Novi Sad J. Math.
Vol. 41, No. 2, 2011, 81-87

GENERALIZED SEMI-IDEALS IN TERNARY SEMIRINGS

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Abstract. We introduce the notion of a generalized semi-ideal in a ternary semiring. Various examples to establish a relationship between ideals, bi-ideals, quasi-ideals and generalized semi-ideals are furnished. A criterion for a commutative ternary semiring without any divisors of zero to a ternary division semiring is given.

AMS Mathematics Subject Classification (2010): 16Y60, 16Y99

Key words and phrases: Ternary semiring, generalized semi-ideals and ideals in ternary semirings, ternary division semiring

1. Introduction

Ternary rings and their structures were investigated by Lister [4] in 1971. In fact, Lister characterized those additive subgroups of rings which are closed under the triple product. In 2003, T. K. Dutta and S. Kar [3] introduced the notion of a ternary semiring as a generalization of a ternary ring. A ternary semiring arises naturally as follows. Consider the subset $\mathbb{Z}^-$ of all negative integers of $\mathbb{Z}$. Then $\mathbb{Z}^-$is an additive semigroup which is closed under the triple product. $\mathbb{Z}^-$ is a ternary semiring. Note that $\mathbb{Z}^-$ does not form a semiring. In [3] T. K. Dutta and S. Kar introduced the notions of left/right/lateral ideals of ternary semirings and also characterized regular ternary semirings. In 2005, S. Kar [1] introduced the notions of quasi-ideals and bi-ideals in a ternary semiring. The notion of a generalized semi-ideal in a ring has been introduced and studied by T. K. Dutta in [2]. In this paper we introduce the notion of a generalized semi-ideal in a ternary semiring and study them. Also, we establish a relationship between generalized semi-ideals, ideals, bi-ideals, etc. in a ternary semiring to study some properties of a generalized semi-ideals in ternary semirings.

2. Preliminaries

For preliminaries we refer to [1] and [3].

Definition 2.1. An additive commutative semigroup S, together with a ternary multiplication denoted by $[\ ]$ is said to be a ternary semiring if

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i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,

ii) $[(a+b)cd] = [acd] + [bcd]$,

iii) $[a(b+c)d] = [abd] + [acd]$,

iv) $[ab(c+d)] = [abc] + [abd]$ for all $a, b, c, d, e \in S$.

Throughout, S will denote a ternary semiring unless otherwise stated.

Definition 2.2. If there exists an element $0 \in S$ such that $0 + x = x$ and $[0xy] = [xy0] = [x0y] = 0$ for all $x, y \in S$, then 0 is called the zero element of S. In this case we say that S is a ternary semiring with zero.

Definition 2.3. S is called a commutative ternary semiring if $[abc] = [bac] = [bca]$, for all $a, b, c \in S$.

Definition 2.4. An additive subsemigroup T of S is called a ternary subsemiring of S if $[t_1t_2t_3] \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.5. An element a in S is called regular if there exists an element $x \in S$ such that $[axa] = a$. S is called regular if all of its elements are regular.

Definition 2.6. S is said to be zero-divisor free (ZDF) if for $a, b, c \in S$, $[abc] = 0$ implies that $a = 0$ or $b = 0$ or $c = 0$.

Definition 2.7. S with $|S| \geq 2$ is called a ternary division semiring if for any non-zero element a of S , there exists a non-zero element $b \in S$ such that $[abx] = [bax] = [xab] = [xba] = x$, for all $x \in S$.

Definition 2.8. A left (right/lateral) ideal I of S is an additive subsemigroup of S such that $[s_1s_2i] \in I$ ($[is_1s_2] \in I/[s_1is_2] \in I$) for all $i \in I$, for all $s_1, s_2 \in S$. If I is a left, a right and a lateral ideal of S, then I is called an ideal of S.

Definition 2.9. An additive subsemigroup Q of S is called a quasi-ideal of S if $[QSS] \bigcap ([SQS] + [SSQSS]) \bigcap [SSQ] \subseteq Q$.

Definition 2.10. A ternary subsemiring B of S is called a bi-ideal of S if $[BSBSB] \subseteq B$.

3. Generalized semi-ideals in ternary semirings

Generalized semi-ideals in semirings are introduced and studied by T .K. Dutta in [1]. As a generalization, we define generalized semi-ideals in ternary semirings.

Definition 3.1. A non-empty subset A of S satisfying the condition $a + b \in A$, for all $a, b \in A$ is called

i) generalized left semi-ideal of S if $[[xxx]xa] \in A$ for all $a \in A$ for all $x \in S$,

ii) generalized right semi-ideal of S if $[axx]xx] \in A$ for all $a \in A$, for all $x \in S$,

iii) generalized lateral semi-ideal of S if $[xxa]xx] \in A$ for all $a \in A$, for all $x \in S$,

iv) generalized semi-ideal of S if it is a generalized left semi-ideal, a generalized right semi-ideal and a generalized lateral semi-ideal of S.

Example 3.2. Consider a ternary semiring $\mathbb{Z}$ of all integers. The subset A of $\mathbb{Z}$ containing all non-negative integers and the set B of all non-positive integers are generalized semi-ideals of $\mathbb{Z}$.

Remark 3.3. The concepts of generalized semi-ideal and ternary subsemiring are independent in S. This means that is every ternary subsemiring of S need not be a generalized semi-ideal of S and every generalized semi-ideal of S need not be a ternary subsemiring of S. For this, consider the following examples.

Example 3.4. Let $S = M_2\ (\mathbb{Z}_0^-)$ be the ternary semiring of the set of all 2×2 square matrices over $\mathbb{Z}_0^-$, the set of all non-positive integers.

Let $T = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} / a \in \mathbb{Z}_0^-\}$. T is a ternary subsemiring of S, but it is not a generalized semi-ideal of S.

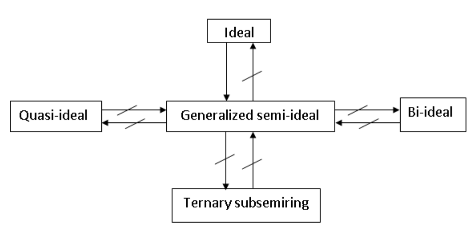
Example 3.5. Let $S = \{\ldots, -2i, -i, 0, i, 2i, \ldots\}$ be a ternary semiring with respect to addition and complex triple multiplication. Let $A = \{0, i, 2i, \ldots\}$. A is a generalized semi-ideal of S, but not a ternary subsemiring of S.

Every ideal of S is a generalized semi-ideal of S but the converse need not be true.

Example 3.6. Every quasi-ideal need not be a generalized semi-ideal and every generalized semi-ideal need not be a quasi-ideal of S. In Example 3.4), T is a quasi-ideal of S, but it is not a generalized semi-ideal of S. In Example 3.5, A is a generalized semi-ideal of S, but not a quasi-ideal of S.

Every quasi-ideal is a bi-ideal in S [2]. Hence, the bi-ideals and generalized semi-ideals in S are independent concepts.

The flow-chart of the relationship between the ideals, bi-ideals, quasi-ideals, ternary subsemiring and generalized semi-ideals in a ternary semiring is given below.



4. Properties of generalized semi-ideals

The intersection of an arbitrary collection of generalized semi-ideals of a ternary semiring is generalized semi-ideal of S. But, the union of two generalized semi-ideals of S may not be a generalized semi-ideal of S. This we establish in the following example.

Let $S = \{\dots, -2i, -i, 0, i, 2i, \dots\}$ be a ternary semiring with respect to addition and complex triple multiplication. Then $I = \{\dots, -4i, -2i, 0, 2i, \dots\}$ and $J = \{\dots, -10i, -5i, 0, 5i, 10i, \dots\}$ are two generalized semi-ideals of S, but $I \bigcup J$ is not a generalized semi-ideal of S.

Theorem 4.1. *Let A be a generalized semi-ideal of S and let T be a ternary subsemiring of S. If $A \bigcap T \neq \emptyset$, then $A \bigcap T$ is a generalized semi-ideal of T.*

Proof. Let $a, b \in A \bigcap T$. Then $a + b \in A \bigcap T$. For $x \in T$ and $a \in A \bigcap T$ we have

$$[[xxx]xa] \in A \bigcap T, [[axx]xx] \in A \bigcap T, [[xxa]xx] \in A \bigcap T.$$

Hence $A \bigcap T$ is generalized semi-ideal of S. □

Theorem 4.2. *If A and B are generalized semi-ideals of S, then $A + B = \{a + b / a \in A, b \in B\}$ is a generalized semi-ideal of S.*

Proof. Let $x, y \in A + B$. Hence $x = a + b, y = c + d$, for $a, c \in A$ and $b, d \in B$. Then $x + y = (a + b) + (c + d) = (a + c) + (b + d) \in A + B$. Let $t \in S$ and $x \in A + B$, hence $x = a + b$ for some $a \in A$ and $b \in B$. Therefore

$$[[ttt]tx] = [[ttt]t(a + b)] = [[ttt](ta)] + [[ttt]tb] \in A + B.$$

Similarly, we have

$$[[ttx]tt] = [[tt(a + b)]tt] = [([tta] + [ttb])tt] = [[tta]tt] + [[ttb]tt] \in A + B$$

and

$$[[xtt]tt] = [[(a + b)tt]tt] = [([att] + [btt])tt] = [[att]tt] + [[btt]tt] \in A + B.$$

Thus $A + B$ is a generalized semi-ideal of S. □

Theorem 4.3. *Let S be a ternary semiring with zero. Let A and B be two generalized semi-ideals of S containing zero. Then $A+B$ is the smallest generalized semi-ideal of S containing both A and B.*

Proof. From Theorem 4.2, $A + B$ is a generalized semi-ideal of S. Since $0 \in A$, $0 \in B$ we get $0 \in A + B$ and for $a \in A, a = a + 0 \in A + B$. Hence $A \subseteq A + B$. Similary, $B \subseteq A+B$. Let I be any other generalized semi-ideal containing both A and B. Let $x \in A + B$. Then $x = a + b$, for some $a \in A$ and $b \in B$. Hence, $x = a + b \in I$. Therefore, $A + B \subseteq I$. Thus, $A + B$ is the smallest generalized semi-ideal containing both A and B. □

If A, B, C are subsets of S, then by $[ABC]$ we mean the set of all finite sums of the form $\sum[a_ib_ic_i]$, where $a_i \in A, b_i \in B, c_i \in C$ ([2]).

Theorem 4.4. *Let A be a generalized left semi-ideal of S. Then $[ABC]$ is a generalized left semi-ideal for any non-empty subsets B and C of S.*

Proof. For $x, y \in [ABC]$, let $x = \sum_{i=1}^{n}[a_ib_ic_i]$ and $y = \sum_{j=1}^{m}[a_ib_ic_i]$. Obviously $x + y$ is a finite sum of the form $\sum[a_ib_ic_i]$. Hence $x + y \in [ABC]$. For $t \in S$, we have

$$\begin{aligned} [[ttt]tx] &= [[ttt]t\sum_{i=1}^{n}[a_ib_ic_i]] \\ &= \sum_{i=1}^{n}[[ttt]t[a_ib_ic_i]] \\ &= \sum_{i=1}^{n}[[[ttt]ta_i]b_ic_i] \in [ABC]. \end{aligned}$$

Since A is generalized left semi-ideal S, $[ABC]$ is a generalized left semi-ideal of S. □

Theorem 4.5. *Let A be a generalized left (right) semi-ideal and B be a bi-ideal of S. Then $[ABB]$ ($[BBA]$) is generalized left(right) semi-ideal as well as bi-ideal of S.*

Proof. Let $x, y, z \in [ABB]$. Hence $x = \sum_{i=1}^{n}[a_ib_ic_i]$, $y = \sum_{i=n+1}^{m}[a_ib_ic_i]$, $z = \sum_{i=m+1}^{p}[a_ib_ic_i]$ for all $a_i \in A$ and $b_i, c_i \in B$. Thus $x+y$ is a finite sum of the form $\sum[a_ib_ic_i]$. Hence, $x + y \in [ABB]$. Let $t \in S$ and $x = \sum_{i=1}^{n}[a_ib_ic_i] \in [ABB]$. Then

$$\begin{aligned} [[ttt]tx] &= [[ttt]t\sum_{i=1}^{n}[a_ib_ic_i]] \\ &= \sum_{i=1}^{n}[[[ttt]ta_i]b_ic_i] \in [ABB]. \end{aligned}$$

Hence $[ABB]$ is generalized left semi-ideal of S. Now

$$\begin{aligned} [[ABB][ABB][ABB]] &= [A[[B[BAB]B]AB]B] \\ &\subseteq [A[BSBSB]B] \\ &\subseteq [ABB]. \end{aligned}$$

(Since $[BAB] \subseteq S$ and B is a bi-ideal). This shows that $[ABB]$ is a ternary subsemiring of S.

Again,

$$\begin{aligned} [[ABB]S[ABB]S[ABB]] &= [A[B[BSA]B[BSA]B]B] \\ &\subseteq [A[BSBSB]B] \\ &\subseteq [ABB], \end{aligned}$$

since B is a bi-ideal. Hence $[ABB]$ is a bi-ideal of S. □

Theorem 4.6. *Let A and B be ternary subsemirings of S such that $A^3 = A$ and A be a left ideal of B and B be a generalized left semi-ideal of S. Then A is a generalized left semi-ideal of S.*

Proof. Let $a \in A$, therefore $a = [a_1a_2a_3]$, where $a_1, a_2, a_3 \in A$. Now, for any $x \in S$,

$$\begin{aligned} [xxx]xa] &= [[xxx]x[a_1a_2a_3]] \\ &= [[[xxx]xa_1]a_2a_3] \in [Ba_2a_3] \\ &\subseteq A \end{aligned}$$

(since A is a left ideal of B, $a_1 \in A \subset B$, B is a generalized left semi-ideal of S). Therefore, A is a generalized left semi-ideal of S. □

Theorem 4.7. *If G is a generalized left(right) semi-ideal of S and T_1, T_2 are two ternary subsemirings of S, then $[GT_1T_2]$ ($[T_1T_2G]$) is a generalized left(right) semi-ideal of S.*

Proof. For any $a, b \in [GT_1T_2], a = \sum_{i=1}^{n}[g_it_it_i^{'}]$ and $b = \sum_{i=n+1}^{m}[g_it_it_i^{'}]$, for $g_i \in G, t_i \in T_1, t_i^{'} \in T_2$. Therefore, $a+b$ is a finite sum of the form $\sum[g_it_it_i^{'}]$ will imply $a+b \in [GT_1T_2]$. Let $a = \sum_{i=1}^{n}[g_it_it_i^{'}] \in [GT_1T_2]$ and let $x \in S$. Then

$$\begin{aligned} [[xxx]xa] &= [[xxx]x\sum_{i=1}^{n}[g_it_it_i^{'}]] \\ &= \sum_{i=1}^{n}[[[xxx]xg_i]t_it_i^{'}] \in [GT_1T_2]. \end{aligned}$$

Hence $[GT_1T_2]$ is a generalized left semi-ideal of S. □

A necessary and sufficient condition for a commutative ternary semiring S without any divisors of zero to be a ternary division semiring is given in the following theorem.

Theorem 4.8. *A commutative ternary semiring S without any divisors of zero will be a ternary division semiring iff for any generalized semi-ideal A, $a \in S\backslash A$ (the complement of A in S) and $x(\neq 0) \in S$ implies $[[xxx]xa] \in S\backslash A$.*

Proof. Suppose a commutative ternary semiring S without any divisor of zero will be a ternary division semiring. Let A be a generalized semi-ideal of S. Select $a \in S\backslash A$ and $x(\neq 0) \in S$. Hence $\exists y(\neq 0) \in S$ such that $[xyz] = [yxz] = [zxy] = [zxy] = z$, for all $z \in S$. Therefore, $[xya] = [yxa] = [axy] = [ayx] = a$. This proves that $[[xxx]xa] \in S\backslash A$. Assume that $[[xxx]xa] = x^4a \in A$. Therefore, $a = [[yxy]^4ax^4] \in A$. (Since S is commutative, A is a generalized semi-ideal), which is a contradiction. Hence, $[[xxx]xa] \in S\backslash A$.

Conversely, suppose that for any generalized semi-ideal A, $a \in S\backslash A$ and $x \neq 0 \in S$ implies $[xxx]xa] \in S\backslash A$. To prove that S is a ternary division semiring, that is to prove that for $a(\neq 0) \in S$ $\exists b(\neq 0) \in S$ such that $[abS] = S$. If possible, let $[abS] \neq S$ and $y \in S\backslash[abS]$, then $[[aaa]ay] = [a^3ay] = [aa^3y] = [aby] \in [abS]$, where $b = a^3(\neq 0) \in S$, which is a contradiction because $[a^3ay] \in S\backslash[abS]$. Hence $[abS] = S$. Therefore S is a ternary division semiring. □

Suppose A is a generalized semi-ideal of a commutative ternary semiring S. Let $\beta(A)$ denote the set of all those elements a for which there exists a nonzero element $x \in S$ such that $[[xxx]xa] \in A$. It is then clear that $A \subseteq \beta(A)$. Further we have

Theorem 4.9. *Let S be a commutative ternary semiring without any divisors of zero. If A is a generalized semi-ideal of S, then $\beta(A)$ is also a generalized semi-ideal of S.*

Proof. Let $a, b \in \beta(A)$. So there exist non-zero elements $x, y \in S$ such that $p = [[xxx]xa] \in A$,$q = [[yyy]yb] \in A$.

Now

$$\begin{aligned}\varepsilon &= [[xxx]x[yyy]y(a+b)]\\ &= [[xxx]x[yyy]ya] + [[xxx]x[yyy]yb]\\ &= [[yyy]y[[xxx]xa]] + [[xxx]x[[yyy]yb]]\\ &= [[yyy]yp] + [[xxx]xq] \in A.\end{aligned}$$

For $z(\neq 0) \in S$, $[[zzz]z\varepsilon] \in A$ (since A is a generalized semi-ideal of S). Therefore $[[[xyz][xyz][xyz]][xyz](a+b)] \in A$. Hence $(a+b) \in \beta(A)$.

For $a \in \beta(A)$, $[[xxx]xa] \in A$. Let $z \in S$, hence

$$[[xxx]x[[zzz]za]] = [[zzz]z[[xxx]xa]] \in A.$$

Therefore, $[[zzz]za] \in \beta(A)$ for all $z \in S$. Hence $\beta(A)$ is a generalized semi-ideal of S. □

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Received by the editors May 18, 2009