or $Q_1 = R_1$. Suppose that $Q_2 = R_2$. Then, $Q = Q_1 \times R_2$ is a ϕ-primary ideal of R, where Q_1 is a ψ_1-primary ideal of R_1. Now assume that $\psi_2(R_2) \neq R_2$. Let $a_1, b_1 \in R_1$ be such that $a_1 b_1 \in Q_1$. Then, $(a_1, 1)(b_1, 1) = (a_1 b_1, 1) \in Q_1 \times R_2 - \psi_1(Q_1) \times \psi_2(R_2) = Q - \phi(Q)$ since $1 \notin \psi_2(R_2)$. As $Q = Q_1 \times R_2$ is ϕ-primary we get $(a_1, 1) \in Q_1 \times R_2$ or $(b_1, 1) \in \sqrt{Q_1 \times R_2} = \sqrt{Q_1} \times R_2$. So, either $a_1 \in Q_1$ or $b_1 \in \sqrt{Q_1}$, and this implies that Q_1 is primary. If the latter case holds, that is if $Q_2 \neq \psi_2(Q_2)$, one can show that $Q = R_1 \times Q_2$ is ϕ-primary, where Q_2 is ψ_2-primary which must be primary if $\psi_1(R_1) \neq R_1$.

Next we show that an ideal of R having one of these three types is ϕ-primary. In case (1) we have $\phi(Q) = \psi_1(Q_1) \times \psi_2(Q_2) = Q_1 \times Q_2 = Q$. So, obviously Q is ϕ-primary. If the case (2) holds, and if Q_1 is primary, then $Q = Q_1 \times R_2$ is a primary ideal of R and so it is ϕ-primary. So, assume that Q_1 is ψ_1-primary and $\psi_2(R_2) = R_2$. Let $(a_1, a_2), (b_1, b_2) \in R$ be such that $(a_1 b_1, a_2 b_2) = (a_1, a_2)(b_1, b_2) \in Q - \phi(Q) = Q_1 \times R_2 - \psi_1(Q_1) \times \psi_2(R_2) = (Q_1 - \psi_1(Q_1)) \times R_2$. Then, $a_1 b_1 \in Q_1 - \psi_1(Q_1)$ gives $a_1 \in Q_1$ or $b_1 \in \sqrt{Q_1}$. Thus, either $(a_1, a_2) \in$

$Q_1 \times R_2$ or $(b_1, b_2) \in \sqrt{Q_1} \times R_2 = \sqrt{Q_1 \times R_2}$. Hence, Q is ϕ-primary. The proof for the case (3) is similar. □

Theorem 2.14. *(1) Let T and S be commutative rings and let I be a weakly primary ideal of T. Then $J = I \times S$ is a ϕ-primary ideal of $R = T \times S$ for each ϕ with $\phi_\omega \leq \phi \leq \phi_1$.*

(2) Let R be a commutative ring and let J be a finitely generated proper ideal of R. Suppose that J is ϕ-primary where $\phi \leq \phi_3$. Then, either J is weakly primary or $J^2 \neq 0$ is idempotent and R decomposes as $T \times S$ where $S = J^2$ and $J = I \times S$ where I is weakly primary. Hence J is ϕ-primary for each ϕ with $\phi_\omega \leq \phi \leq \phi_1$.

Proof.

(1) Let I be a weakly primary ideal of T, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function with $\phi_\omega \leq \phi$. If I is actually primary, then J is primary and hence is ϕ-primary for all ϕ. So, assume that I is not primary. Then $I^2 = 0$ by [5, Theorem 2.2]. So $J^2 = 0 \times S$. It follows that $\phi_\omega(J) = 0 \times S$. So, $J - \phi_\omega(J) = I \times S - 0 \times S = (I - \{0\}) \times S$. Assume that $(x_1, x_2), (y_1, y_2) \in R$ are such that $(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2) \in J - \phi_\omega(J)$. Then, $x_1 y_1 \in I - \{0\}$. So, either $x_1 \in I$ or $y_1 \in \sqrt{I}$ since I is a weakly primary ideal of T. Therefore, either $(x_1, x_2) \in J$ or $(y_1, y_2) \in \sqrt{I} \times S = \sqrt{I \times S}$. Therefore, J is ϕ_ω-primary and so it is ϕ-primary.

(2) If J is primary, then J is weakly primary. So we may assume that J is not primary. Then, by Theorem 2.11, $J^2 \subseteq \phi(J)$ and hence $J^2 \subseteq \phi(J) \subseteq \phi_3(J) = J^3$. So $J^2 = J^3$. Hence, J^2 is idempotent. Since J^2 is finitely generated, we have $J^2 = Re$ for some idempotent element $e \in R$. Consider the two cases $J^2 = 0$ and $J^2 \neq 0$. If the former case holds, then $\phi(J) \subseteq J^3 = 0$. So $\phi(J) = 0$ and hence J is weakly primary. In the latter case, put $S = J^2 = Re$ and $T = R(1-e)$. Then, $R = T \times S$. Let $I = J(1-e)$. Then, $J = I \times S$ where $I^2 = (J(1-e))^2 = J^2(1-e)^2 = (e)(1-e) = 0$. We show that I is weakly primary. Suppose that $xy \in I - \{0\}$. Then $(x, 1)(y, 1) = (xy, 1) \in I \times S - (I \times S)^2 = I \times S - 0 \times S \subseteq J - \phi(J)$ since $\phi \leq \phi_3$. implies $\phi(J) \subseteq J^3 = (I \times S)^3 = 0 \times S$. Hence, $(x, 1) \in J$ or $(y, 1) \in \sqrt{J}$ since J is assumed to be ϕ-primary. Therefore, $x \in I$ or $y \in \sqrt{I}$, that is I is weakly primary.

□

Corollary 2.15. *Let R be an indecomposable commutative ring and J a finitely generated ϕ-primary ideal of R where $\phi \leq \phi_3$. Then J is weakly primary. If, further, R is an integral domain, J is actually primary.*

Corollary 2.16. *Let R be a Noetherian integral domain. A proper ideal J of R is primary if and only if $xy \in J - J^3$ implies $x \in J$ or $y \in \sqrt{J}$.*

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