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ANALYTIC REPRESENTATION FOR A PRODUCT OF A REAL ANALYTIC AND AN L^1-FUNCTION IN $\mathbb{R}^n$

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Abstract. We give an elementary proof of the known result that $P(z)\hat{f}(z)$ is an analytic representation for $f(x)P(x)$, where P is real analytic and $f \in L^1(\mathbb{R}^n)$ and $\hat{f}$ is the analytic representation of f.

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1. Introduction

Recall that the Cauchy kernel in $\mathbb{R}^n$ has the form

$$K(t-z) = \frac{1}{(2\pi i)^n} \prod_{j=1}^{n} \frac{1}{t_j - z_j},$$

where $z = (z_1, \ldots, z_n)$, $t = (t_1, \ldots, t_n)$ and $z_j = x_j + iy_j$, $y_j \neq 0$, $j = 1, 2, \ldots, n$. If $f \in L^1(\mathbb{R}^n)$, one has that

$$f^*(x_1 + iy_1, \ldots, x_n + iy_n) \to f(x)$$

as $y_j \to 0$, in the Schwartz space $D'(R^n)$, where

$$f^*(z) = \hat{f}(z_1, \ldots, z_n) - \hat{f}(\bar{z_1}, z_2, \ldots, z_n) + \ldots + (-1)^n \hat{f}(\bar{z_1}, \ldots, \bar{z_n}),$$

$$\hat{f}(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} \frac{f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n}{(t_1 - z_1) \ldots (t_n - z_n)}. \tag{1}$$

We refer to [1, 2, 3, 4] for the classical results (see also [5]).

In this paper we will define an appropriate boundary value representation for a function $F(x) = F(x_1, \ldots, x_n)$, $x \in \mathbb{R}^n$ of the form

$$F(x) = f(x)\, P(x), \tag{2}$$

where $f \in L^1(\mathbb{R}^n)$ and P is a real analytic function on $\mathbb{R}^n$.

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2. Main assertion

Denote by $\Lambda \subset C^n$ the domain of analytic extension of P. Using the convergence of the power series of P in the circle $C(x,r)$ for every $x \in \mathbb{R}^n$, $r = r_x > 0$, one obtains this extension. The domain Λ contains $\mathbb{R}^n$.

We recall a result of the theory for analytic representation of distributions:

If $f \in L^1(\mathbb{R}^n)$, then the function

$$\hat{f}(z_1,\ldots,z_n) = \frac{1}{(2\pi i)^n}\int_{\mathbb{R}^n} \frac{f(t_1,\ldots,t_n)}{(t_1-z_1)\ldots(t_n-z_n)}dt_1\ldots dt_n$$

where $z_i = x_i + iy_i$, $i = 1,\ldots,n$ and $Im\, z_i \neq 0$, is the Cauchy representation of the corresponding distribution f; this means, for every $\varphi \in D(\mathbb{R}^n)$,

$$\lim_{y\to 0+}\int_{-\infty}^{\infty}\left[\hat{f}(x+iy) - \hat{f}(x-iy)\right]\varphi(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx. \tag{3}$$

Now we formulate our main theorem

Theorem 1. *Let $f \in L^1(\mathbb{R}^n)$ and P be as above. Then*

$$\begin{aligned}
&P(z_1,z_2,\ldots,z_n)\,\hat{f}(z_1,\ldots,z_n)\\
-&P(\bar{z_1},z_2,\ldots,z_n)\,\hat{f}(\bar{z_1},z_2,\ldots,z_n)\\
+&P(\bar{z_1},\bar{z_2},\ldots,z_n)\,\hat{f}(\bar{z_1},\bar{z_2},\ldots,z_n)\\
&\ldots\\
+&(-1)^n P(\bar{z_1},\bar{z_2},\ldots,\bar{z_n})\,\hat{f}(\bar{z_1},\bar{z_2},\ldots,\bar{z_n})
\end{aligned}$$

converges to $F(x) = f(x)\,P(x)$ in $D'(\mathbb{R}^n)$ as $y_1,\ldots,y_n \to 0+$, where $z = x+iy, \bar{z} = x - iy \in \Lambda$.

Proof. For simplicity we give the proof for two dimensions.

Let $\varphi \in D\left(\mathbb{R}^2\right)$, then we consider the following integral

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^2} \left[P(z_1,z_2)\,\hat{f}(z_1,z_2) - P(\bar{z_1},z_2)\,\hat{f}(\bar{z_1},z_2)\right]\varphi(x_1,x_2)\,dx \\
&\quad + \iint\limits_{\mathbb{R}^2} \left[P(z_1,\bar{z_2})\,\hat{f}(z_1,\bar{z_2}) - P(\bar{z_1},\bar{z_2})\,\hat{f}(\bar{z_1},\bar{z_2})\right]\varphi(x_1,x_2)\,dx \\
&= \iint\limits_{\mathbb{R}^2} P(z_1,z_2)\,[\hat{f}(z_1,z_2) - \hat{f}(\bar{z_1},z_2) + \hat{f}(z_1,\bar{z_2}) - \hat{f}(\bar{z_1},\bar{z_2})]\varphi(x_1,x_2)\,dx_1dx_2 \\
&\quad + \iint\limits_{\mathbb{R}^2}[P(z_1,z_2) - P(\bar{z_1},z_2)]\hat{f}(\bar{z_1},z_2)]\varphi(x_1,x_2)\,dx_1dx_2 \\
&\quad + \iint\limits_{\mathbb{R}^2}[P(z_1,\bar{z_2}) - P(z_1,z_2)]\hat{f}(z_1,\bar{z_2})]\varphi(x_1,x_2)\,dx_1dx_2 \\
&\quad + \iint\limits_{\mathbb{R}^2}[P(z_1,z_2) - P(\bar{z_1},\bar{z_2})]\hat{f}(\bar{z_1},\bar{z_2})]\varphi(x_1,x_2)\,dx_1dx_2.
\end{aligned}$$

We will consider separately the following four integrals

$$\begin{aligned}
I_1 &= \iint\limits_{\mathbb{R}^2} P(z_1,z_2)\,[\hat{f}(z_1,z_2) - \hat{f}(\bar{z_1},z_2) + \hat{f}(z_1,\bar{z_2}) - \hat{f}(\bar{z_1},\bar{z_2})]\varphi(x_1,x_2)\,dx_1dx_2, \\
I_2 &= \iint\limits_{\mathbb{R}^2}[P(z_1,z_2) - P(\bar{z_1},z_2)]\hat{f}(\bar{z_1},z_2)]\varphi(x_1,x_2)\,dx_1dx_2, \\
I_3 &= \iint\limits_{\mathbb{R}^2}[P(z_1,\bar{z_2}) - P(z_1,z_2)]\hat{f}(z_1,\bar{z_2})]\varphi(x_1,x_2)\,dx_1dx_2, \\
I_4 &= \iint\limits_{\mathbb{R}^2}[P(z_1,z_2) - P(\bar{z_1},\bar{z_2})]\hat{f}(\bar{z_1},\bar{z_2})]\varphi(x_1,x_2)\,dx_1dx_2.
\end{aligned}$$

We write the integral I_1 in the form

$$\begin{aligned}
I_1 &= \iint\limits_{\mathbb{R}^2} P(z_1,z_2)\,f^*(z_1,z_2)\,\varphi(x_1,x_2)\,dx_1dx_2 \\
&= \iint\limits_{\mathbb{R}^2} P(z_1,z_2)\,\frac{y_1y_2}{\pi^2}\iint\limits_{\mathbb{R}^2}\frac{f(t_1,t_2)\,dt_1dt_2}{|t_1-z_1|^2\,|t_2-z_2|^2}\varphi(x_1,x_2)\,dx_1dx_2.
\end{aligned}$$

Since both integrals exist, by Fubini's theorem we may change the order of integration and get

$$\begin{aligned} I_1 &= \iint\limits_{\mathbb{R}^2} f(t_1,t_2)\,dt_1dt_2 \frac{y_1y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1,z_2)}{|t_1-z_1|^2|t_2-z_2|^2}\varphi(x_1,x_2)\,dx_1dx_2 \\ &= \iint\limits_{\mathbb{R}^2} f(t_1,t_2)\,dt_1dt_2 \frac{y_1y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1,z_2)-P(x_1,x_2)}{|t_1-z_1|^2|t_2-z_2|^2}\varphi(x_1,x_2)\,dx_1dx_2 \\ &\quad + \iint\limits_{\mathbb{R}^2} f(t_1,t_2)\,dt_1dt_2 \frac{y_1y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(x_1,x_2)}{|t_1-z_1|^2|t_2-z_2|^2}\varphi(x_1,x_2)\,dx_1dx_2 \\ &= I_1^{'} + I_1^{''}. \end{aligned}$$

Now, consider the integral $I_1^{'}$:

Since $\varphi \in D(\mathbb{R}^2)$, there exists $a > 0$ such that $\sup p\varphi \subset [-a,a]\times[-a,a]$, further since $y_1, y_2 \to 0+$ we have that for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|P(x_1+iy_1, x_2+iy_2) - P(x_1,x_2)| < \varepsilon$ if $\sqrt{y_1^2+y_2^2} < \delta, x_1, x_2 \in [-a,a]\times[-a,a]$.

$$\left|I_1^{'}\right| \leq \varepsilon \|\varphi\| \cdot \|f\|.$$

Consider the integral $I_1^{''}$:

Since $\varphi \in D(\mathbb{R}^2)$, it follows that $\varphi(x_1,x_2)P(x_1,x_2) \in D(\mathbb{R}^2)$ and by (3) we have

$$\begin{aligned} &\lim_{y_1,y_2\to 0+} \iint\limits_{\mathbb{R}^2} P(x_1,x_2) f^*(z_1,z_2)\varphi(x_1,x_2)\,dx_1dx_2 \\ &= \iint\limits_{\mathbb{R}^2} P(x_1,x_2) f(x_1,x_2)\varphi(x_1,x_2)\,dx_1dx_2 \\ &= <F,\varphi> \end{aligned}$$

In the following we estimate the integral I_2.

By the Fubini theorem, we may write

$$\begin{aligned} I_2 &= \iint\limits_{\mathbb{R}^2} [P(z_1,z_2) - P(\bar{z_1},z_2)]\hat{f}(\bar{z_1},z_2)]\varphi(x_1,x_2)\,dx_1dx_2 \\ &= \iint\limits_{\mathbb{R}^2} f(t_1,t_2)\,dt_1dt_2 \frac{y_1y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1,z_2)-P(\bar{z_1},z_2)}{(t_1-\bar{z_1})(t_2-z_2)}\varphi(x_1,x_2)\,dx_1dx_2 \end{aligned}$$

1. If $t_1, t_2 \notin [-a,a]\times[-a,a]$, then since $\operatorname{supp}\varphi \subset [-a,a]\times[-a,a]$,

$$\sqrt{(t_1-x_1)^2+(t_2-x_2)^2} \geq l, x_1, x_2 \in \operatorname{supp}\varphi \text{ and } t_1, t_2 > 0.$$

We have that

$$\left|\frac{P(z_1,z_2)-P(\bar{z_1},z_2)}{(t_1-\bar{z_1})(t_2-z_2)}\right| \leq \frac{\varepsilon}{l^2}$$

and

$$|I_2| \leq \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} |f(t_1,t_2)|\, dt_1 dt_2 \iint_{\mathbb{R}^2} \frac{\varepsilon\, |\varphi(x_1,x_2)|}{l^2} dx_1 dx_2 \leq \frac{a^2\, \|\varphi\|}{l^2}.$$

Thus, the second integral tends to zero as $y_1, y_2 \to 0+$.

2. Let $(t_1,t_2) \in [-a,a] \times [-a,a]$ and $t_1, t_2 \in \operatorname{supp}\varphi$. In that case we write

$$\begin{aligned} I_2 &= \frac{1}{(2\pi i)^2} \iint_{\mathbb{R}^2} f(t_1,t_2)\, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint_{\mathbb{R}^2} \frac{P(z_1,z_2) - P(\bar{z_1},z_2)}{(t_1-\bar{z_1})(t_2-z_2)} [\varphi(x_1,x_2) - \varphi(t_1,t_2)] dx_1 dx_2 \\ &\quad + \frac{1}{(2\pi i)^2} \iint_{\mathbb{R}^2} f(t_1,t_2)\, \varphi(t_1,t_2)\, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint_{\mathbb{R}^2} \frac{P(z_1,z_2) - P(\bar{z_1},z_2)}{(t_1-\bar{z_1})(t_2-z_2)} dx_1 dx_2 \\ &= I_2^{'} + I_2^{''}. \end{aligned}$$

Since P is continuous for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sqrt{y_1^2 + y_2^2} < \delta$ and

$$\left| \frac{[P(z_1,z_2) - P(\bar{z_1},z_2)]\,[\varphi(x_1,x_2) - \varphi(t_1,t_2)]}{(t_1-\bar{z_1})(t_2-z_2)} \right| \leq \varepsilon \left| \frac{\varphi(x_1,x_2) - \varphi(t_1,t_2)}{(t_1-x_1)(t_2-x_2)} \right|.$$

Since the function $\left| \frac{\varphi(x_1,x_2)-\varphi(t_1,t_2)}{(t_1-x_1)(t_2-x_2)} \right|$ is integrable we may apply the Lebesgue dominant theorem and obtain for $I_2^{'}$ that

$$\begin{aligned} &\lim_{y_1,y_2\to 0+} \frac{1}{(2\pi i)^2} \iint_{\mathbb{R}^2} f(t_1,t_2)\, dt_1 dt_2 \\ &\qquad \frac{y_1 y_2}{\pi^2} \iint_{\mathbb{R}^2} \frac{P(z_1,z_2) - P(\bar{z_1},z_2)}{(t_1-\bar{z_1})(t_2-z_2)} [\varphi(x_1,x_2) - \varphi(t_1,t_2)] dx_1 dx_2 = 0. \end{aligned}$$

We consider the integral

$$\begin{aligned} I_2^{''} = &\frac{1}{(2\pi i)^2} \iint_{\mathbb{R}^2} f(t_1,t_2)\, \varphi(t_1,t_2)\, dt_1 dt_2 \\ &\frac{y_1 y_2}{\pi^2} \iint_{\mathbb{R}^2} \frac{P(z_1,z_2) - P(\bar{z_1},z_2)}{(t_1-\bar{z_1})(t_2-z_2)} dx_1 dx_2. \end{aligned}$$

Since P is continuous for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|P(x_1 + iy_1, x_2 + iy_2) - P(x_1,x_2)| < \varepsilon$$

if $\sqrt{y_1^2 + y_2^2} < \delta$, $x_1, x_2 \in [-a, a] \times [-a, a]$ and

$$\left| \int_{-a}^{a} \int_{-a}^{a} \frac{P(z_1, z_2) - P(\bar{z_1} - z_2)}{(t_1 - \bar{z_1})(t_2 - z_2)} dx_1 dx_2 \right| \leq \varepsilon M.$$

This proves that $I_2^{''} \to 0$ as $y_1, y_2 \to 0+$.

We consider the third integral

$$I_3 = \iint\limits_{\mathbb{R}^2} [P(z_1, \bar{z_2}) - P(z_1, z_2)] \hat{f}(z_1, \bar{z_2})] \varphi(x_1, x_2) \, dx_1 dx_2.$$

Since both integrals exist we may change the order of integration and obtain

$$\begin{aligned} I_3 &= \iint\limits_{\mathbb{R}^2} [P(z_1, \bar{z_2}) - P(z_1, z_2)] \hat{f}(z_1, \bar{z_2})] \varphi(x_1, x_2) \, dx_1 dx_2 \\ &= \iint\limits_{\mathbb{R}^2} f(t_1, t_2) \, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1, \bar{z_2}) - P(z_1, z_2)}{(t_1 - z_1)(t_2 - \bar{z_2})} \varphi(x_1, x_2) \, dx_1 dx_2. \end{aligned}$$

1. If $t_1, t_2 \notin [-a, a] \times [-a, a]$, then since $\operatorname{supp} \varphi \subset [-a, a] \times [-a, a]$,

$$\sqrt{(t_1 - x_1)^2 + (t_2 - x_2)^2} \geq l,$$

$x_1, x_2 \in \operatorname{supp} \varphi$ and $t_1, t_2 > 0$.

This implies

$$\left| \frac{P(z_1, \bar{z_2}) - P(z_1, z_2)}{(t_1 - z_1)(t_2 - \bar{z_2})} \right| \leq \frac{\varepsilon}{l^2}$$

and

$$|I_3| \leq \frac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} |f(t_1, t_2)| \, dt_1 dt_2 \iint\limits_{\mathbb{R}^2} \frac{\varepsilon \, |\varphi(x_1, x_2)|}{l^2} dx_1 dx_2 \leq \frac{a^2 \, \|\varphi\|}{l^2}.$$

Thus, the third integral tends to zero as $y_1, y_2 \to 0+$.

2. Let $(t_1, t_2) \in [-a, a] \times [-a, a]$ and $t_1, t_2 \in \operatorname{supp} \varphi$. In that case we write

$$\begin{aligned} I_3 &= \frac{1}{(2\pi i)^2} \iint\limits_{\mathbb{R}^2} f(t_1, t_2)\, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1, \bar{z}_2) - P(z_1, z_2)}{(t_1 - z_1)(t_2 - \bar{z}_2)} [\varphi(x_1, x_2) - \varphi(t_1, t_2)] dx_1 dx_2 \\ &\quad + \frac{1}{(2\pi i)^2} \iint\limits_{\mathbb{R}^2}^{\infty} f(t_1, t_2)\, \varphi(t_1, t_2)\, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1, \bar{z}_2) - P(z_1, z_2)}{(t_1 - z_1)(t_2 - \bar{z}_2)} dx_1 dx_2 = I_3^{'} + I_3^{''}. \end{aligned}$$

Since P is continuous for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sqrt{y_1^2 + y_2^2} < \delta$ and

$$\left| \frac{P(z_1, \bar{z}_2) - P(z_1, z_2) [\varphi(x_1, x_2) - \varphi(t_1, t_2)]}{(t_1 - z_1)(t_2 - \bar{z}_2)} \right| \leq \varepsilon \left| \frac{\varphi(x_1, x_2) - \varphi(t_1, t_2)}{(t_1 - x_1)(t_2 - x_2)} \right|$$

Since the function

$$\left| \frac{\varphi(x_1, x_2) - \varphi(t_1, t_2)}{(t_1 - x_1)(t_2 - x_2)} \right|$$

is integrable we may apply the Lebesgue dominant theorem and obtain for $I_3^{'}$ that

$$\begin{aligned} &\lim_{y_1, y_2 \to 0+} \frac{1}{(2\pi i)^2} \iint\limits_{\mathbb{R}^2} f(t_1, t_2)\, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1, \bar{z}_2) - P(z_1, z_2)}{(t_1 - z_1)(t_2 - \bar{z}_2)} [\varphi(x_1, x_2) - \varphi(t_1, t_2)] dx_1 dx_2 = 0. \end{aligned}$$

We consider the integral

$$I_3^{''} = \frac{1}{(2\pi i)^2} \iint\limits_{\mathbb{R}^2} f(t_1, t_2)\, \varphi(t_1, t_2)\, dt_1 dt_2 \frac{y_1 y_2}{\pi^2} \iint\limits_{\mathbb{R}^2} \frac{P(z_1, \bar{z}_2) - P(z_1, z_2)}{(t_1 - z_1)(t_2 - \bar{z}_2)} dx_1 dx_2$$

Since P is continuous for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|P(x_1 + iy_1, x_2 - iy_2) - P(x_1 + iy_1, x_2 + iy_2)| < \varepsilon$$

if $\sqrt{y_1^2 + y_2^2} < \delta$, $x_1, x_2 \in [-a, a] \times [-a, a]$ and

$$\left| \int_{-a}^{a} \int_{-a}^{a} \frac{P(z_1, \bar{z}_2) - P(z_1, z_2)}{(t_1 - z_1)(t_2 - \bar{z}_2)} dx_1 dx_2 \right| \leq \varepsilon M.$$

This proves that $I_3^{''} \to 0$ as $y_1, y_2 \to 0+$.

Finally, we consider the fourth integral. Since both integrals exist we may change the order of integration and get

$$\begin{aligned} I_4 &= \iint_{\mathbb{R}^2} [P(z_1, z_2) - P(\bar{z_1}, \bar{z_2})] \hat{f}(\bar{z_1}, \bar{z_2})] \varphi(x_1, x_2)\, dx_1 dx_2 \\ &= \iint_{\mathbb{R}^2} f(t_1, t_2)\, dt_1 dt_2 \frac{y_1 y_2}{\pi^2} \iint_{\mathbb{R}^2} \frac{P(z_1, z_2) - P(\bar{z_1}, \bar{z_2})}{(t_1 - \bar{z_1})(t_2 - \bar{z_2})} \varphi(x_1, x_2)\, dx_1 dx_2 \end{aligned}$$

In this case we consider two subcases:

1. If $t_1, t_2 \notin [-a, a] \times [-a, a]$, then since $\operatorname{supp} \varphi \subset [-a, a] \times [-a, a]$,

$$\sqrt{(t_1 - x_1)^2 + (t_2 - x_2)^2} \geq l,$$

$x_1, x_2 \in \sup p\varphi$ and $t_1, t_2 > 0$, it follows that

$$\left| \frac{P(z_1, z_2) - P(\bar{z_1}, \bar{z_2})}{(t_1 - \bar{z_1})(t_2 - \bar{z_2})} \right| \leq \frac{\varepsilon}{l^2}$$

and

$$|I_4| \leq \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} |f(t_1, t_2)|\, dt_1 dt_2 \iint_{\mathbb{R}^2} \frac{\varepsilon\, |\varphi(x_1, x_2)|}{l^2} dx_1 dx_2 \leq \frac{a^2 \|\varphi\| \|f\|_1}{l^2}$$

Thus the fourth integral tends to zero as $y_1, y_2 \to 0+$.

2. Let $(t_1, t_2) \in [-a, a] \times [-a, a]$ and $t_1, t_2 \in \operatorname{supp} \varphi$.
In that case we write

$$\begin{aligned} I_4 &= \frac{1}{(2\pi i)^2} \iint_{\mathbb{R}^2} f(t_1, t_2)\, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint_{\mathbb{R}^2} \frac{P(z_1, z_2) - P(\bar{z_1}, \bar{z_2})}{(t_1 - \bar{z_1})(t_2 - \bar{z_2})} [\varphi(x_1, x_2) - \varphi(t_1, t_2)] dx_1 dx_2 \\ &\quad + \frac{1}{(2\pi i)^2} \iint_{\mathbb{R}^2} f(t_1, t_2)\, \varphi(t_1, t_2)\, dt_1 dt_2 \\ &\quad \frac{y_1 y_2}{\pi^2} \iint_{\mathbb{R}^2} \frac{P(z_1, z_2) - P(\bar{z_1}, \bar{z_2})}{(t_1 - \bar{z_1})(t_2 - \bar{z_2})} dx_1 dx_2 \\ &= I_4^{'} + I_4^{''}. \end{aligned}$$

Since P is continuous for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sqrt{y_1^2 + y_2^2} < \delta$ and

$$\left| \frac{P(z_1, z_2) - P(\bar{z_1}, \bar{z_2}) [\varphi(x_1, x_2) - \varphi(t_1, t_2)]}{(t_1 - \bar{z_1})(t_2 - \bar{z_2})} \right| \leq \varepsilon \left| \frac{\varphi(x_1, x_2) - \varphi(t_1, t_2)}{(t_1 - x_1)(t_2 - x_2)} \right|.$$

Since the function $\left|\frac{\varphi(x_1,x_2)-\varphi(t_1,t_2)}{(t_1-x_1)(t_2-x_2)}\right|$ is integrable we may apply the Lebesgue dominant theorem and give for $I_4^{'}$ that

$$\lim_{y_1,y_2\to 0+}\frac{1}{(2\pi i)^2}\iint\limits_{\mathbb{R}^2} f(t_1,t_2)\,dt_1dt_2$$
$$\frac{y_1y_2}{\pi^2}\iint\limits_{\mathbb{R}^2}\frac{P(z_1,z_2)-P(\bar{z_1},\bar{z_2})}{(t_1-\bar{z_1})(t_2-\bar{z_2})}[\varphi(x_1,x_2)-\varphi(t_1,t_2)]dx_1dx_2=0.$$

Finally, we consider the last integral

$$I_4^{''} = \frac{1}{(2\pi i)^2}\iint\limits_{\mathbb{R}^2} f(t_1,t_2)\,\varphi(t_1,t_2)\,dt_1dt_2$$
$$\frac{y_1y_2}{\pi^2}\iint\limits_{\mathbb{R}^2}\frac{P(z_1,z_2)-P(\bar{z_1},\bar{z_2})}{(t_1-\bar{z_1})(t_2-\bar{z_2})}dx_1dx_2.$$

Since P is continuous for a given $\varepsilon>0$ there exists $\delta>0$ such that

$$|P(x_1+iy_1,x_2+iy_2)-P(x_1-iy_1,x_2-iy_2)|<\varepsilon$$

if $\sqrt{y_1^2+y_2^2}<\delta$, $x_1,x_2\in[-a,a]\times[-a,a]$ and

$$\left|\int_{-a}^{a}\int_{-a}^{a}\frac{P(z_1,z_2)-P(\bar{z_1},\bar{z_2})}{(t_1-\bar{z_2})(t_2-\bar{z_2})}dx_1dx_2\right|\le\varepsilon M.$$

This proves; that $I_4^{''}\to 0$ as $y_1,y_2\to 0+$. □

Remark 1. From the consideration given above it follows that

$$\begin{aligned}&P(z_1,\ldots,z_n)\,\hat{f}(z_1,\ldots,z_n)\\&\quad-P(\bar{z_1},\ldots,z_n)\,\hat{f}(\bar{z_1},\ldots,z_n)\\&\quad\ldots\\&\quad+(-1)^n\,P(\bar{z_1},\ldots,\bar{z_n})\,f(\bar{z_1},\ldots,\bar{z_n})\end{aligned}$$

converges to $F(x)=f(x)\,P(x)$ in $D^{'}(\mathbb{R}^n)$ as $y_1,..,y_n\to 0+$

If $f\in L^1(\mathbb{R}^n)$ and if P is an entire function, then the function

$$F(z)=P(z)\,\hat{f}(z)$$

is an analytic representation of the distribution $F(x)=P(x)\,\hat{f}(x)$.

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