Novi Sad J. Math.
Vol. 41, No. 2, 2011, 99-109

IDEMPOTENT ELEMENTS OF $WP_G(2,2) \cup \{\sigma_{id}\}$

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Abstract. A generalized hypersubstitution of type $\tau = (2,2)$ is a mapping σ which maps the binary operation symbols f and g to terms $\sigma(f)$ and $\sigma(g)$ which does not necessarily preserve arities. Any generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma}$ on the set of all terms of type $\tau = (2,2)$. A binary operation on $Hyp_G(2,2)$ the set of all generalized hypersubstitutions of type $\tau = (2,2)$ can be defined by using this extension. The set $Hyp_G(2,2)$ together with the identity hypersubstitution σ_{id} which maps f to $f(x_1, x_2)$ and maps g to $g(x_1, x_2)$ forms a monoid. The concept of an idempotent element plays an important role in many branches of mathematics, for instance, in semigroup theory and semiring theory. In this paper we characterize the idempotent generalized hypersubstitutions of $WP_G(2,2) \cup \{\sigma_{id}\}$ a submonoid of $\underline{Hyp_G(2,2)}$.

AMS Mathematics Subject Classification (2010): 08B15, 20M07

Key words and phrases: generalized hypersubstitution, projection generalized hypersubstitution, weak projection generalized hypersubstitution, idempotent element

1. Introduction

The concept of a hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel, and D. Schweigert [1]. In [5], the author and K. Denecke introduced the concept of generalized hypersubstitutions, strong hyperidentities and strongly solid varieties. Let $\{f_i \mid i \in I\}$ be an indexed set of operation symbols of type τ, where f_i is n_i-ary, $n_i \in \mathbb{N}$. Let $W_\tau(X)$ be the set of all terms of type τ built up by the operation symbols from $\{f_i \mid i \in I\}$ and variables from an alphabet $X := \{x_1, x_2, x_3, \ldots\}$. A generalized hypersubstitution is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity. Any generalized hypersubstitution σ can be uniquely extended to a mapping $\hat{\sigma}$ on $W_\tau(X)$ the set of all terms of the given type. To define the extension $\hat{\sigma}$ of σ, we defined inductively the concept of superposition of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$,

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(iii) if $t = f_i(s_1, \dots, s_{n_i})$, then

$$S^m(t, t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m)).$$

The generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ on the set of all terms of type τ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i supposed that $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Let $Hyp_G(\tau)$ be the set of all generalized hypersubstitutions of type τ. We can define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. This means $\sigma_1 \circ_G \sigma_2$ is the generalized hypersubstitution which maps each fundamental operation symbol f_i to the term $\hat{\sigma}_1[\sigma_2(f_i)]$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$. Then we have the following proposition.

Proposition 1.1. ([5]) *For arbitrary terms $t, t_1, \dots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \dots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

It turns out that $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and σ_{id} is the identity element (see [5]).

An identity $s \approx t$ of a variety V is called a *strong hyperidentity* if for every generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in V. If M is a submonoid of $\underline{Hyp_G(\tau)}$, then $s \approx t$ is called an *M-strong hyperidentity* if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities for every $\sigma \in M$. A variety V is called *M-strongly solid* if every identity satisfied in V is an M-strong hyperidentity, and in case of $M = Hyp_G(\tau)$, we will say V is *strongly solid.* For more details on generalized hypersubstitutions and strongly solid varieties see [3], [4], [5].

2. Weak Projection Generalized Hypersubstitutions

In [2], K. Denecke and Sh.L. Wismath studied M-hyperidentities and M-solid varieties based on submonoids M of the monoid $\underline{Hyp(\tau)}$. They defined a number of natural such monoids based on various properties of hypersubstitutions. In [4] the author extended these concepts to generalized hypersubstitutions. In a similar way, we can define these monoids for the type $\tau = (2, 2)$.

Definition 2.1. Let $\tau = (2, 2)$ be a type with the binary operation symbols f and g. Any generalized hypersubstitution σ of type $\tau = (2, 2)$ is determined by the terms t_1, t_2 in $W_{(2,2)}(X)$ to which it maps the binary operation symbols f and g and we denote σ_{t_1,t_2}, i.e. $\sigma_{t_1,t_2}(f) = t_1$ and $\sigma_{t_1,t_2}(g) = t_2$.

(i) A generalized hypersubstitution σ of type $\tau = (2,2)$ is called a *projection generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are variables, i.e. $\{\sigma(f), \sigma(g)\} \subseteq \{x_i \in X \mid i \in \mathbb{N}\}$. We denote the set of all projection generalized hypersubstitutions of type $\tau = (2,2)$ by $P_G(2,2)$, i.e. $P_G(2,2) := \{\sigma_{x_i,x_j} \mid i,j \in \mathbb{N}, x_i, x_j \in X\}$.

(ii) A generalized hypersubstitution σ of type $\tau = (2,2)$ is called a *weak projection generalized hypersubstitution* if the term $\sigma(f)$ or $\sigma(g)$ belongs to $\{x_i \in X \mid i \in \mathbb{N}\}$. We denote the set of all weak projection generalized hypersubstitutions of type $\tau = (2,2)$ by $WP_G(2,2)$.

In [4] the author proved that for any type τ, the set $P_G(\tau) \cup \{\sigma_{id}\}$ is a submonoid of $\underline{Hyp_G(\tau)}$. It is easy to see that $WP_G(\tau) \cup \{\sigma_{id}\}$ is a submonoid of $\underline{Hyp_G(\tau)}$, and $P_G(\tau) \cup \{\sigma_{id}\}$ forms a submonoid of $WP_G(\tau) \cup \{\sigma_{id}\}$. It is obvious that every projection generalized hypersubstitution is idempotent and σ_{id} is also idempotent. Next, we will consider a necessary condition for weak projection generalized hypersubstitutions to be idempotent.

3. Idempotent Elements of $WP_G(2,2) \setminus P_G(2,2)$

In this section, we consider especially the idempotent elements of $WP_G(2,2) \setminus P_G(2,2)$. For any semigroup $S, x \in S$ is called an *idempotent element* of S if and only if $xx = x$. Then we have the following proposition.

Proposition 3.1. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. Then σ_{t_1,t_2} is idempotent if and only if $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.*

Proof. Assume that σ_{t_1,t_2} is idempotent, i.e. $\sigma^2_{t_1,t_2} = \sigma_{t_1,t_2}$. Then

$$\hat{\sigma}_{t_1,t_2}[t_1] = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(f)] = \sigma^2_{t_1,t_2}(f) = \sigma_{t_1,t_2}(f) = t_1.$$

Similarly, we get $\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(g)] = \sigma^2_{t_1,t_2}(g) = \sigma_{t_1,t_2}(g) = t_2$.

Conversely, let $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Since $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, then

$$(\sigma_{t_1,t_2} \circ_G \sigma_{t_1,t_2})(f) = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(f)] = \hat{\sigma}_{t_1,t_2}[t_1] = t_1 = \sigma_{t_1,t_2}(f).$$

Similarly, since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, then

$$(\sigma_{t_1,t_2} \circ_G \sigma_{t_1,t_2})(g) = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(g)] = \hat{\sigma}_{t_1,t_2}[t_2] = t_2 = \sigma_{t_1,t_2}(g).$$

Thus $\sigma^2_{t_1,t_2} = \sigma_{t_1,t_2}$. □

For the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$, where $t_1, t_2 \in W_{(2,2)}(X)$, we have exactly the following cases:

(i) $t_1 \in X, t_2 \notin X$ and $op(t_2) = 1$,

(ii) $t_2 \in X, t_1 \notin X$ and $op(t_1) = 1$,

(iii) $t_1 \in X, t_2 \notin X$ and $op(t_2) > 1$,

(iv) $t_2 \in X, t_1 \notin X$ and $op(t_1) > 1$,

where $op(t_1)$ and $op(t_2)$ denote the numbers of all operation symbols occurring in the terms t_1 and t_2, respectively.

Lemma 3.2. *Let $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$ be idempotent. Then we have:*

(i) *if $t_1 \in X$ and $op(t_2) = 1$, then the binary operation symbol occurring in t_2 is g, and*

(ii) *if $t_2 \in X$ and $op(t_1) = 1$, then the binary operation symbol occurring in t_1 is f.*

Proof. (i) Let $t_1 \in X$ and $op(t_2) = 1$. By Proposition 3.1, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$ and since $op(t_2) = 1$, then the term t_2 begins with exactly one binary operation symbol either f or g. In case of $t_2 = f(x_i, x_j)$, where $i, j \in \mathbb{N}, x_i, x_j \in X$, since σ_{t_1,t_2} maps f to a variable, we have $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$, which is a contradiction. Thus the binary operation symbol occurring in t_2 is g.

(ii) The proof is similar to the proof of (i). □

Proposition 3.3. *Let $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$. Then we have:*

(i) *if $t_1 \in X, t_2 \notin X$ and $op(t_2) = 1$, then σ_{t_1,t_2} is idempotent if and only if $t_2 \in \{g(x_i, x_j) \mid i, j \in \mathbb{N}, x_i, x_j \in X\} \setminus (\{g(x_2, x_i) \mid i \in \mathbb{N}, i \neq 2\} \cup \{g(x_i, x_1) \mid i \in \mathbb{N}, i \neq 1, 2\})$,*

(ii) *if $t_2 \in X, t_1 \notin X$ and $op(t_1) = 1$, then σ_{t_1,t_2} is idempotent if and only if $t_1 \in \{f(x_i, x_j) \mid i, j \in \mathbb{N}, x_i, x_j \in X\} \setminus (\{f(x_2, x_i) \mid i \in \mathbb{N}, i \neq 2\} \cup \{f(x_i, x_1) \mid i \in \mathbb{N}, i \neq 1, 2\})$.*

Proof. (i) Assume that σ_{t_1,t_2} is idempotent. Since $op(t_2) = 1$, then by Lemma 3.2(i)

$$t_2 \in \{g(x_i, x_j) \mid i, j \in \mathbb{N}, x_i, x_j \in X\}.$$

If $t_2 = g(x_2, x_i)$, where $i \in \mathbb{N}$ and $i \neq 2$,

$$\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[g(x_2, x_i)] = g(x_i, x_i) \text{ for some } i \in \mathbb{N}, i \neq 2.$$

It contradicts to Proposition 3.1.

If $t_2 = g(x_2, x_2), \hat{\sigma}_{t_1,t_2}[g(x_2, x_2)] = g(x_2, x_2)$.

In case of $t_2 = g(x_i, x_j)$, where $i, j \in \mathbb{N}$ and $i \neq 1, 2$,

$$\hat{\sigma}_{t_1,t_2}[g(x_i, x_j)] = \begin{cases} g(x_i, x_i) & ; \quad j = 1, \\ g(x_i, x_j) & ; \quad j \geq 2. \end{cases}$$

If $t_2 = g(x_1, x_j), \hat{\sigma}_{t_1,t_2}[g(x_1, x_j)] = g(x_1, x_j)$.

Conversely, we can check easily that if $t_2 \in \{g(x_i, x_j) \mid i, j \in \mathbb{N}, x_i, x_j \in X\} \setminus (\{g(x_2, x_i) \mid i \in \mathbb{N}, i \neq 2\} \cup \{g(x_i, x_1) \mid i \in \mathbb{N}, i \neq 1, 2\})$ and $t_1 \in X$ then σ_{t_1,t_2} is idempotent.

(ii) The proof is similar to the proof of (i). □

Next, let F be a variable over the two-element alphabet $\{f, g\}$. For an arbitrary term t of type $\tau = (2,2)$, we define two semigroup words $Lp(t)$ and $Rp(t)$ over $\{f, g\}$ inductively as follows:

(i) if $t = F(x_i, t_2), i \in I\!N, x_i \in X, t_2 \in W_{(2,2)}(X)$, then $Lp(t) := F$,

(ii) if $t = F(t_1, x_i), t_1 \in W_{(2,2)}(X), i \in I\!N, x_i \in X$, then $Rp(t) := F$,

(iii) if $t = F(t_1, t_2), t_1, t_2 \in W_{(2,2)}(X) \setminus X$, then $Lp(t) := F(Lp(t_1))$ and $Rp(t) := F(Rp(t_2))$.

Instead of $F_1(F_2(...F_n)...))$ it will be used: $F_1F_2...F_n...$ for the semigroup words: $LP(t)$ and $Rp(t)$.

As an example, let $t, t_1, t_2 \in W_{(2,2)}(X)$, where $t_1 = f(x_1, g(x_4, x_3)), t_2 = g(f(x_2, x_1), f(x_1, x_5))$ and $t = f(t_1, t_2)$, as shown by a tree below in Figure 1,

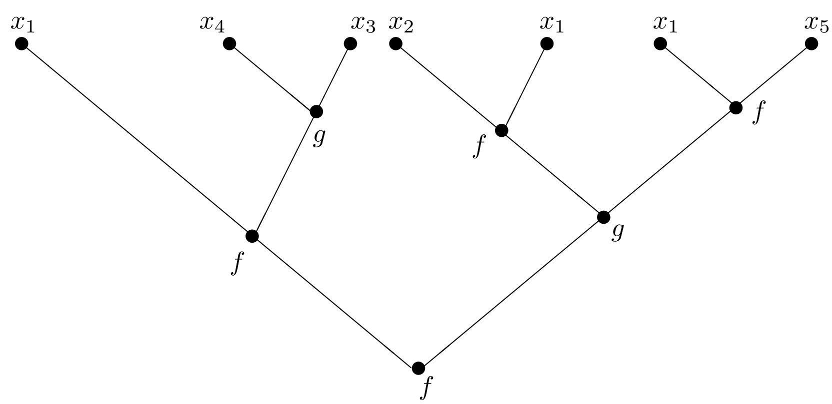

Figure 1:

then $Lp(t_1) = f, Rp(t_1) = fg, Lp(t_2) = gf, Rp(t_2) = gf, Lp(t) = ff$ and $Rp(t) = fgf$.

Notice that $Lp(t)$ is the left path from the root to the leaf which is labelled by the leftmost variable in t and $Rp(t)$ is the right path from the root to the leaf which is labelled by the rightmost variable in t.

We denote the sets of all operation symbols occurring in $Lp(t)$ and $Rp(t)$ by $ops(Lp(t))$ and $ops(Rp(t))$, respectively. So, from the previous example we have $ops(Lp(t)) = \{f\}, ops(Rp(t)) = \{f, g\}$.

Next, we consider in case (iii) $t_1 \in X, t_2 \notin X$ and $op(t_2) > 1$, and case (iv) $t_2 \in X, t_1 \notin X$ and $op(t_1) > 1$. For the case $t_1 \in X, t_2 \notin X$ and $op(t_2) > 1$, we have the following propositions.

Proposition 3.4. *For the generalized hypersubstitution* $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$, *where* $t_1, t_2 \in W_{(2,2)}(X)$. *If* $t_1 = x_1, t_2 \notin X, op(t_2) > 1$ *and if* $Lp(t_2) =$

$F_1 ... F_n$, where $F_j \in \{f, g\}; j = 1, \ldots, n$, then σ_{t_1,t_2} is idempotent if and only if there exists $i \in \{1, \ldots, n\}$ such that $F_i = g$ with the subterm t_2' of t_2, where $t_2' = g(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$, and the following conditions are satisfied:

(i) *if $x_1 \in var(t_2)$ and $s_1 \in X$, then $s_1 = x_1$,*

(ii) *if $x_2 \in var(t_2)$ and $s_2 \in X$, then $s_2 = x_2$,*

(iii) *if $x_1 \in var(t_2)$ and $s_1 \notin X$, then $ops(Lp(s_1)) = \{f\}$ and $leftmost(s_1) = x_1$,*

(iv) *if $x_2 \in var(t_2)$ and $s_2 \notin X$, then $ops(Lp(s_2)) = \{f\}$ and $leftmost(s_2) = x_2$,*

where $leftmost(s_1)$ and $leftmost(s_2)$ is the first variable (from the left) which occur in s_1 and s_2, respectively.

Proof. Assume that σ_{t_1,t_2} is idempotent and let $Lp(t_2) = F_1 \ldots F_n$, where $F_j \in \{f, g\}; j = 1, \ldots, n$, then there must exist the least $i \in \{1, \ldots, n\}$ such that $F_i = g$ since otherwise $\hat{\sigma}_{t_1,t_2}[t_2] \in X$ which contradicts to $t_2 \notin X$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Thus, there exists the least $i \in \{1, \ldots, n\}$ such that $F_i = g$ with the subterm t_2' of t_2, where $t_2' = g(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$. Since σ_{t_1,t_2} is idempotent and $F_1 \ldots F_{i-1} \in \{f\}$, then $t_2 = \hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[t_2'] = \hat{\sigma}_{t_1,t_2}[g(s_1, s_2)] = S^2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2]) = S^2(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2])$.

(i) If $x_1 \in var(t_2)$ and $s_1 \in X$, then we have to replace x_1 in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_1]$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, then $\hat{\sigma}_{t_1,t_2}[s_1]$ must be x_1. Thus $s_1 = x_1$.

(ii) If $x_2 \in var(t_2)$ and $s_2 \in X$, then we have to replace x_2 in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_2]$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, then $\hat{\sigma}_{t_1,t_2}[s_2]$ must be x_2. Thus $s_2 = x_2$.

(iii) If $x_1 \in var(t_2)$ and $s_1 \notin X$, then $s_1 = f(r_1, r_2)$ or $g(r_1, r_2)$, where $r_1, r_2 \in W_{(2,2)}(X)$. We will prove by induction on the number of operation symbols which occur in $Lp(s_1)$ that $ops(Lp(s_1)) = \{f\}$. If the number of operation symbols which occur in $Lp(s_1)$ is 1, then $s_1 = f(x_j, r_2)$ or $g(x_j, r_2), j \in \mathbb{N}, x_j \in X$. If $s_1 = g(x_j, r_2)$, then $t_2' = g(g(x_j, r_2), s_2)$. Consider

$$\begin{aligned} \hat{\sigma}_{t_1,t_2}[t_2'] &= S^2(\sigma_{t_1,t_2}(g), S^2(\sigma_{t_1,t_2}(g), x_j, \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]) \\ &= S^2(t_2, S^2(t_2, x_j, \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]). \end{aligned}$$

Since $x_1 \in var(t_2)$, we have to replace x_1 in the term t_2. After replacing, the term $S^2(t_2, S^2(t_2, x_j, \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2])$ must be longer than the term t_2, implying that $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$, which is a contradiction. Thus $s_1 = f(x_j, r_2)$. Assume that $ops(Lp(s)) = \{f\}$ if the number of operation symbols which occur in $Lp(s)$ is $n - 1$. Consider $s_1 = g(s, r_2)$, then the number of operation symbols which occur in $Lp(s_1)$ is n and by the same argument as before, we have a contradiction. Thus $s_1 = f(s, r_2)$.

By the induction hypothesis, $ops(Lp(s_1)) = \{f\}$. Next, assume that $leftmost(s_1) = x_2$, and since $ops(Lp(s_1)) = \{f\}$, then

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2'] \\ &= S^2(\sigma_{t_1,t_2}(g), S^2(\sigma_{t_1,t_2}(f), x_2, \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]) \\ &= S^2(t_2, x_2, \hat{\sigma}_{t_1,t_2}[s_2]). \end{aligned}$$

Since $x_1 \in var(t_2)$, we have to replace x_1 in the term t_2 by x_2. This contradicts to $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. In case of $leftmost(s_1) = x_j$, where $j \in I\!N, j > 2$. Because of $ops(Lp(s_1)) = \{f\}$, then

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2'] \\ &= S^2(\sigma_{t_1,t_2}(g), S^2(\sigma_{t_1,t_2}(f), x_j, \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]) \\ &= S^2(t_2, x_j, \hat{\sigma}_{t_1,t_2}[s_2]). \end{aligned}$$

Since $x_1 \in var(t_2)$, we have to replace x_1 in the term t_2 by x_j. This contradicts to $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Thus $leftmost(s_1) = x_1$.

(iv) The proof is similar to (iii).

Conversely, it is left to prove that in each of the conditions (i) - (iv) the generalized hypersubstitution σ_{t_1,t_2} is idempotent. But this is a routine work. □

Note that, for the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$, where $t_1, t_2 \in W_{(2,2)}(X), t_1 = x_1, t_2 \notin X, op(t_2) > 1$ and $Lp(t_2) = F_1...F_n$, where $F_j \in \{f, g\}; j = 1, \ldots, n$. If there exists the least $i \in \{1, \ldots, n\}$ such that $F_i = g$ with the subterm t_2' of t_2, where $t_2' = g(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$ and $x_1, x_2 \notin var(t_2)$ then σ_{t_1,t_2} is idempotent because we have nothing to substitute in $\hat{\sigma}_{t_1,t_2}[t_2]$, so $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, and it is obvious that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$.

Proposition 3.5. *For the generalized hypersubstitution* $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$, *where* $t_1, t_2 \in W_{(2,2)}(X)$. *If* $t_1 = x_2, t_2 \notin X, op(t_2) > 1$ *and if* $Rp(t_2) = F_1...F_n$, *where* $F_j \in \{f, g\}; j = 1, \ldots, n$, *then* σ_{t_1,t_2} *is idempotent if and only if there exists* $i \in \{1, \ldots, n\}$ *such that* $F_i = g$ *with the subterm* t_2' *of* t_2, *where* $t_2' = g(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$, *and the following conditions are satisfied:*

(i) *if* $x_1 \in var(t_2)$ *and* $s_1 \in X$, *then* $s_1 = x_1$,

(ii) *if* $x_2 \in var(t_2)$ *and* $s_2 \in X$, *then* $s_2 = x_2$,

(iii) *if* $x_1 \in var(t_2)$ *and* $s_1 \notin X$, *then* $ops(Rp(s_1)) = \{f\}$ *and* $rightmost(s_1) = x_1$,

(iv) *if* $x_2 \in var(t_2)$ *and* $s_2 \notin X$, *then* $ops(Rp(s_2)) = \{f\}$ *and* $rightmost(s_2) = x_2$,

where $rightmost(s_1)$ *and* $rightmost(s_2)$ *are the last variables which occur in* s_1 *and* s_2, *respectively.*

Proof. The proof is similar to Proposition 3.4. □

Note that, for the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2)\backslash P_G(2,2)$, where $t_1, t_2 \in W_{(2,2)}(X), t_1 = x_2, t_2 \notin X, op(t_2) > 1$ and $Rp(t_2) = F_1...F_n$, where $F_j \in \{f, g\}; j = 1, \dots, n$. If there exists the least $i \in \{1, \dots, n\}$ such that $F_i = g$ with the subterm t'_2 of t_2, where $t'_2 = g(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$ and $x_1, x_2 \notin var(t_2)$ then σ_{t_1,t_2} is idempotent because we have nothing to substitute in $\hat{\sigma}_{t_1,t_2}[t_2]$, so $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, and it is obvious that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$.

Lemma 3.6. *For the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2)\backslash P_G(2,2)$, where $t_1, t_2 \in W_{(2,2)}(X)$. If $t_1 = x_i, i \in \mathbb{N}, i > 2, t_2 \notin X, op(t_2) > 1$ and σ_{t_1,t_2} is idempotent, then $firstop(t_2) = g$, where $firstop(t_2)$ is the first operation symbol (from the left) occurring in t_2.*

Proof. Assume that $firstop(t_2) = f$. Let $t_2 = f(s_1, s_2)$, where $s_1, s_2 \in W_{(2,2)}(X)$ and either s_1 or s_2 can be a variable but not both. Consider $t_2 = \hat{\sigma}_{t_1,t_2}[t_2] = S^2(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2]) = S^2(x_i, \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2]) = x_i \in X$. This contradicts to $t_2 \notin X$. Thus $firstop(t_2) = g$. □

Lemma 3.7. *For the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2)\backslash P_G(2,2)$, where $t_1, t_2 \in W_{(2,2)}(X)$. If $t_2 = x_i, i \in \mathbb{N}, i > 2, t_1 \notin X, op(t_1) > 1$ and σ_{t_1,t_2} is idempotent, then $firstop(t_1) = f$, where $firstop(t_1)$ is the first operation symbol (from the left) occurring in t_1.*

Proof. Assume that $firstop(t_1) = g$. Let $t_1 = g(s_1, s_2)$, where $s_1, s_2 \in W_{(2,2)}(X)$ and either s_1 or s_2 can be a variable but not both. Consider $t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2]) = S^2(x_i, \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2]) = x_i \in X$. This contradicts to $t_1 \notin X$. Thus $firstop(t_1) = f$. □

Proposition 3.8. *For the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$, where $t_1, t_2 \in W_{(2,2)}(X)$. If $t_1 = x_i, i > 2, t_2 \notin X, op(t_2) > 1$, then σ_{t_1,t_2} is idempotent if and only if $t_2 = g(s_1, s_2)$ where $s_1, s_2 \in W_{(2,2)}(X)$ and either s_1 or s_2 can be a variable but not both, and the following conditions are satisfied:*

(i) *if $x_1 \in var(t_2)$ and $s_1 \in X$, then $s_1 = x_1$,*

(ii) *if $x_2 \in var(t_2)$ and $s_2 \in X$, then $s_2 = x_2$.*

Proof. Assume that σ_{t_1,t_2} is idempotent. By Lemma 3.6 and since $t_2 \notin X$, $op(t_2) > 1$, then $t_2 = g(s_1, s_2)$, where $s_1, s_2 \in W_{(2,2)}(X)$ and either s_1 or s_2 can be a variable, but not both. Consider

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[g(s_1, s_2)] \\ &= S^2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2]) \\ &= S^2(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \hat{\sigma}_{t_1,t_2}[s_2]). \end{aligned}$$

(i) If $x_1 \in var(t_2)$ and $s_1 \in X$, then we have to replace x_1 in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_1]$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, then $\hat{\sigma}_{t_1,t_2}[s_1]$ must be x_1. Thus $s_1 = x_1$.

(ii) If $x_2 \in var(t_2)$ and $s_2 \in X$, then we have to replace x_2 in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_2]$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, then $\hat{\sigma}_{t_1,t_2}[s_2]$ must be x_2. Thus $s_2 = x_2$.

Conversely, it is left to prove that in each of the conditions (i) - (ii) the generalized hypersubstitution is idempotent. But, it is a routine work. □

Proposition 3.9. *For the generalized hypersubstitution* $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$*, where* $t_1, t_2 \in W_{(2,2)}(X), t_1 = x_i, i > 2, t_2 \notin X, op(t_2) > 1, t_2 = g(s_1, s_2)$*, where* $s_1, s_2 \in W_{(2,2)}(X)$ *and either* s_1 *or* s_2 *can be a variable but not both. The following conditions are satisfied:*

(i) *if* $x_1 \in var(t_2)$ *and* $s_1 \notin X$*, then* σ_{t_1,t_2} *is not idempotent,*

(ii) *if* $x_2 \in var(t_2)$ *and* $s_2 \notin X$*, then* σ_{t_1,t_2} *is not idempotent.*

Proof. (i) Since $s_1 \notin X$, then $s_1 = f(r_1, r_2)$ or $s_1 = g(r_1, r_2)$, where $r_1, r_2 \in W_{(2,2)}(X)$.

If $s_1 = f(r_1, r_2), t_2 = g(f(r_1, r_2), s_2)$. Consider

$$\begin{aligned}
\hat{\sigma}_{t_1,t_2}[t_2] &= S^2(\sigma_{t_1,t_2}(g), S^2(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[r_1], \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]) \\
&= S^2(t_2, S^2(x_i, \hat{\sigma}_{t_1,t_2}[r_1], \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]) \\
&= S^2(g(f(r_1, r_2), s_2), x_i, \hat{\sigma}_{t_1,t_2}[s_2]).
\end{aligned}$$

Since $x_1 \in var(t_2)$, then x_1 must occur in the term r_1, or r_2, or s_2. We have to replace x_1 by x_i and thus $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$. This implies that σ_{t_1,t_2} is not idempotent.

If $s_1 = g(r_1, r_2), t_2 = g(g(r_1, r_2), s_2)$. Consider

$$\begin{aligned}
\hat{\sigma}_{t_1,t_2}[t_2] &= S^2(\sigma_{t_1,t_2}(g), S^2(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[r_1], \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]) \\
&= S^2(t_2, S^2(t_2, \hat{\sigma}_{t_1,t_2}[r_1], \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]).
\end{aligned}$$

Since $x_1 \in var(t_2)$, then we have to replace x_1 in the term t_2. After replacing, the term $S^2(t_2, S^2(t_2, \hat{\sigma}_{t_1,t_2}[r_1], \hat{\sigma}_{t_1,t_2}[r_2]), \hat{\sigma}_{t_1,t_2}[s_2]) \neq t_2$. This implies that σ_{t_1,t_2} is not idempotent.

(ii) We can proof in the similar way as the proof of (i). □

For the case $t_2 \in X, t_1 \notin X$ and $op(t_1) > 1$, we have also the following propositions and these propositions can be proved in the same manner as in the case $t_1 \in X, t_2 \notin X, op(t_2) > 1$.

Proposition 3.10. *For the generalized hypersubstitution* $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$*, where* $t_1, t_2 \in W_{(2,2)}(X)$*. If* $t_2 = x_1, t_1 \notin X, op(t_1) > 1$ *and if* $Lp(t_1) = F_1...F_n$*, where* $F_j \in \{f, g\}; j = 1, \ldots, n$*, then* σ_{t_1,t_2} *is idempotent if and only if there exists* $i \in \{1, \ldots, n\}$ *such that* $F_i = f$ *with the subterm* t_1' *of* t_1*, where* $t_1' = f(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$*, and the following conditions are satisfied:*

(i) *if* $x_1 \in var(t_1)$ *and* $s_1 \in X$*, then* $s_1 = x_1$*,*

(ii) *if* $x_2 \in var(t_1)$ *and* $s_2 \in X$*, then* $s_2 = x_2$*,*

(iii) *if* $x_1 \in var(t_1)$ *and* $s_1 \notin X$, *then* $ops(Lp(s_1)) = \{g\}$ *and* $leftmost(s_1) = x_1$,

(iv) *if* $x_2 \in var(t_1)$ *and* $s_2 \notin X$, *then* $ops(Lp(s_2)) = \{g\}$ *and* $leftmost(s_2) = x_2$.

Note that, for the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2)\setminus P_G(2,2)$ where $t_1, t_2 \in W_{(2,2)}(X), t_2 = x_1, t_1 \notin X, op(t_1) > 1$ and $Lp(t_1) = F_1...F_n$ where $F_j \in \{f, g\}; j = 1, \ldots, n$. If there exists the least $i \in \{1, \ldots, n\}$ such that $F_i = f$ with the subterm t'_1 of t_1 where $t'_1 = f(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$ and $x_1, x_2 \notin var(t_1)$ then σ_{t_1,t_2} is idempotent because we have nothing to substitute in $\hat{\sigma}_{t_1,t_2}[t_1]$, so $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and it is obvious that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.

Proposition 3.11. *For the generalized hypersubstitution* $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$ *where* $t_1, t_2 \in W_{(2,2)}(X)$. *If* $t_2 = x_2, t_1 \notin X, op(t_1) > 1$ *and if* $Rp(t_1) = F_1...F_n$ *where* $F_j \in \{f, g\}; j = 1, \ldots, n$, *then* σ_{t_1,t_2} *is idempotent if and only if there exists* $i \in \{1, \ldots, n\}$ *such that* $F_i = f$ *with the subterm* t'_1 *of* t_1 *where* $t'_1 = f(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$ *and the following conditions are satisfied:*

(i) *if* $x_1 \in var(t_1)$ *and* $s_1 \in X$, *then* $s_1 = x_1$,

(ii) *if* $x_2 \in var(t_1)$ *and* $s_2 \in X$, *then* $s_2 = x_2$,

(iii) *if* $x_1 \in var(t_1)$ *and* $s_1 \notin X$, *then* $ops(Rp(s_1)) = \{f\}$ *and* $rightmost(s_1) = x_1$,

(iv) *if* $x_2 \in var(t_1)$ *and* $s_2 \notin X$, *then* $ops(Rp(s_2)) = \{f\}$ *and* $rightmost(s_2) = x_2$.

Note that for the generalized hypersubstitution $\sigma_{t_1,t_2} \in WP_G(2,2)\setminus P_G(2,2)$, where $t_1, t_2 \in W_{(2,2)}(X), t_2 = x_2, t_1 \notin X, op(t_1) > 1$ and $Rp(t_1) = F_1...F_n$, where $F_j \in \{f, g\}; j = 1, \ldots, n$. If there exists the least $i \in \{1, \ldots, n\}$ such that $F_i = f$, with the subterm t'_1 of t_1, where $t'_1 = f(s_1, s_2); s_1, s_2 \in W_{(2,2)}(X)$ and $x_1, x_2 \notin var(t_1)$, then σ_{t_1,t_2} is idempotent because we have nothing to substitute in $\hat{\sigma}_{t_1,t_2}[t_1]$, so $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, and it is obvious that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.

Proposition 3.12. *For the generalized hypersubstitution* $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$, *where* $t_1, t_2 \in W_{(2,2)}(X)$. *If* $t_2 = x_i, i > 2, t_1 \notin X, op(t_1) > 1$, *then* σ_{t_1,t_2} *is idempotent if and only if* $t_1 = f(s_1, s_2)$ *where* $s_1, s_2 \in W_{(2,2)}(X)$ *and either* s_1 *or* s_2 *can be a variable but not both, and the following conditions are satisfied:*

(i) *if* $x_1 \in var(t_1)$ *and* $s_1 \in X$, *then* $s_1 = x_1$,

(ii) *if* $x_2 \in var(t_1)$ *and* $s_2 \in X$, *then* $s_2 = x_2$.

Proposition 3.13. *For the generalized hypersubstitution* $\sigma_{t_1,t_2} \in WP_G(2,2) \setminus P_G(2,2)$, *where* $t_1, t_2 \in W_{(2,2)}(X), t_2 = x_i, i > 2, t_1 \notin X, op(t_1) > 1, t_1 = f(s_1, s_2)$, *where* $s_1, s_2 \in W_{(2,2)}(X)$ *and either* s_1 *or* s_2 *can be a variable but not both. The following conditions are satisfied:*

(i) *if* $x_1 \in var(t_1)$ *and* $s_1 \notin X$, *then* σ_{t_1,t_2} *is not idempotent,*

(ii) *if* $x_2 \in var(t_1)$ *and* $s_2 \notin X$, *then* σ_{t_1,t_2} *is not idempotent.*

Acknowledgements. The author expresses his thanks to the referee for his valuable comments. This research was supported by the Faculty of Science, Chiang Mai University, Thailand.

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Received by the editors October 8, 2009