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BOUNDED LINEAR OPERATORS IN TRANSVERSAL FUNCTIONAL PROBABILISTIC SPACE

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Abstract. The purpose of this paper is two fold. Firstly, we define strongly B-bounded and strongly C-bounded operators and discuss their relationship. Further, we provide examples to show that there is no direct relation between strongly B-bounded and strongly C-bounded operators in transversal functional probabilistic spaces.

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1. Introduction

Transversal spaces were introduced by Milan R. Taskovic [1]. The notion of transversal functional probabilistic metric spaces (lower and upper) was introduced in [3] as a natural extension of Metric spaces, probabilistic spaces and Fuzzy metric spaces. We define strongly B-bounded and strongly C-bounded operators and also we discuss their relationship in lower and upper transversal functional probabilistic spaces. Further we provide examples to show that there is no direct relation between strongly B-bounded and strongly C-bounded operators in transversal functional probabilistic spaces.

Definition ([1])**.** Let X be a nonempty set and let $P := (P, \preceq)$ be a partially ordered set. The function $\rho : X \times X \to P$ is called upper ordered transverse on X if $\rho(x, y) = \rho(y, x)$, and if there exists an upper bisection function $g : P \times P \to P$ such that

$$\rho(x, y) \preceq \sup\{\rho(x, z), \rho(z, y), g(\rho(x, z), \rho(z, y))\}$$

for all $x, y, z \in X$. An upper ordered transversal space is a triple (X, ρ, g).

Definition ([1])**.** The function $\rho : X \times X \to P$ is called lower ordered transverse on X if $\rho(x, y) = \rho(y, x)$, and if there exists an upper bisection function d : $P \times P \to P$ such that

$$\inf\{\rho(x, z), \rho(z, y), g(\rho(x, z), \rho(z, y))\} \preceq \rho(x, y)$$

for all $x, y, z \in X$. A lower ordered transversal space is a triple (X, ρ, g).

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For $P = [0, +\infty)$ the spaces (X, ρ, g) and (X, ρ, d) we will call upper and lower transversal space.

For $P = [a, b]$, $0 < a < b$ these spaces we will call the upper or lower transversal interval spaces. Especially, for $a = 0$ and $b = 1$ we will call these spaces upper and lower transversal probabilistic spaces.

Definition ([3])**.** Let X be a nonempty set. The symmetric function $\rho : X \times X \times [0, +\infty) \to [0, 1]$ is called upper functional probabilistic transverse on X if there exists a function $g : [0, 1] \times [0, 1] \to [0, 1]$, called an upper probabilistic transverse on X if there exists a function $d : [0, 1] \times [0, 1] \to [0, 1]$, called a lower probabilistic bisection function, such that

$$\rho(p, q)(x) \geq \min\{\rho(p, s)(x), \rho(s, q)(x), d(\rho(p, s)(x), \rho(s, q)(x))\}$$

for all $p, q, s \in X$ and for each $x \in [0, +\infty)$. The triple (X, ρ, d) we will call lower transversal functional probabilistic space.

Definition ([3])**.** Let (X, ρ, d) be a lower transversal functional probabilistic space.

(a) A sequence $(p_n)_{n \in N}$ in (X, ρ, d) converges to a point $p \in x$ if for any $\varepsilon > 0$ and any $\lambda \in (0, 1)$ there exists an integer n_0 such that $\rho(p, p_n)(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

(b) A sequence $(p_n)_{n \in N}$ is said to be Cauchy if for any $\varepsilon > 0$ and any $\lambda \in (0, 1)$ there exists an integer n_0 such that $\rho(p_m, p_n)(\varepsilon) > 1 - \lambda$ for all $m, n \geq n_0$.

(c) A lower transversal probabilistic space will be called complete if every Cauchy sequence is convergent in X.

Throughout this paper we consider lower transversal functional probabilistic spaces with the lower functional probabilistic transverse $\rho(p, q)(x)$ which satisfies the following conditions

(T1) $\rho(p, q)(x)$ is a left-continuous function for $x \in (0, +\infty)$ and right-continuous at the point $x = 0$,

(T2) $\rho(p, q)(x) = 1$ for all $x > 0$ iff $p = q$,

(T3) $\rho(p, q)(x)$ is a non-decreasing function,

(T4) $\lim_{x \to +\infty} \rho(p, q)(x) = 1$ for all $p, q \in X$.

Also, we assume that the lower probabilistic bisection function $d(x, y)$ satisfies:

(B1) $d(x, y)$ is a non-decreasing and continuous function,

(B2) $d(x, x) \geq x$,

(B3) $\lim_{x \to 1} d(a, x) = a$.

Definition ([3])**.** Let (X, ρ, d) be a lower transversal functional probabilistic space. A subset $F \subseteq X$ will be called closed if for every sequence $(p_n)_{n \in N} \subseteq F$ such that $p_n \to p_0$ as $n \to \infty$ it follows that $p_0 \in F$. The minimal closed set containing F will be called the closure of F and it will be denoted by $\bar{F}$.

Definition ([3])**.** Let (X, ρ, d) be a lower transversal functional probabilistic space. A collection of sets $\{F_n\}_{n\in N}$ is said to have lower transversal diameter zero iff for each pair $\lambda \in (0,1)$ and $x > 0$ there exists $n \in N$ such that $\rho(p,q) > 1-\lambda$ for all $p, q \in F_n$.

We give the following definition of bounded set in transversal functional probabilistic space (X, ρ, d).

Definition 2.1. Let A be a non-empty set in lower transversal functional probabilistic space (X, ρ, d). Then

(i) A is certainly bounded if and only if, $\psi_A(x_0) = 1$ for some $x_0 \in (0, +\infty)$

(ii) A is perhaps bounded if and only if $\psi_A(x_0) < 1$ for every $x_0 \in (0, +\infty)$ and $\ell^- \psi_a(+\infty) = 1$;

(iii) A is perhaps unbounded if and only if $\ell^- \psi_A(+\infty) \in (0,1)$;

(iv) A is certainly unbounded if and only if, $\ell^- \psi_A(+\infty) = 0$ i.e

$$\psi_A(x) = 0,$$

where $\psi_A(x) = \inf\{\rho(p,q)(x); p, q \in A\}$ and $\ell^- \psi_A(x) = \lim_{t\to x^-} \psi_A(t)$. Moreover, A will be said to be D-bounded if either (i) or (ii) holds.

Definition 2.2. Let (X, ρ, d) and (X', ρ', d') be lower transversal functional probabilistic spaces. A linear map $T : X \to X'$ is said to be

(i) Certainly bounded if every certainly bounded set A of the space (X, ρ, d) has as image by T a certainly bounded set TA of the space (X', ρ', d'), i.e. if there exists $x_0 \in (0, \infty)$ such that $\rho(p,q)(x_0) = 1$ for all $p, q \in A$, then there exists $x_1 \in (0, \infty)$ such that $\rho'(Tp, Tq)(x_1) = 1$ for all $p, q \in A$.

(ii) Bounded if it maps every D-bounded set of X into a D-bounded set of X' i.e., if and only if, it satisfies the implication

$\lim_{x\to+\infty} \psi_A(x) = 1 \Rightarrow \lim_{x\to+\infty} \psi_{TA}(x) = 1$ for every non-empty subset A of V.

(iii) Strongly B-bounded if there exists a constant $k > 0$ such that, for every $p, q \in X$ and for every $x > 0$, $\rho'(Tp, Tq)(x) \geq \rho(p,q)\left(\frac{x}{k}\right)$ or equivalently if there exists a constant $h > 0$ such that, for every $p, q \in X$ and for every $x > 0$,

$$\rho'(Tp, Tq)(hx) \geq \rho(p,q)(x)$$

(iv) Strongly C-bounded if there exists a constant $h \in (0,1)$ such that, for every $p, q \in V$ and for every $x > 0$,

$$\rho(p,q)(x) > 1 - x \Rightarrow \rho'(Tp, Tq)(hx) > 1 - hx$$

Theorem 2.3. *The identity map I between lower transversal functional probabilistic space (X, ρ, d) into itself is strongly C-bounded.*

Result 2.4. When $k = 1$, then the identity map I between (X, ρ, d) into itself is a strongly B-bounded operator.

In the following example we will introduce a strongly C-bounded operator, which is not strongly B-bounded.

Example 2.5. Let X be a vector space and $p, q \neq 0$, if for every $p, q \in X$ and $x \in R$,

$$\rho(p,q)(x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases}, \qquad \rho'(p,q)(x) = \begin{cases} \frac{1}{2} & x \leq 1 \\ \frac{5}{7} & 1 < x < \infty \\ 1 & x = \infty \end{cases}$$

And for $p = q \neq 0$,

$$\rho(p,q)(x) = \rho'(p,q)(x) = 1$$

and

$$\begin{aligned} d(a,b) &= \min\{a,b\} \\ d'(a,b) &= \min\{a,b\} \end{aligned}$$

Then (X, ρ, d) and (X', ρ', d') are lower transversal functional probabilistic spaces.

Now let $I : (X, \rho, d) \to (X', \rho', d')$ be the identity operator, then I is strongly C-bounded but not strongly B-bounded, bounded and certainly bounded it is clear that I is not certainly bounded and is not bounded. I is not strongly B-bounded, because for every $k > 0$ and for $x = \max\left\{3, \dfrac{1}{k}\right\}$

$$\rho'(Ip, Iq)(kx) = \frac{5}{7} < 1 = \rho(p,q)(x)$$

But I is strongly C-bounded, because for every $p, q > 0$ and for every $x > 0$, this condition

$\rho(p,q)(x) > 1 - x$ is satisfied only if $x > 1$ now if $h = \dfrac{4}{7x}$ then

$$\begin{aligned} \rho'(Ip, Iq)(hx) &= \rho'(Ip, Iq)\left(\frac{4x}{7x}\right) \\ &= \rho'(p,q)\left(\frac{4}{7}\right) \\ &= \frac{1}{2} > \frac{3}{7} \end{aligned}$$

$$\begin{aligned} &= 1 - \frac{4}{7} \\ &= 1 - \left(\frac{4}{7x}\right)x \\ &= 1 - hx \end{aligned}$$

Remark 2.6. We have noted in the above example that there is an operator, which is strongly C-bounded but it is not strongly B-bounded. Moreover, we are going to give the example of an operator which is strongly B-bounded, but is not strongly C-bounded.

Example 2.7. Let $X - X' = R$ and for $x > 0$, let

$$\rho(p,q)(x) = G\left(\frac{x}{|p-q|}\right), \qquad \rho'(p,q)(x) = U\left(\frac{x}{|p-q|}\right)$$

where

$$G(x) = \begin{cases} \frac{1}{2}, & 0 < x \le 2, \\ 1, & 2 < x \le +\infty, \end{cases}, \qquad U(x) = \begin{cases} \frac{1}{2}, & 0 < x \le \frac{3}{2} \\ 1 & \frac{3}{2} < x \le +\infty \end{cases}$$

Consider the identity map $I : (R, \rho, d) \to (R, \rho', d')$. Now

(i) I is a strongly B-bounded operator, such that for every $p, q \in R$ and every $x > 0$ then

$$\begin{aligned} \rho'(Ip, Iq)\left(\frac{3}{4}x\right) &= U\left(\frac{3}{4}\frac{x}{|p-q|}\right) \\ &= \begin{cases} \frac{1}{2} & 0 < x \le 2|p-q| \\ 1 & 2|p-q| < x \le +\infty \end{cases} \\ &= G\left(\frac{x}{|p-q|}\right) \\ &= \rho(p,q)(x) \end{aligned}$$

(ii) I is not a strongly C-bounded operator, such that for every $h \in (0,1)$ Let $x = \frac{3}{8h}$, $|p-q| = \frac{1}{4}$. If $x > 2|p-q|$, then the condition $\rho(p,q)(x) > 1 - x$ will be satisfied, but we note that

$$\begin{aligned} \rho'(Ip, Iq)(hx) &= \rho'(p,q)h(x) \\ &= U\left(\frac{hx}{|p-q|}\right) \\ &= U\left(\frac{3}{2}\right) \\ &= \frac{1}{2} < \frac{5}{8} \\ &= 1 - h\left(\frac{3}{8h}\right) \\ &= 1 - hx. \end{aligned}$$

Definition 2.8. Let (X, ρ, d) be lower transversal functional probabilistic space, then we define

$$B(p,q) = \inf\{h \in R; \rho(p,q)(hx) > 1 - h\}.$$

Lemma 2.9. Let $T : (X, \rho, d) \to (X', \rho', d')$ be strongly B-bounded linear operator, for every p, q in X and let $\rho'(Tp, Tq)(x)$ be strictly increasing on $[0, 1]$ then $B(Tp, Tq) < B(p, q) \ \forall \ \rho, q \in X$.

Proof. Let $\eta \in \left(0, \dfrac{1-\gamma}{\gamma} B(p, q)\right)$, where $B(p, q) > \gamma[B(p, q) + \eta]$ and so

$$\rho'(Tp, Tq)(B(p, q)) > \rho'(Tp, Tq)(\gamma[B(p, q) + n])$$

and where $\rho'(Tp, Tq)$ is strictly increasing on [0, 1], then

$$\begin{aligned} \rho'(Tp, Tq)(\gamma[B(p, q) + \eta]) &\geq \rho(p, q)(B(p, q) + \eta) \\ &\geq \rho(p, q)(B(p, q)) \\ &> 1 - B(p, q) \end{aligned}$$

We conclude that

$$B(Tp, Tq) = \inf\{B(p, q); \rho'(Tp, Tq)(B(p, q)^+) > 1 - B(p, q)\}$$

So $B(Tp, Tq) < B(p, q) \ \forall \ p, q \in X$. □

Theorem 2.10. *Let $T : (X, \rho, d) \to (X'\rho', d')$ be a strongly B-bounded linear operator, and let $\rho'(Tp, Tq)$ be strictly increasing on $[0, 1]$. Then, T is a strongly C-bounded linear operator,*

Proof. Let T be a strictly B-bounded operator. Since by the above result, $B(Tp, Tq) < B(p, q)$, $\forall \ p, q \in V$ there exists $\gamma_{p,q} \in (0, 1)$ such that

$$B(Tp, Tq) < \gamma_{p,q} B(p, q)$$

This means that

$$\begin{aligned} &\inf\{h \in R; \rho'(Tp, Tq)(h^+) > 1 - h\} \\ &\quad \leq \ \gamma \inf\{h \in R; \rho(p, q)(h^+) > 1 - h\} \\ \\ &\quad = \ \inf\{\gamma h \in R; \rho(p, q)(h^+) > 1 - h\} \\ &\quad = \ \inf\left\{h \in R; \rho(p, q)\left(\frac{h^+}{\gamma}\right) > 1 - \frac{h}{\gamma}\right\} \end{aligned}$$

We conclude that

$$\rho(p, q)\left(\frac{h}{\gamma}\right) > 1 - \left(\frac{h}{\gamma}\right) \quad \Rightarrow \quad \rho'(Tp, Tq)(h) > 1 - h.$$

Now if $x = \dfrac{h}{\gamma}$ then $\rho(p, q)(x) > 1 - x \Rightarrow \rho'(Tp, Tq)(xh) > 1 - xh$, so, T is strongly C-bounded operator. □

From the above theorem, we have that under some condition every strongly B-bounded operator is a strongly C-bounded operator.

Example 2.11. Let $(X\|\cdot\|)$ be a normed space and G be a non-decreasing function from $[0,\infty)$ to $[0,1]$ such that $G(+\infty)=1$ and $G(0)=0$ and

$$\rho(p,q)(x) = \begin{cases} 1 & \text{if } \ p=q \\ G\left(\frac{x}{\|p-q\|^\alpha}\right) & \text{if } \ p\neq q \end{cases}$$

where $\alpha \geq 0$ and $d(a,b)=\min\{a,b\}$

(X,ρ,d) become a lower Transversal functional probability space induced by the $\|\cdot\|$, and denote this space by $(X,\|\cdot\|,\alpha)$.

Theorem 2.12. *Let G be strictly increasing on $[0,1]$, then*

$T:(X,\|\cdot\|,\alpha)\to(X',\|\cdot\|,\alpha)$ is a strongly B-bounded operator if and only if T is a bounded linear operator in normed space.

Proof. Let $k>0$ and $x>0$. Then for every $p,q\in X$.

$$\begin{aligned} G\left(\frac{kx}{\|Tp-Tq\|^\alpha}\right) &= \rho(Tp,Tq)(kx) \\ &\geq \rho(p,q)(x) \\ &= G\left(\frac{x}{\|p-q\|^\alpha}\right) \end{aligned}$$

iff

$$\begin{aligned} & \frac{kx}{\|Tp-Tq\|^\alpha} \geq \frac{x}{\|p-q\|^\alpha} \\ \Leftrightarrow \quad & \|T(p-q)\|^\alpha \leq k\|p-q\|^\alpha \\ \Leftrightarrow \quad & \|T(p-q)\| \leq k^{1/\alpha}\|p-q\| \end{aligned}$$

Put $p-q=x \Rightarrow \|T(x)\|\leq k^{1/\alpha}\|x\|$

Thus, T is a bounded linear operator in normed space. □

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