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THE WEAK CONGRUENCE REPRESENTABILITY OF SUBLATTICES AND SUBORDERS OF REPRESENTABLE LATTICES

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Abstract. In this paper we introduce a new direction in the investigations of the open problem of the representation of algebraic lattices by the weak congruence lattices. In some cases the representability of some lattice follows from the representability of another lattice. We give some results.

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1. Introduction

The weak congruence lattice of an algebra is an important lattice containing lots of information about the algebra. If $\mathcal{A}$ is an algebra, we denote by $Cw\mathcal{A}$ the week congruence lattice of $\mathcal{A}$. If A is the support of the algebra, the relation $\Delta = \{(x, x) \mid x \in A\}$ is an element of $Cw\mathcal{A}$. The ideal $\downarrow\Delta$ of $Cw\mathcal{A}$ is isomorphic to the subalgebra lattice of $\mathcal{A}$, and filter $\uparrow\Delta$ is the congruence lattice of $\mathcal{A}$. $Cw\mathcal{A}$ is a complete, algebraic lattice. It also contains the congruence lattices of all the subalgebras of $\mathcal{A}$ as interval sublattices. Given an algebraic lattice $\mathcal{L}$ and its element a, we say that a lattice $\mathcal{L}$ together with its element a is weak congruence representable, if there is an algebra $\mathcal{A}$ such that $Cw\mathcal{A}$ is isomorphic to $\mathcal{L}$ under an isomorphism mapping Δ to a. For such an element a of $\mathcal{L}$ we say that it is Δ-suitable in $\mathcal{L}$.

In certain cases the representability of a lattice causes representability of another related lattice. This could be a sublattice, or a lattice which is a suborder of the initial lattice, or in another way related to the initial lattice. Ideals and filters are special cases of sublattices, for which we will investigate whether their representability follows from the representability of the initial lattice. We will discuss some other cases of suborders and sublattices. In cases in which a proof is given that the representability of a lattice $\mathcal{L}'$ follows from the representability of another lattice $\mathcal{L}$, the proof also provides an algorithm, by which we get a representation of the lattice $\mathcal{L}'$ from any representation of $\mathcal{L}$.

Since Δ is a codistributive element of $Cw\mathcal{A}$, any codistributive element has to be Δ-suitable. Some known necessary conditions for a codistributive element of an algebraic lattice to be Δ-suitable are given in [10] and [5]. Some of them

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are generalized in [7]. Those criteria will help us prove that in some cases the representability of a lattice does not imply the representability of some related lattices.

2. Preliminaries

We denote by $Sub\mathcal{A}$ the set of all the subalgebras of an algebra $\mathcal{A}$, as well as the lattice it forms under inclusion. By $Con\mathcal{A}$ we denote the set of all congruences, as well as the corresponding lattice. We define a notion more general than congruence:

Definition 1. ([4]) Let $\mathcal{A}$ be an algebra with the support A, and ρ a relation on A. We say that ρ is a weak congruence of $\mathcal{A}$ if it is symmetric, transitive and compatible with all the operations of $\mathcal{A}$, including nullary ones.

The set of all weak congruences of $\mathcal{A}$ we denote by $Cw\mathcal{A}$. Note that $\Delta = \{(x,x) \mid x \in A\}$ is a weak congruence.

Theorem 2. ([5]) *The collection $Cw\mathcal{A}$ of weak congruences on an algebra $\mathcal{A}$ is an algebraic lattice under inclusion.*

Note that the congruences are reflexive weak congruences, thus $Con\mathcal{A} \subseteq Cw\mathcal{A}$. Moreover, $Con\mathcal{A}$ is a sublattice of $Cw\mathcal{A}$, i.e. the ideal $\uparrow\Delta$ of $Cw\mathcal{A}$. The next theorem reveals more of the structure of the weak congruence lattice. We need the following notation:

$\Delta_B = \{(x,x) \mid x \in B\}$, for any $B \subseteq A$.

Notice that Δ_B is a weak congruence of $\mathcal{A}$ if and only if B is a subuniverse of $\mathcal{A}$.

Theorem 3. ([9]) *If $Cw\mathcal{A}$ is the lattice of weak congruences of an algebra $\mathcal{A}$, then:*

(i) *for every subalgebra $\mathcal{B}$ of $\mathcal{A}$, $Con\mathcal{B}$ is the interval sublattice $[\Delta_B, B^2]$, in particular $Con\mathcal{A} =\uparrow\Delta$;*

(ii) *the lattice $Sub\mathcal{A}$ of the subuniverses of $\mathcal{A}$ is isomorphic with the principal ideal $\downarrow\Delta$, under $\mathcal{B} \mapsto \Delta_B$;*

(iii) *the map $m_\Delta : \rho \mapsto \rho \wedge \Delta$ is a homomorphism from $Cw\mathcal{A}$ onto $\downarrow\Delta$.*

Some properties of the weak congruence lattice $Cw\mathcal{A}$ and its element Δ imply some necessary conditions for an element a of a lattice to be Δ-suitable. Since Δ is codistributive, we have that any Δ-suitable element of a lattice is codistributive.

A codistributive element of an algebraic lattice fulfills the following theorem:

Theorem 4. ([5]) *If an element a of an algebraic lattice $\mathcal{L} = (L, \vee, \wedge)$ is codistributive, then for every $b \in\downarrow a$, the family $\{x \in L \mid a \wedge x = b\}$ has the top element.*

If $\mathcal{L}$ is an algebraic lattice and $x \in L$, we denote the top element of the family $\{y \in L \mid a \wedge y = a \wedge x\}$ by $\overline{x}$.

Some further conditions a codistributive element of an algebraic lattice must fulfil in order to be Δ-suitable are given in the following proposition and theorem, and they are based on the properties of the weak congruence lattice:

Proposition 5. ([7]) *If a is a Δ-suitable element of a lattice $\mathcal{L}$ and $x, y \in L$, such that $x \prec y \leq a$ and $\bar{x} \neq 0$, then the interval $[\bar{x}, \bar{y})$ has the top element.*

Theorem 6. ([7]) *A Δ-suitable element $a \in L$ satisfies the following:*
(1) *if $x \wedge y \neq \mathbf{0}$ then $\overline{x \vee y} = \overline{x} \vee \overline{y}$;*
(2) *if $\overline{x} \neq \mathbf{0}$ and $\overline{x} < y$, then $\overline{y \wedge a} \neq y \wedge a$;*
(3) *If $\overline{x} \neq 0$ and $x < y \leqslant a$, then $[y \vee \overline{x}, \overline{y}) \setminus \bigcup_{z \in (x,y)} [y \vee \overline{z}, \overline{y})$ is either the empty set, or has the top element;*
(4) *If $\overline{x} \neq 0$ and $x < y \leqslant a$, there is a mapping $\varphi : [x, y) \to [y, \overline{y})$, such that:*
- for all $t \in [x, \overline{x}]$ and $u \in [x, y)$, the set $\{c \in \mathrm{Ext}(t) \mid c \leqslant \varphi(u)\}$ is either empty or has the top element, and
- for all $t \in [x, \overline{x}]$, the set $\{c \in \mathrm{Ext}(t) \mid (\forall u \in [x, y))(c \nleqslant \varphi(u))\}$ is an antichain (possibly empty), where

$$\mathrm{Ext}(t) := \{w \in [y, \overline{y}] \mid w \cap \overline{x} = t\}.$$

3. The representability of filters, ideals and convex sublattices

3.1. Filters of the representable lattices

A first problem we are going to discuss is: if a lattice is representable by the weak congruence lattice of an algebra, the question is whether a filter of the lattice is representable. Recall that a filter of a lattice is principal if it is of the form $\uparrow f$, for an element f of the lattice. Since a representable filter must be an algebraic lattice and thus complete, it must be principal. The problem is equivalent to the following: if $\mathcal{L} = (L, \vee, \wedge)$ is a representable lattice, for what kind of element $l \in L$ is the lattice $\uparrow l$ representable? If a is a Δ-suitable element of the lattice, what element of the lattice $\uparrow l$ must be Δ-suitable? If $l \leqslant a$, the only question making sense is whether a must be Δ-suitable in the lattice $\uparrow l$, since the equivalence classes under the equivalence defined by $x \sim y \Leftrightarrow a \wedge x = a \wedge y$ could be changed drastically if we replace a with any other element; but for $a = l$ we only lose some classes, and the remaining ones are unchanged. And the answer is yes:

Proposition 7. *If a is a Δ-suitable element of a lattice $\mathcal{L}$, then it is also Δ-suitable in the lattice $\uparrow b$, for any $b \leqslant a$.*

Proof. If b is the bottom element in the lattice, the proposition is obviously true, so we can assume $b \neq 0$. Let $\mathcal{A} = (A, H)$ be an algebra, whose weak congruence lattice is isomorphic to $\mathcal{L}$ under an isomorphism ϕ mapping Δ_A to a. The same isomorphism maps Δ_B to b, for a subuniverse B of the algebra $\mathcal{A}$.

Now, if $\mathcal{A}' = (A, F)$, where $F = H \cup B$ (F is made by adding all the elements of B as nullary operations to the set H of the operations of $\mathcal{A}$), then $Sub\mathcal{A}' \cong [b, a]$ and $Cw\mathcal{A}' \cong \uparrow b$. □

Now, if a is a Δ-suitable element of the lattice $\mathcal{L}$ and $l \not\leq a$, then $a \notin \uparrow l$ and the question making sense for the filter $\uparrow l$ is whether the element $l \vee a$ is Δ-suitable in general. The answer here is negative, as it is shown by the lattice in Figure 1. It is representable, i.e. its element a is Δ-suitable. Let $A = \{a, b, c, d, e, f, g\}$ and $\mathcal{A} = (A, \{*, e, f\})$, where * is given by the table below:

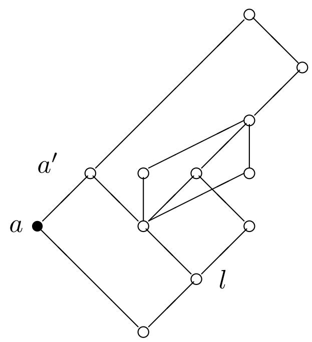


Figure 1

$*$	a	b	c	d	e	f	g
a	b	a	c	d	d	d	g
b	b	b	b	d	d	d	g
c	c	c	c	d	d	d	g
d	d	d	d	c	a	a	c
e	e	e	e	c	b	a	d
f	f	f	f	c	a	b	e
g	c	a	b	d	e	f	f

Now, $Cw\mathcal{A}$ is isomorphic to the lattice in Figure 1, its diagonal element corresponding to the element a of the same lattice, thus a is Δ-suitable. But this does not hold for the element $a' = a \vee l$ in the lattice $\uparrow l$, since the condition (4) of Theorem 6 is not satisfied.

Therefore, the suitability of the element a in a lattice $\mathcal{L}$ in general does not imply the suitability of the lattice $\uparrow l$ in case when $l \notin \downarrow a$. This could be explained by the fact that the equivalence classes of $\mathcal{L}$ in the equivalence given by: $x \sim y \Leftrightarrow a \wedge x = a \wedge y$, intersected with $\uparrow l$, generally change, even if we replace a with $a' = a \vee l$, as it is shown in the given example in which, passing from a to a', we got one equivalence class more - from two classes, we got three. This will not be the case in the analogous situation with the principal ideals.

3.2. Ideals of the representable lattices

An ideal in a lattice has to be principal, as it is case with the filters, to be representable: every representable lattice, since it is algebraic, has to be complete and to have a top element, which is at the same time the supremum of the ideal in the initial lattice. Conversely, any principle ideal of a representable (and thus algebraic) lattice has to be algebraic. So the representability problem in case of an ideal is confined to the question: for what kind of an element x of a lattice $\mathcal{L}$, representability of the ideal $\downarrow x$ follows from the representability of $\mathcal{L}$. More precisely: if a is a Δ-suitable element of the lattice $\mathcal{L} = (L, \vee, \wedge)$, which element of the ideal $\downarrow x$ could be Δ-suitable? If we take $a \wedge x$ to represent the diagonal element, unlike the case of the filter $\uparrow x$, the equivalence classes of elements of L which are less than or equal to x - in the equivalence defined by $y \sim z \Leftrightarrow a \wedge y = a \wedge z$ - intersected with $\downarrow x$ are exactly the equivalences of $\downarrow x$ - in the equivalence defined by $y \sim z \Leftrightarrow (a \wedge x) \wedge y = (a \wedge x) \wedge z$. Any other choice of an element to represent the diagonal element could result in a complete change in the mentioned equivalence classes of the initial lattice, so it is hard to speak of the representability of the ideal $\downarrow x$ in general, in the way described above.

Therefore, a question we are going to discuss is the following: if a is a Δ-suitable element of a lattice $\mathcal{L} = (L, \vee, \wedge)$ and $x \in L$, is $a \wedge x$ Δ-suitable in lattice $\downarrow x$? The answer is positive, if x is the top element of the class it belongs to, i.e. if $x = \overline{x} = \overline{x \wedge a}$:

Proposition 8. *If a is a Δ-suitable element of a lattice $\mathcal{L} = (L, \vee, \wedge)$ and x an element of L such that $x = \overline{x}$, then $a \wedge x$ is Δ-suitable in the lattice $\downarrow x$.*

Proof. Let $\mathcal{A} = (A, H)$ be an algebra such that $Cw\mathcal{A}$ is isomorphic to $\mathcal{L}$ under a lattice isomorphism π, mapping Δ to a. Since $a \wedge x \leqslant a$, $a \wedge x$ corresponds to Δ_B, for a subalgebra $\mathcal{B} = (B, H)$ of algebra $\mathcal{A}$. Now, $Sub\mathcal{B}$ is isomorphic to $\downarrow a \wedge x$ under the isomorphism $\pi|_{\downarrow \Delta_B}$ and $Cw\mathcal{B}$ is isomorphic to $\downarrow x$ under the isomorphism $\pi|_{\downarrow B^2}$ that maps the diagonal relation Δ_B on $a \wedge x$. Therefore, $a \wedge x$ is Δ-suitable in the lattice $\downarrow x$. □

However, if x is not the top element of its class, i.e. $\overline{x} \neq x$, element $a \wedge x$ does not have to be Δ-suitable in the lattice $\downarrow x$. Take, for example, the lattice in Figure 2. It is representable, i.e. its element a is Δ-suitable: let $A = \{a, b, c, d, e, f\}$ and $\mathcal{A} = (A, \{*, b\})$, where * is defined by the table below:

$*$	a	b	c	d	e	f
a	a	a	b	b	a	f
b	b	a	a	b	b	a
c	d	d	e	e	e	f
d	d	d	e	e	e	a
e	d	d	c	c	e	a
f	b	d	a	a	c	f

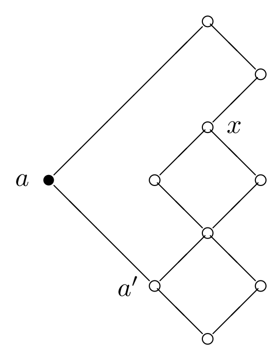
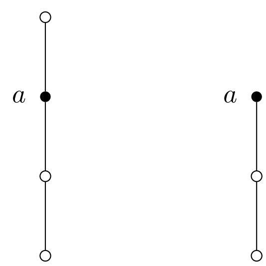


Figure 2

Notice that the lattice $Cw\mathcal{A}$ is isomorphic to the lattice in Figure 2, whose element a is thus Δ-suitable, because it is related to Δ_A in the isomorphism. Nevertheless $a \wedge x$ is not Δ-suitable in lattice $\downarrow x$, due to the fact that the condition from Proposition 5 is not satisfied in that case. This also means that none of conditions (3) and (4) of Theorem 6 is not satisfied, because these two conditions are certain generalizations of Proposition 5. Thus, conditions (3) and (4) of Theorem 6 do not have to be preserved when passing from a lattice to its ideal $\downarrow x$, if $x \neq \overline{x}$.

Condition (2) of Theorem 6 also may not be preserved in a principle ideal of a lattice, as it is shown in Figure 3: the presented lattice is representable, for example by the groupoid $\mathcal{A} = (A, *)$, where $A = \{a, b\}$, and $*$ is given by the table in the same figure; however, the ideal $\downarrow a$ of the lattice is not representable, i.e. a is not Δ-suitable in this case, for it does not satisfy condition (2) of Theorem 6.



$*$	a	b
a	a	b
b	b	a

Figure 3

Finally, even condition (1) of Theorem 6 does not have to be preserved in the ideal $\downarrow x$, when $x \neq \overline{x}$. Look at the lattice in Figure 4. It is representable by the algebra $\mathcal{A} = (A, *, f)$, where $A = \{a, b, c, d, e\}$ and operations $*$ and f are represented by the table below:

$*$	a	b	c	d	e
a	a	a	a	a	a
b	b	c	c	e	d
c	c	b	b	e	e
d	e	b	b	d	d
e	e	c	b	e	d

f	a	b	c	d	e
	a	a	a	a	a

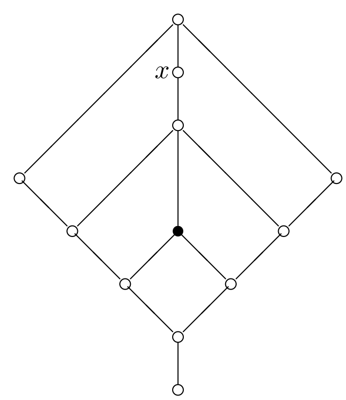


Figure 4

However, the ideal $\downarrow x$ is not representable, for condition (1) of Theorem 6 is not fulfilled in it.

Thus, not only that the ideal $\downarrow x$ of a representable lattice does not have to be representable, but it also does not have to preserve any of the known criteria for a Δ-suitable element, contained in Theorem 6, if $x \neq \overline{x}$.

3.3. Convex sublattices of the representable lattices

We can now face the problem of representability of a convex sublattice of a representable lattice $\mathcal{L}$. In order to be representable, a convex sublattice $\mathcal{M}$ must be algebraic, and thus have a smallest and a top element. If b is the smallest and c is the top element of $\mathcal{M}$, we have $\mathcal{M} = [b, c]$, so the problem of representability of a convex sublattice of $\mathcal{L}$ is equivalent to the problem of representability of an interval of $\mathcal{L}$. By Proposition 8 and Proposition 7 we have the following:

Proposition 9. *If a is a Δ-suitable element of a lattice $\mathcal{L} = (L, \vee, \wedge)$, $b \in\downarrow a$ and $c \in\uparrow b$ such that $c = \overline{c}$, then $a \wedge c$ is Δ-suitable in the interval sublattice [b,c].*

Proof. By Proposition 7 we have that a is Δ-suitable in the filter $\mathcal{B} =\uparrow b$. Since $c = \overline{c}$ in the lattice $\mathcal{B}$, by Proposition 8 we have that $a \wedge c$ is Δ-suitable in the ideal $\downarrow c$ of $\mathcal{B}$. But, that ideal is the interval sublattice [b,c] of $\mathcal{L}$ and the proposition follows. □

On the basis of the above considerations we can conclude immediately that the representability of the intervals of any different type from those given in Proposition 9, does not follow from the representability of that lattice. This does not, of course, mean that an interval of a different kind will not be representable in a particular case.

4. Representability of suborders and sublattices of a representable lattice

By certain simplification of a representable lattice we can produce another representable lattice:

Theorem 10. *If a is a Δ-suitable element of a lattice $\mathcal{L}$, and b a compact element of $\downarrow a$, then a is a Δ-suitable element of the lattice $\mathcal{L}'$ with the universe $L' = L \setminus \cup\{(c, \bar{c}) | c \in [b, a]\}$ and the same order as that of L ($\mathcal{L}'$ is a subposet of $\mathcal{L}$).*

Proof. Let $\mathcal{A} = (A, H)$ be an algebra representing $\mathcal{L}$, such that Δ_A represents a, and Δ_B represents b. Since b is compact, the algebra B is generated by a finite set $X = \{b_1, b_2, ..., b_n\}$. Let $\mathcal{A}' = (A, H \cup \{f_a | a \in A\})$, where f_a is $(n + 2)$-ary operation defined in the following way:

$$f_a(x_1, x_2, \ldots, x_{n+2}) = \begin{cases} a, & X \subseteq \{x_1, x_2, \ldots, x_{n+2}\} \wedge (x_1 = a) \\ x_2, & \text{else} \end{cases}$$

Notice that any subset of A is closed for any operation f_a, so that any subuniverse of $\mathcal{A}$ is also a subuniverse of $\mathcal{A}'$, and vice versa. We also have the following:

$$[\Delta_C, C^2]_{Cw\mathcal{A}} = [\Delta_C, C^2]_{Cw\mathcal{A}'}, \text{ for all subuniverses } C \text{ of } \mathcal{A},\ C \not\supseteq B.$$

Let C be a superset of B which is also a subuniverse of $\mathcal{A}$; with $f|_C$ we denote the restriction of f to C, and with $F|_C$ the set of the restrictions of the functions in set F to C; let $\rho \in Con\mathcal{C}'$, where $\mathcal{C}' = (C, H|_C \cup \{f_a|_C\ |a \in A\})$ then $\rho = \Delta_C$ or $\rho = C^2$:

If $\rho \neq \Delta_C$, then there exist $x, y \in C$, such that $x \neq y$ and $x\rho y$. But then $\rho = C^2$, because $b_1, b_2, \ldots, b_n \in C$ and for all $z \in C$:

$$f_x(x, z, b_1, b_2, \ldots, b_n)\rho f_x(y, z, b_1, b_2, \ldots, b_n), \text{ i.e. } x\rho z.$$

So, $\mathcal{L}' \cong Cw\mathcal{A}'$. □

Taking $b = a$ in the above theorem, we get the following corollary:

Corollary 11. *If a is a Δ-suitable and compact element of a lattice $\mathcal{L}$, then a is a Δ-suitable element of the lattice $\mathcal{L}'$ with the universe $L' = L \setminus (a, \bar{a})$, which is a subposet of L.*

This is important in connection with the representability of an algebraic lattice by the weak equivalence lattice of a simple algebra. Namely, for an algebraic lattice $\mathcal{L} = (L, \vee, \wedge)$, to be representable as the weak congruence lattice of a simple algebra, so that $a \in L$ is Δ-suitable, it is necessary that $(a, 1) = \emptyset$. On the basis of the above corollary we get a sufficient condition for an algebraic lattice $\mathcal{L} = (L, \vee, \wedge)$ to be representable by a simple algebra: it suffices enough that there exists a representable lattice $\mathcal{L}' = (L', \vee, \wedge)$, such that $L' \supset L$ and $L' \setminus (a, 1) = L$, and the order on L is the same as on L'. This is also a sufficient condition, for if a lattice is representable by a simple algebra, then for $L' = L$ we get a representable lattice $\mathcal{L}'$, where $L' \setminus (a, 1) = L' = L$.

The above theorem and its corollary could be generalized. To prove the next theorem we introduce the following definition:

Definition 12. If f is a function $f : S \to S$ and ρ an equivalence relation on S, we say that f is a ρ-projection if it satisfies the condition:
$(\forall x, y \in S)[x\rho y \Rightarrow (f(y) = f(x) \text{ and } f(x)\rho x)]$

Theorem 13. *If a is a Δ-suitable element of a lattice $\mathcal{L}$ and b a compact element of $\downarrow a$ and $d \in [b, \bar{b}]$, then a is a Δ-suitable element of the lattice $\mathcal{L}'$ with the universe $L' = L \setminus \cup\{(c, \bar{c}) \setminus [d \vee c, \bar{c}] | c \in [b, a]\}$ and of the same order as that of $\mathcal{L}$ ($\mathcal{L}'$ is a subposet of $\mathcal{L}$).*

Proof. Let $\mathcal{A} = (A, H)$ be an algebra whose weak congruence lattice is isomorphic to $\mathcal{L}$, such that Δ_A represents a, Δ_B represents b, and d is represented by $\rho \in ConB$. Since b is compact, the algebra B is generated by a finite set $X = \{b_1, b_2, ..., b_n\}$.

Let $F = \{f : A \to A \mid f_{|B}$ be a ρ-projection and $f(x) = x$ for $x \in A \setminus B\}$

For $f \in F$ we define an operation on A of arity $n+3$ as follows:

$$g_f(x_1, x_2, \ldots, x_{n+3}) = \begin{cases} f(x_1), & X \subseteq \{x_1, x_2, \ldots, x_{n+3}\} \wedge (x_2 = x_3) \\ x_1, & \text{else} \end{cases}$$

Now, let $\mathcal{A}' = (A, H \cup \{g_f \mid f \in F\})$. We prove that $Cw\mathcal{A}' = \mathcal{L}'$.

Notice that any subuniverse of $\mathcal{A}$ is closed under any operation g_f, for any subuniverse containing X contains B as well and is closed under any ρ-projection, and under any $f \in F$. Therefore, any subuniverse of $\mathcal{A}$ is also a subuniverse of $\mathcal{A}'$, and vice versa. We also have the following:

$[\Delta_C, C^2]_{Cw\mathcal{A}} = [\Delta_C, C^2]_{Cw\mathcal{A}'}$, for all subuniverses C of $\mathcal{A}$, $C \not\supseteq B$.

If C is a superset of B and subuniverse of $\mathcal{A}$, and $\tau \in Con\mathcal{C}'$, where $\mathcal{C}' = (C, H|_C \cup \{g_f|_C \mid f \in F\})$ then $\tau = \Delta_C$ or $\tau \supset \rho$:

If $\tau \neq \Delta_C$, then there exist $x, y \in C$, such that $x \neq y$ and $x\tau y$. But then $\tau \supset \rho$:

$c\rho d \Rightarrow f(c) = d$ for some $f \in F \Rightarrow g_f(c, x, y, b_1, \ldots, b_n)\tau g_f(c, x, x, b_1, \ldots, b_n)$
$\Rightarrow c\tau d.$

Conversely, if $\tau \in Con\mathcal{C}$ and $\tau \supset \rho$, then τ is compatible with all the operations on F, and so $\tau \in Con\mathcal{C}'$.

So, $\mathcal{L}' \cong Cw\mathcal{A}'$. □

Here it was not necessary to prove that $\mathcal{L}'$ is a lattice, because it follows from the fact that the corresponding weak congruences of algebra $\mathcal{A}$ are exactly all the weak congruences of $\mathcal{A}'$, which form a lattice under the same order. Thus we have proved that $\mathcal{L}'$, as a poset, is a subposet of $\mathcal{L}$, whose representability as a lattice follows from the representability of $\mathcal{L}$. Generally, this lattice does not have to be a sublattice of $\mathcal{L}$. Let us take for example the lattice $\mathcal{L}$ in Figure 5, and the corresponding lattice $\mathcal{L}'$, according to the previous theorem:

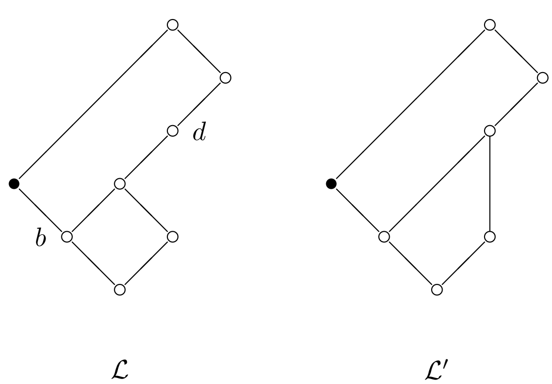


Figure 5

The lattice $\mathcal{L}'$ is, as an ordered set, a suborder of the lattice $\mathcal{L}$, observed as an ordered set, but it is not a sublattice of $\mathcal{L}$. However, if we take from the lattice $\mathcal{L}'$ all the intervals of the form $[c, \overline{c}]$, where $c \in \downarrow a \setminus \uparrow b$, we will get a representable lattice, which is a sublattice of $\mathcal{L}$. Namely, from the previous theorem we derive the following corollary:

Corollary 14. *If a is a Δ-suitable element of a lattice $\mathcal{L}$, $b \leq a$ and $d \in [b, \overline{b}]$, then a is Δ-suitable in the sublattice $\mathcal{L}'$ of $\mathcal{L}$, the support of which is $L' = [b, a] \cup \uparrow d$.*

Proof. If $\mathcal{A}' = (A, H \cup B)$, then, on the basis of Proposition 7, the element a is Δ-suitable in the filter $\uparrow b$. Now we apply the previous theorem to the lattice $\mathcal{L}_1$, which is equal to $\uparrow b$, and to the elements a and b. Since b is the least element of $\mathcal{L}_1$, it is compact in $\mathcal{L}_1$, so that if $d \in [b, \overline{b}]$, a is Δ-suitable element of $\mathcal{L}'$ with the universe $L' = L_1 \setminus \cup\{(c, \overline{c}) \setminus [d \vee c, \overline{c}] \mid c \in [b, a]\}$ - where L_1 is the carrier of the lattice $\mathcal{L}_1$ - and the order equal to the order $\mathcal{L}_1$. Obviously, $L' = [b, a] \cup \uparrow d$. The set L' is closed for operations $\vee$ and $\wedge$ of lattice $\mathcal{L}_1$ and $\mathcal{L}$, so that $\mathcal{L}'$ is a sublattice of the lattices $\mathcal{L}_1$ i $\mathcal{L}$. □

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