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ENTIRE FUNCTIONS THAT SHARE RATIONAL FUNCTIONS WITH THEIR DERIVATIVES[1]

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Abstract. In this paper, we use the idea of normal family to deal with the uniqueness problems of entire functions that share a rational function with its derivative and get a uniqueness theorem. The conclusions in this paper can be used to improve several known results. Some examples are provided to show that the results presented in this paper are possible.

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1. Introduction and main results

In this article, by meromorphic functions we shall always mean the meromorphic functions in the complex plane. We are going to mainly use the basic notation of Nevanlinna Theory (see [7], [18], [19]), such as $T(r, f)$, $N(r, f)$, $m(r, f)$, $\overline{N}(r, f)$ and $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set of finite measure. Let f and g denote two non-constant meromorphic functions, and let R be a rational function. If $f - R$ and $g - R$ have the same zeros with the same multiplicities (ignoring multiplicities), then we say that f and g share R CM (IM), and denote it by $f = R \rightleftharpoons g = R$ ($f = R \Leftrightarrow g = R$). In this paper, we also need the following two definitions.

Definition 1.1. Let f be a non-constant entire function, the order of f, denoted $\sigma(f)$, being defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

where, and in the sequel, $M(r, f) = \max_{|z|=r}\{|f(z)|\}$.

Definition 1.2. Let f be a nonconstant meromorphic function, the hyper order of f, denoted $\sigma_2(f)$, is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

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In 1977, Rubel and Yang [15] proved the well-known theorem.

Theorem A. *Let a and b be two complex numbers such that $b \neq a$, and let f be a nonconstant entire function. If f and f' share values a and b CM, then $f \equiv f'$.*

This result has undergone various extensions and improvements. Mues and Steinmetz [12] proved the following theorem.

Theorem B. *Let a and b be two complex numbers such that $b \neq a$, and let f be a non-constant entire function. If f and f' share values a and b IM, then $f \equiv f'$.*

Ang Chen, et al [2] got the following theorem, in which the second relationship between f and f' is $f = R_2 \Rightarrow f' = R_2$, here for the definition of R_2 see Theorem C.

Theorem C. *Let $R_1 = P_1(z)e^{Q(z)}, R_2 = P_2(z)e^{Q(z)}$, where $Q(z)$ is a polynomial and $P_1(z), P_2(z)$ are rational functions, be two functions and $R_2(\not\equiv R_1, 0)$. Let f be a nonconstant meromorphic function with finitely many poles. If $f = R_1 \rightleftharpoons f' = R_1$, $f = R_2 \Rightarrow f' = R_2$. If f, R_1 have no common poles and the order of R_1 is less than the order of f, then one of the following cases must occur:*
(1) $f \equiv f'$.
(2) $f = R_2 + Ce^{\lambda z}$ and $(\lambda - 1)R_1 = \lambda R_2 - R_2'$, where C, λ are two nonzero constants. In fact, R_1, R_2 are two polynomials.

On the other hand, there were also many improvements of Theorem B by assuming the second relationship between f and f' is $f' = R_2 \Rightarrow f = R_2$, here R_2 can be a constant(see Theorem D), or be a polynomial(see Theorem F). In 2006, Li and Yi [9] gave an example to show the condition that f and f' have two shared values in Theorems B is necessary. They also thought about whether the condition can be changed to some extent and gave an affirmative answer as follows.

Theorem D. *Let a and b be two complex numbers such that $b \neq a,\ 0$, and let f be a non-constant entire function. If $f = a \rightleftharpoons f' = a$ and $f' = b \Rightarrow f = b$, then $f \equiv f'$.*

Remark 1.1. In the same paper, authors [9] gave an example to show that $b \neq 0$ cannot be omitted in Theorem D.

In 2007, Li and Yi [10] proved the following result.

Theorem E. *Let f be a non-constant entire function of hyper-order $\sigma_2(f) < \frac{1}{2}$ and let Q be a non-constant polynomial. If $f = Q \rightleftharpoons f' = Q$, then*

$$\frac{f' - Q}{f - Q} \equiv c$$

for some constant $c \neq 0$.

In 2009, Qi, Lü and Chen [14] improved Theorem D and got the following result.

Theorem F. *Let $Q_1(z) = a_1 z^p + a_{1,p-1} z^{p-1} + \cdots + a_{1,0}$ and $Q_2(z) = a_2 z^p + a_{2,p-1} z^{p-1} + \cdots + a_{2,0}$ be two polynomials such that $\deg Q_1 = \deg Q_2 = p$ (where p is a non-negative integer) and a_1, $a_2 (a_2 \neq 0)$ are two distinct complex numbers. Let f be a transcendental entire function. If $f = Q_1 \rightleftharpoons f' = Q_1$ and $f' = Q_2 \Rightarrow f = Q_2$, then $f \equiv f'$.*

Naturally, we ask what will happen if the polynomials Q_1, Q_2 are replaced by the rational functions R_1, R_2? In this paper, we consider the above question and use the idea of normal families to obtain a uniqueness theorem.

We set $R(z) = P_1(z)/P_2(z)$, where $P_1,\ P_2$ are relatively prime polynomials. In this paper, deviating from the usual definition of the degree of a rational function, $\deg(P_1) - \deg(P_2)$ is called the degree of $R(z)$ and denoted by $\deg(R)$.

Theorem 1.1. *Let $R_1(z)$ and $R_2(z)$ be two non-zero rational functions such that $\lim_{z\to\infty} \frac{R_2(z)}{R_1(z)} \neq 1$ and $\deg(R_1) = \deg(R_2)$, and let f be a transcendental entire function. If $f = R_1 \rightleftharpoons f' = R_1$ and $f' = R_2 \Rightarrow f = R_2$, then one of the following cases must occur:*
(i) $f \equiv f'$;
(ii) $f' = R_2 + C\lambda e^{\lambda z}$ and $(\lambda - 1)R_1' = \lambda R_2 - R_2'$, where C, $\lambda \neq 1$ are two non-zero constants. In fact, R_1, R_2 are two polynomials.

Remark 1.2. The following shows the hypothesis that f is transcendental cannot be omitted in Theorem 1.1.

Example 1. Let $f(z) = z^4$, $R_1(z) = 2z^4 - 4z^3$ and $R_2(z) = z^4$. Then

$$\frac{f'(z) - R_1(z)}{f(z) - R_1(z)} = 2 \quad \text{and} \quad f'(z) = R_2(z) \Rightarrow f(z) = R_2(z).$$

Whereas it does not satisfy the result of Theorem 1.1.

Remark 1.3. We add an example to point out that the case (ii) in Theorem 1.1 cannot be deleted.

Example 2. Let $f = 2e^{\frac{z}{2}} + \frac{1}{2}z^2$, $R_1 = 2z - \frac{1}{2}z^2$ and $R_2 = z$. Then

$$\frac{f' - R_1}{f - R_1} = \frac{1}{2} \quad \text{and} \quad f' \neq R_2.$$

Thus, it satisfies the assumption of Theorem 1.1.

Remark 1.4. Obviously, when R_1, R_2 can be two polynomials defined as in Theorem F, it is easy to see Theorem 1.1 improves Theorem F and Theorem D.

In order to prove Theorem 1.1, we need the following result which is of independent interest.

Theorem 1.2. *Let $R_1(z)$ and $R_2(z)$ be two non-zero rational functions such that $\lim_{z\to\infty} \frac{R_2(z)}{R_1(z)} \neq 1$ and $\deg(R_1) = \deg(R_2)$, and let f be a non-constant entire function. If $f = R_1 \Rightarrow f' = R_1$ and $f' = R_2 \Rightarrow f = R_2$, then f is of order at most one.*

2. Some Lemmas

In order to prove our theorems, we need the following lemmas. Let h be a meromorphic function in $\mathbb{C}$. h is called a normal function if there exists a positive M such that $h^\sharp(z) \leq M$ for all $z \in \mathbb{C}$, where

$$h^\sharp(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D, see [16]. Normal families, in particular, of entire functions often appear in operator theory on spaces of analytic functions, for instance, see, Lemma 3 in [8] and Lemma 4 in [17].

The following lemma is the famous Marty's criterion.

Lemma 2.1. *[16] A family $\mathcal{F}$ of meromorphic functions on a domain D is normal if and only if for each compact subset $K \subseteq D$, there exists a constant M such that $f^\sharp(z) \leq M$ for each $f \in \mathcal{F}$ and $z \in K$.*

The well-known Zalcman's lemma is a very important tool in the study of normal families. It has also undergone various extensions and improvements. The following is one up-to-date local version, which is due to Pang and Zalcman [13](cf. [3], [4], [20], [21]).

Lemma 2.2. *Let $\mathcal{F}$ be a family of analytic functions in the unit disc $\triangle$ with the property that for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f(z) = 0$. If $\mathcal{F}$ is not normal in Δ, then for $0 \leq \alpha \leq k$, there exists*

1. *a number $r \in (0, 1)$;*
2. *a sequence of complex numbers z_n, $|z_n| < r$;*
3. *a sequence of functions $f_n \in \mathcal{F}$;*
4. *a sequence of positive numbers $\rho_n \to 0$,*

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta)$ converges locally uniformly (with respect to the spherical metric) to a non-constant entire function $g(\zeta)$ on $\mathbb{C}$. Moreover, the zeros of $g(\zeta)$ are of multiplicities at least k, $g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1$.

The next result is due to Clunie and Hayman [5].

Lemma 2.3. *A normal meromorphic function has order at most two. A normal holomorphic function is of exponential type, and thus has order at most one.*

Lemma 2.4. *[11] Let $R(z)(\not\equiv 0)$ and $H(z)(\not\equiv 0)$ be two rational functions; let $Q(z)$ be a polynomial; and let $F(z)$ be a transcendental meromorphic function with finite order. If $F(z)$ is a solution of the following differential equation*

$$F'(z) - R(z)e^{Q(z)}F(z) = H(z), \tag{2.1}$$

then $Q(z)$ is a constant. In particular, if $R(z) = \frac{1}{P(z)}$, where $P(z)$ is a polynomial, then $R(z)$ is also a constant.

Lemma 2.5. *[1] Let f be a meromorphic funtion on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most one.*

Lemma 2.6. *[7], [19] Let $f(z)$ be a meromorphic function, and let $a_1(z)$, $a_2(z)$, $a_3(z)$ be three distinct meromorphic functions satisfying $T(r,a_i) = S(r,f), (i = 1,2,3)$. Then*

$$T(r,f) \le \overline{N}\left(r,\frac{1}{f-a_1}\right) + \overline{N}\left(r,\frac{1}{f-a_2}\right) + \overline{N}\left(r,\frac{1}{f-a_3}\right) + S(r,f).$$

3. Proof of Theorem 1.2

We assume $R_1 = \frac{Q_1}{Q_2}$, $R_2 = \frac{Q_3}{Q_4}$, here $Q_i (i = 1,2,3,4)$ are polynomials. Let $P_1 = Q_1Q_4$, $P_2 = Q_2Q_3$.

Since $R_1 \not\equiv 0$ and $\deg(R_1) = \deg(R_2)$, it is easy to see $\deg(P_1) = \deg(P_2)$. Now we consider the function $F = \frac{f}{R_1} - 1$. In the following, we will distinguish two cases for discussion.

Case 1. F has a bounded spherical derivative.

Then by Lemma 2.5, F is of order at most one. Hence $f = (F+1)R_1$ is of order at most one as well. Thus, the conclusion of Theorem 1.2 is revealed.

Case 2. F has unbounded spherical derivative.

Then there exists a sequence $(w_n)_n$ such that $\lim\limits_{n\to\infty} F^\sharp(w_n) = \infty$. Since $F^\sharp$ is continuous, hence bounded in every compact set, we have $w_n \to \infty$ as $n \to \infty$.

Since R_1 is a rational function, there exists an r_1 such that for all $z \in \mathbb{C}$ satisfying $|z| \ge r_1$, we have

$$0 \leftarrow \left|\frac{R_1'(z)}{R_1(z)}\right| \le \frac{M_1}{|z|} < 1,\ R_1(z) \ne 0. \tag{3.1}$$

Let $r > r_1$, and $D = \{z:\ |z| \ge r\}$, then F is analytic in D. Without loss of generality, we may assume $|w_n| \ge r+1$ for all n. We define $D_1 = \{z:\ |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{f(w_n+z)}{R_1(w_n+z)} - 1. \tag{3.2}$$

From (3.2), if $F(w_n+z)=0$, thus $f(w_n+z)=R_1(w_n+z)$. Noting that $f=R_1\Rightarrow f'=R_1$, then by (3.1), we obtain the following: if $F_n(z)=0$ and n is large enough, then

(3.3)
$$|F_n'|=\left|\left(\frac{f(w_n+z)}{R_1(w_n+z)}\right)'\right|\le\left|\frac{f'(w_n+z)}{R_1(w_n+z)}\right|+\left|\frac{f(w_n+z)}{R_1(w_n+z)}\right|\left|\frac{R_1'(w_n+z)}{R_1(w_n+z)}\right|\le 2.$$

Obviously, $F_n(z)$ are analytic in D_1 and $F_n^{\#}(0)=F^{\#}(w_n)\to\infty$ as $n\to\infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z=0$. In the following we will obtain a contradiction by proving that $(F_n)_n$ is normal at $z=0$.

In view of (3.3), we can apply Lemma 2.2 with ($\alpha=k=1$ *and* $A=2$). Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ such that $z_n\to 0$ and $\rho_n\to 0$, and that the sequence $(g_n)_n$ defined by

$$g_n(\zeta)=\rho_n^{-1}F_n(z_n+\rho_n\zeta)=\rho_n^{-1}\left\{\frac{f(w_n+z_n+\rho_n\zeta)}{R_1(w_n+z_n+\rho_n\zeta)}-1\right\}\to g(\zeta) \tag{3.4}$$

converges locally and uniformly in $\mathbb{C}$, where $g(\zeta)$ is a nonconstant entire function and $g^{\#}(\zeta)\le g^{\#}(0)=3$. By Lemma 2.3, the order of $g(\zeta)$ is at most 1.

Firstly, we claim that
$$g=0\Rightarrow g'=1.$$

Set $G_n(\zeta)=\frac{f'(w_n+z_n+\rho_n\zeta)}{R_1(w_n+z_n+\rho_n\zeta)}$, then from (3.4) and $\frac{R'(w_n+z_n+\rho_n\zeta)}{R(w_n+z_n+\rho_n\zeta)}\to 0$, we get

(3.5)
$$G_n(\zeta)=\frac{f'(w_n+z_n+\rho_n\zeta)}{R_1(w_n+z_n+\rho_n\zeta)}=g_n'(\zeta)+\frac{(\rho_n g_n(\zeta)+1)R_1'(w_n+z_n+\rho_n\zeta)}{R_1(w_n+z_n+\rho_n\zeta)}\to g'(\zeta)$$

locally and uniformly in $\mathbb{C}$.

Suppose that there exists a point ζ_0 such that $g(\zeta_0)=0$. Then by Hurwitz's Theorem, there exists $\zeta_n,\ \zeta_n\to\zeta_0$ as $n\to\infty$, such that (for n sufficiently large)

$$g_n(\zeta_n)=\rho_n^{-1}(F_n(z_n+\rho_n\zeta_n))=0.$$

Thus $F_n(z_n+\rho_n\zeta_n)=0$ and $f(w_n+z_n+\rho_n\zeta_n)=R_1(w_n+z_n+\rho_n\zeta_n)$, by the assumption we have
$$\frac{f'(w_n+z_n+\rho_n\zeta_n)}{R_1(w_n+z_n+\rho_n\zeta_n)}=1.$$

Then, by (3.5) we derive that

$$g'(\zeta_0)=\lim_{n\to\infty}\frac{f'(w_n+z_n+\rho_n\zeta_n)}{R_1(w_n+z_n+\rho_n\zeta_n)}=1.$$

Thus $g(\zeta)=0\Rightarrow g'(\zeta)=1$. Then our claim holds.

Since $\deg(P_1)=\deg(P_2)$, we assume $P_1=a_1z^n+a_{1,n-1}z^{n-1}+\cdots+a_{1,0}$ and $P_2=a_2z^n+a_{2,n-1}z^{n-1}+\cdots+a_{2,0}$. In the following, we will prove $g'(\zeta)\ne\frac{a_2}{a_1}$ on $\mathbb{C}$.

Suppose that there exists a point ζ_0 such that $g'(\zeta_0) = \frac{a_2}{a_1}$. If $g'(\zeta) \equiv \frac{a_2}{a_1}$, then $g(\zeta) = \frac{a_2}{a_1}\zeta + c_0$, where c_0 is a constant, which together with the fact $g = 0 \to g' = 1$ gives $a_2 = a_1$. This contradicts to the assumption $\lim_{z\to\infty} \frac{R_2(z)}{R_1(z)} \neq 1$. Thus $g'(\zeta) \not\equiv \frac{a_2}{a_1}$.

Since $G_n(\zeta) - \frac{R_2(w_n+z_n+\rho_n\zeta)}{R_1(w_n+z_n+\rho_n\zeta)} \to g'(\zeta) - \frac{a_2}{a_1}$ as $n \to \infty$ and $g'(\zeta_0) = \frac{a_2}{a_1}$, by Hurwitz's theorem, there exists $\zeta_n,\ \zeta_n \to \zeta_0$ as $n \to \infty$, such that (for n sufficiently large)

(3.6)
$$G_n(\zeta_n) - \frac{R_2(w_n + z_n + \rho_n\zeta_n)}{R_1(w_n + z_n + \rho_n\zeta_n)} = 0 \Rightarrow f'(w_n + z_n + \rho_n\zeta_n) = R_2(w_n + z_n + \rho_n\zeta_n).$$

Since $a_1 \neq a_2$ and $\rho_n \to 0$, noting $f' = R_2 \Rightarrow f = R_2$, from (3.4), (3.6) we get

$$g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_n) = \rho_n^{-1}\left(\frac{R_2(w_n + z_n + \rho_n\zeta_n)}{R_1(w_n + z_n + \rho_n\zeta_n)} - 1\right) = \infty,$$

which contradicts to $g'(\zeta_0) = \frac{a_2}{a_1}$. This shows that $g'(\zeta) \neq \frac{a_2}{a_1}$ on $\mathbb{C}$.

Since g is of the order at most one, and so is g', hence it follows that

$$g'(\zeta) = \frac{a_2}{a_1} + e^{c_1 + c_2\zeta}, \tag{3.7}$$

where $c_1,\ c_2$ are finite constants. We divide it into two subcases as follows.

Subcase 2.1. If $c_2 = 0$, from (3.7) we have

$$g(\zeta) = \left(\frac{a_2}{a_1} + e^{c_1}\right)\zeta + c_0, \tag{3.8}$$

where c_0 is a constant. Since $g = 0 \to g' = 1$, from (3.8) we have $\frac{a_2}{a_1} + e^{c_1} = 1$. By a simple calculation we get $g^\sharp(0) \leq \frac{1}{1+|c_0|^2} < 3$, which is a contradiction.

Subcase 2.2. If $c_2 \neq 0$, by (3.7) we obtain

$$g(\zeta) = \frac{a_2}{a_1}\zeta + \frac{1}{c_2}e^{c_1 + c_2\zeta} + B, \tag{3.9}$$

where B is a constant. Obviously, $g(\zeta) = 0$ has infinitely many solutions. Suppose that there exists a point ζ_0 such that $g(\zeta_0) = 0$. Then by (3.7)(3.9) and $g = 0 \Rightarrow g' = 1$, we can get $\zeta_0 = \frac{a_2 - a_1 - c_2 B a_1}{c_2 a_2}$. This is also a contradiction. These contradictions show that Case 2 cannot occur and hence the proof of Theorem 1.2 is complete.

4. Proof of Theorem 1.1

By Theorem 1.2, we get f is of the order at most 1. Since $f = R_1 \rightleftharpoons f' = R_1$, we deduce that

$$e^\alpha = \frac{f' - R_1}{f - R_1}. \tag{4.1}$$

where α is an entire function. Noting $\sigma(f) \leq 1$, from (4.1), we have $\sigma(e^{\alpha}) \leq \sigma(f) \leq 1$. Therefore we can set $e^{\alpha} = C_1 e^{C_2 z}$, where C_1, C_2 are two constants. Let $F = f - R_1$ and $A = R_1 - R_1'$, we see that $A(\not\equiv 0)$ is a rational function and

$$F' - A = C_1 e^{C_2 z} F.$$

By Lemma 2.4, we deduce $C_2 = 0$. Thus $e^{\alpha} = \lambda$, here λ is a constant. From (4.1), we have

$$f' = \lambda f + (1-\lambda) R_1. \tag{4.2}$$

If $\lambda = 1$, we have that $f \equiv f'$, which is (i).

In the following, we assume that $\lambda \neq 1$. Since f is an entire function, R_1 is a rational function, from (4.2) it is easy to see that f and R_1 are two entire functions. So, R_1 is a polynomial. From the integral of (4.2), we have

$$f = Ce^{\lambda z} + h(z), \tag{4.3}$$

where C is a non-zero constant and $h(z)$ is a polynomial. Thus, we have

$$f' = C\lambda e^{\lambda z} + h'(z). \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2), we deduce that

$$(\lambda - 1) R_1 - (\lambda h - h') \equiv 0. \tag{4.5}$$

Next, we will prove that $h'(z) \equiv R_2(z)$. Suppose that $h'(z) \not\equiv R_2(z)$, then

$$\overline{N}\left(r, \frac{1}{f' - R_2}\right) = \overline{N}\left(r, \frac{1}{C\lambda e^{\lambda z} + h'(z) - R_2}\right). \tag{4.6}$$

Since $f(z)$ is a transcendental entire function and $h'(z) - R_2(z)$ is a rational function, we deduce $T(r, h'(z) - R_2(z)) = S(r, f)$. Moreover, it is well known that 0 and ∞ are Picard values of $e^{\lambda z}$. Then by Lemma 2.6, we obtain

$$T(r, C\lambda e^{\lambda z}) \leq \overline{N}\left(r, \frac{1}{C\lambda e^{\lambda z} + h'(z) - R_2}\right) + S(r, f). \tag{4.7}$$

By the Nevanlinna First Fundamental Theorem, we immediately obtain

$$\overline{N}\left(r, \frac{1}{C\lambda e^{\lambda z} + h'(z) - R_2}\right) \leq T(r, C\lambda e^{\lambda z}) + S(r, f). \tag{4.8}$$

Combining with (4.7) and (4.8), we obtain

$$\overline{N}\left(r, \frac{1}{C\lambda e^{\lambda z} + h'(z) - R_2}\right) = T(r, C\lambda e^{\lambda z}) + S(r, f) \neq S(r, f). \tag{4.9}$$

Suppose that z_0 is a zero of $f' - R_2$, by the assumption we have $f(z_0) = R_2(z_0)$. By putting z_0 into (4.3) and (4.4) we have

$$(\lambda - 1) R_2(z_0) = \lambda h(z_0) - h'(z_0).$$

If $(\lambda-1)R_2-(\lambda h-h')\not\equiv 0$, we have

$$\overline{N}\left(r,\frac{1}{f'-R_2}\right)\leq\overline{N}\left(r,\frac{1}{(\lambda-1)R_2-(\lambda h-h')}\right)$$
$$=O(\log r)=S(r,f),$$

which contradicts with (4.9). Hence,

$$(\lambda-1)R_2-(\lambda h-h')\equiv 0.$$

Comparing it to (4.5), we have $R_1=R_2$, which is a contradiction. Thus, we obtain $h'(z)=R_2(z)$. Then, from (4.4) and (4.5), we have

$$f'=C\lambda e^{\lambda z}+R_2(z),$$

and

$$(\lambda-1)R_1'=\lambda R_2-R_2',$$

which is (ii). Thus Theorem 1.1 is completely proved.

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