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ON GENERALIZED n-INNER PRODUCT SPACES

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Abstract. The primary purpose of this paper is to derive a generalized $(n-k)$ inner product with $n \geq 2$, from the generalized n-inner product, which is a generalization of the definition of Misiak [3] of the n-inner product for each $k \in \{1, 2 \ldots, n-1\}$ and also provide results related to the n-normed product induced by generalized n-inner product.

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1. Introduction

Misiak [3] has introduced an n-norm and n-inner product by the following definitions.

Definition 1.1. Let $n \in N$ (natural numbers) and X be a real linear space of dimension greater than or equal to n. A real valued function $\|\bullet, \ldots, \bullet\|$ on $X \times \cdots \times X = X^n$ satisfying the following four properties:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if any only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,

(iii) $\|x_1, x_2, \ldots, ax_n\| = |a| \, \|x_1, x_2, \ldots, x_n\|$, for any $a \in R$ (real),

(iv) $\|x_1, x_2, \ldots, x_{n-1}, y+z\| = \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$ is called an n-norm on X and the pair $(X, \|\bullet, \ldots, \bullet\|)$ is called n-normed linear space.

Definition 1.2. Assume that n is a positive integer and X is a real vector space such that $\dim X \geq n$ and $(\bullet, \bullet | \bullet, \underset{n-1}{\ldots}, \bullet)$ is a real function defined on X^{n+1} such that:

(i) $(x_1, x_1 | x_2, \ldots, x_n) \geq 0$, for any $x_1, x_2, \ldots, x_n \in X$ and $(x_1, x_1 | x_2, \ldots, x_n) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors;

(ii) $(a, b | x_1, \ldots, x_{n-1}) = (\varphi(a), \varphi(b) | \pi(x_1), \ldots, \pi(x_{n-1}))$, for any a, b, x_1, $x_2, \ldots, x_{n-1} \in X$ and for any bijections $\pi : \{x_1, x_2, \ldots, x_{n-1}\} \to \{x_1, x_2, \ldots, x_{n-1}\}$ and $\varphi : \{a, b\} \to \{a, b\}$;

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(iii) If $n > 1$, then $(x_1, x_1|x_2, \dots, x_n) = (x_2, x_2|x_1, x_3, \dots, x_n)$, for any $x_1, x_2, \dots, x_n \in X$;

(iv) $(\alpha a, b|x_1, \dots, x_{n-1}) = \alpha(a, b|x_1, \dots, x_{n-1})$, for any $a, b, x_1, \dots, x_{n-1} \in X$ and any scalar $\alpha \in R$;

(v) $(a + a_1, b|x_1, \dots, x_{n-1}) = (a, b|x_1, \dots, x_{n-1}) + (a_1, b|x_1, \dots, x_{n-1})$, for any $a, b, a_1, x_1, \dots, x_{n-1} \in X$.

Then $(\bullet, \bullet|\underbrace{\bullet, \dots, \bullet}_{n-1})$ is called n-inner product and $(X, (\bullet, \bullet|\underbrace{\bullet, \dots, \bullet}_{n-1})$ n-prehilbert space. If $n = 1$, then Definition 1.2 reduces to the ordinary inner product. This n-inner product induces an n-norm [3] by

$$\|x_1, \dots, x_n\| = \sqrt{(x_1, x_1|x_2, \dots, x_n)}.$$

Trencevski and Malceski [4] gave the definition of generalized n-inner product and the Cauchy-Schwarz inequality in this space as

Definition 1.3. Assume that n is a positive integer, X is a real vector space such that $\dim X \geq n$ and $\langle \bullet, \dots, \bullet|\bullet, \dots, \bullet \rangle$ is a real function on X^{2n} such that

(I_1) $\langle a_1, \dots, a_n|a_1, \dots, a_n \rangle > 0$ if $a_1, \dots, a_n$ are linearly independent vectors,

(I_2) $\langle a_1, \dots, a_n|b_1, \dots, b_n \rangle = \langle b_1, \dots, b_n|a_1, \dots, a_n \rangle$ for any $a_1, \dots, a_n, b_1, \dots, b_n \in X$

(I_3) $\langle \lambda a_1, \dots, a_n|b_1, \dots, b_n \rangle = \lambda \langle a_1, \dots, a_n|b_1, \dots, b_n \rangle$ for any scalar $\lambda \in R$ and any $a_1, \dots, a_n, b_1, \dots, b_n \in X$,

(I_4) $\langle a_1, \dots, a_n|b_1, \dots, b_n \rangle = -\langle a_{\sigma(1)}, \dots, a_{\sigma(n)}|b_1, \dots, b_n \rangle$ for any odd permutation σ in the set $\{1, \dots, n\}$ and any $a_1, \dots, a_n, b_1, \dots, b_n \in X$,

(I_5) $\langle a_1 + c, a_2, \dots, a_n|b_1, \dots, b_n \rangle = \langle a_1, a_2, \dots, a_n|b_1, \dots, b_n \rangle + \langle c, a_2, \dots, a_n|b_1, \dots, b_n \rangle$ for any $a_1, \dots, a_n, b_1, \dots, b_n, c \in X$,

(I_6) If $\langle a_1, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n|b_1, \dots, b_n \rangle = 0$ for each $i \in \{1, 2, \dots, n\}$, then $\langle a_1, \dots, a_n|b_1, \dots, b_n \rangle = 0$ for arbitrary vectors $a_1, \dots, a_n$.

Then the function $\langle \bullet, \dots, \bullet|\bullet, \dots, \bullet \rangle$ is called generalized n-inner product and the pair $(X, \langle \bullet, \dots, \bullet|\bullet, \dots, \bullet \rangle)$ is called generalized n-prehilbert space.

The generalized n-inner product on X induces an n-norm [3] by

$$\|x_1, \dots, x_n\| = \sqrt{\langle x_1, \dots, x_n|x_1, \dots, x_n \rangle}\,.$$

And Cauchy-Schwarz inequality in generalized n-inner product on X is given as

$$\langle a_1, \dots, a_n|b_1, \dots, b_n \rangle^2 \leq \langle a_1, \dots, a_n|a_1, \dots, a_n \rangle \langle b_1, \dots, b_n|b_1, \dots, b_n \rangle$$

In [1] we obtain the following identities:

Polarization identity in generalized n-inner product space as

$$4\langle x, x_2, \dots, x_n | y, x_2, \dots, x_n \rangle \\ = \|x + y, x_2, \dots, x_n\|^2 - \|x - y, x_2, \dots, x_n\|^2$$

And parallelogram law in generalized n-inner product space as

$$\|x + y, x_2, \dots, x_n\|^2 + \|x - y, x_2, \dots, x_n\|^2 \\ = 2\|x, x_2, \dots, x_n\|^2 + 2\|y, x_2, \dots, x_n\|^2$$

The classical known example [4] of generalized n-inner product space is

Example 1.4. Let X be a space with inner product $\langle \bullet | \bullet \rangle$. Then

$$\langle a_1, \dots, a_n | b_1, \dots, b_n \rangle = \begin{vmatrix} \langle a_1|b_1\rangle & \langle a_1|b_2\rangle & \cdots & \langle a_1|b_n\rangle \\ \langle a_2|b_1\rangle & \langle a_2|b_2\rangle & \cdots & \langle a_2|b_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_n|b_1\rangle & \langle a_n|b_2\rangle & \cdots & \langle a_n|b_n\rangle \end{vmatrix}$$

defines a generalized n-inner product on X.

Misiak [3] generalized the definition of 2-inner product given by Gahler [4] in n-inner product. Recently, Trencevski and Malceski [4] introduced the concept of generalized n-inner product as the generalization of n-inner product and obtained some related results. In [1], we discussed the weak and strong convergence, and proved some identities in this space. In this paper, we present a simple method to derive a generalized $(n-k)$ inner product with $n \geq 2$, from the generalized n-inner product for each $k \in \{1, 2 \dots, n-1\}$ and also provide results related to n-norm induced by generalized n-inner product.

The notion of orthogonality in a generalized n-inner product space can be developed by using a derived generalized inner product or inner product, just as in [1, 2, 4].

2. Main results

To avoid confusion, we shall sometimes denote a generalized n-inner product by $\langle \cdot, \cdot, \dots, \cdot | \cdot, \cdot, \dots, \cdot \rangle_n$ and an n-norm by $\|\cdot, \cdot, \dots, \cdot\|_n$.

Theorem 2.1. *Let $(X, \langle \cdot, \cdot, \dots, \cdot | \cdot, \cdot, \dots, \cdot \rangle_n)$ be generalized n-inner product space with finite dimension $d \geq n \geq 2$. Take a linearly independent set $\{a_1, a_2, \dots, a_d\}$ and define the following function $\langle \cdot, \dots, \cdot | \cdot, \cdot, \dots, \cdot \rangle_{n-1}$ on $X^{2(n-1)}$ by*

$$\langle x_1, x_2, \dots, x_{n-1} | y_1, y_2, \dots, y_{n-1} \rangle \\ = \sum_{i=1}^{d} \langle x_1, x_2, \dots, x_{n-1}, a_i | y_1, y_2, \dots, y_{n-1}, a_i \rangle \tag{2.1}$$

such that this function satisfies (I_6), then the function $\langle \cdot, \cdot, \dots, \cdot | \cdot, \cdot, \dots, \cdot \rangle_{n-1}$ is a generalized $(n-1)$-inner product on X.

Proof. We will verify that $\langle \cdot, \cdot, \ldots, \cdot | \cdot, \cdot, \ldots, \cdot \rangle_{n-1}$ satisfies the following six properties of a generalized $(n-1)$-inner product.

(i) To verify this property, suppose that $x_1, x_2, \ldots, x_{n-1}$ are linearly dependent. Then $\langle x_1, x_2, \ldots, x_{n-1}, a_i | x_1, x_2, \ldots, x_{n-1}, a_i \rangle = 0$, for every $i \in \{1, 2, \ldots, d\}$ and hence $\langle x_1, x_2, \ldots, x_{n-1} | x_1, x_2, \ldots, x_{n-1} \rangle = 0$.

Conversely, suppose that

$$\langle x_1, x_2, \ldots, x_{n-1} | x_1, x_2, \ldots, x_{n-1} \rangle = 0,$$

then

$$\sum_{i=1}^{n} \langle x_1, x_2, \ldots, x_{n-1}, a_i | x_1, x_2, \ldots, x_{n-1}, a_i \rangle = 0$$

so $\langle x_1, x_2, \ldots, x_{n-1}, a_i | x_1, x_2, \ldots, x_{n-1}, a_i \rangle = 0$ for each $i \in \{1, 2, \ldots, d\}$. Hence by (I_1) $x_1, x_2, \ldots, x_{n-1}$, a_i are linearly dependent for each $i \in \{1, 2, \ldots, d\}$.

By elementary linear algebra, this can only happen if $x_1, x_2, \ldots, x_{n-1}$ are linearly dependent.

(ii) By using (I_2), we have

$$\begin{aligned} &\langle x_1, x_2, \ldots, x_{n-1} | y_1, y_2, \ldots, y_{n-1} \rangle_{n-1} \\ &= \sum_{i=1}^{d} \langle x_1, x_2, \ldots, x_{n-1}, a_i | y_1, y_2, \ldots, y_{n-1}, a_i \rangle \\ &= \sum_{i=1}^{d} \langle y_1, y_2, \ldots, y_{n-1}, a_i | x_1, x_2, \ldots, x_{n-1}, a_i \rangle \\ &= \langle y_1, y_2, \ldots, y_{n-1} | x_1, x_2, \ldots, x_{n-1} \rangle_{n-1} \end{aligned}$$

(iii)

$$\begin{aligned} &\langle \lambda x_1, x_2, \ldots, x_{n-1} | y_1, y_2, \ldots, y_{n-1} \rangle_{n-1} \\ &= \sum_{i=1}^{d} \langle \lambda x_1, x_2, \ldots, x_{n-1}, a_i | y_1, y_2, \ldots, y_{n-1}, a_i \rangle \\ &= \lambda \sum_{i=1}^{d} \langle x_1, x_2, \ldots, x_{n-1}, a_i | y_1, y_2, \ldots, y_{n-1}, a_i \rangle \\ &= \lambda \langle x_1, x_2, \ldots, x_{n-1} | y_1, y_2, \ldots, y_{n-1} \rangle_{n-1} \end{aligned}$$

For any scalar $\lambda \in R$, using (I_3).

(iv)
$$\begin{aligned}
&\langle x_1, x_2, \ldots, x_{n-1}|y_1, y_2, \ldots, y_{n-1}\rangle_{n-1}\\
&= \sum_{i=1}^{d}\langle x_1, x_2, \ldots, x_{n-1}, a_i|y_1, y_2, \ldots, y_{n-1}, a_i\rangle\\
&= -\sum_{i=1}^{d}\langle x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n-1)}, a_i|y_1, y_2, \ldots, y_{n-1}, a_i\rangle\\
&= -\langle x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n-1)}|y_1, y_2, \ldots, y_{n-1}\rangle
\end{aligned}$$

for any odd permutation σ in the set $\{1, \ldots, n\}$ and using (I_4).

(v)
$$\begin{aligned}
&\langle x_1 + z, x_2, \ldots, x_{n-1}|y_1, y_2, \ldots, y_{n-1}\rangle_{n-1}\\
&= \sum_{i=1}^{d}\langle x_1 + z, x_2, \ldots, x_{n-1}, a_i|y_1, y_2, \ldots, y_{n-1}, a_i\rangle\\
&= \sum_{i=1}^{d}\langle x_1, x_2, \ldots, x_{n-1}, a_i|y_1, y_2, \ldots, y_{n-1}, a_i\rangle\\
&\quad + \sum_{i=1}^{d}\langle z, x_2, \ldots, x_{n-1}, a_i|y_1, y_2, \ldots, y_{n-1}, a_i\rangle\\
&= \langle x_1, x_2, \ldots, x_{n-1}|y_1, y_2, \ldots, y_{n-1}\rangle_{n-1}\\
&\quad + \langle z, x_2, \ldots, x_{n-1}|y_1, y_2, \ldots, y_{n-1}\rangle_{n-1}
\end{aligned}$$

(vi) If $\langle x_1, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n-1}|y_1, y_2, \ldots, y_{n-1}\rangle_{n-1} = 0$ for each $j \in \{1, 2, \ldots, n-1\}$

$$\Rightarrow \sum_{i=1}^{d}\langle x_1, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n-1}, a_i|y_1, y_2, \ldots, y_{n-1}, a_i\rangle = 0,$$

hence by the orthonormal basis $\{a_1, a_2, \ldots, a_d\}$ and assumption of the theorem, we have the required condition that $\langle x_1, x_2, \ldots, x_{n-1}|y_1, y_2, \ldots, y_{n-1}\rangle_{n-1} = 0$ for arbitrary vectors $x_2, \ldots, x_{n-1}$.

So, $\langle\cdot, \cdot, \ldots, \cdot|\cdot, \cdot, \ldots, \cdot\rangle_{n-1}$ is a generalized $(n-1)$-inner product on X. □

Corollary 2.2. Every generalized n-inner product space is generalized $(n-k)$-inner product space for all $k = 1, 2, \ldots, n-1$, by induction with generalized $(n-k)$-inner product

$$\begin{aligned}
&\langle x_1, x_2, \ldots, x_{n-k}|y_1, y_2, \ldots, y_n\rangle\\
&= \sum_{i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\}} \langle x_1, x_2, \ldots, x_{n-k}, a_{i_1}, a_{i_2}, \ldots, a_{i_k}|y_1, y_2, \ldots, y_{n-k}, a_{i_1}, a_{i_2}, \ldots, a_{i_k}\rangle_n
\end{aligned}$$

such that this function satisfies (I_6), this condition is necessary for $k = 1, 2, \ldots, n-2$, but for $k = n-1$ it is trivially satisfied. In particular, every generalized n-inner product space induces an inner product space. i.e.

$$\langle x, y\rangle = \sum_{i_1,i_2,\ldots,i_{n-1}\in\{1,2,\ldots,d\}} \langle x, a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}} | y, a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}\rangle_n$$

Corollary 2.3. Let $\|\cdot,\cdot,\ldots,\cdot\|_n$ be the induced n-norm from a generalized n-inner product on X. Then the following function

$$\|x_1, x_2, \ldots, x_{n-1}\|_{n-1} = \left(\sum_{i=1}^{d} \|x_1, x_2, \ldots, x_{n-1}, a_i\|^2\right)^{\frac{1}{2}}$$

is an $(n-1)$-norm that corresponds to $\langle\cdot,\cdot,\ldots,\cdot|\cdot,\cdot,\ldots,\cdot\rangle_{n-1}$ on X, and by induction we have

$$\begin{aligned} &\|x_1, x_2, \ldots, x_{n-k}\|_{n-k} \\ &\quad = \left(\sum_{i_1,i_2,\ldots,i_k\in|1,2,\ldots,d|} \|x_1, x_2, \ldots, x_{n-k}, a_{i_1}, a_{i_2}, \ldots, a_{i_k}\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

In particular, $\|x\| = \left(\sum_{i_1,i_2,\ldots,i_{k-1}\in|1,2,\ldots,d|} \|x, a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}\|^2\right)^{\frac{1}{2}}$ defines a norm that corresponds to the derived generalized inner product or inner product $\langle\cdot,\cdot\rangle$ on X.

2.4. Related results on n-normed space induced from generalized n-inner product space.

Suppose now that $(X, \|\cdot,\cdot,\ldots,\cdot\|_n)$ is an n-normed space and $\{a_1, a_2, \ldots, a_d\}$ is a linearly independent orthonormal set in X. Then we can show that

$$\|x_1, x_2, \ldots, x_{n-1}\|_{n-1} = \left(\sum_{i=1}^{d} \|x_1, x_2, \ldots, x_{n-1}, a_i\|^2\right)^{\frac{1}{2}}$$

defines an $(n-1)$-norm on X. In particular, the triangle inequality can be verified as:

$$\begin{aligned} &\|x+y, x_2, \ldots, x_{n-1}\|_{n-1}^2 \\ &\quad = \langle x+y, x_2, \ldots, x_{n-1} | x+y, x_2, \ldots, x_{n-1}\rangle \\ &\quad = \sum_{i=1}^{d} \langle x+y, x_2, \ldots, x_{n-1}, a_i | x+y, x_2, \ldots, x_{n-1}, a_i\rangle \\ &\quad = \sum_{i=1}^{d} \| x+y, x_2, \ldots, x_{n-1}, a_i\|^2 \\ &\quad \le \sum_{i=1}^{d} (\|x, x_2, \ldots, x_{n-1}, a_i\| + \|y, x_2, \ldots, x_{n-1}, a_i\|)^2 \end{aligned}$$

Thus

$$\begin{aligned}
&\|x+y,x_2,\dots,x_{n-1}\|_{n-1}\\
&\leq \left(\sum_{i=1}^{d}(\|x,x_2,\dots,x_{n-1},a_i\|+\|y,x_2,\dots,x_{n-1},a_i\|)^2\right)^{\frac{1}{2}}\\
&\leq \left(\sum_{i=1}^{d}\|x,x_2,\dots,x_{n-1},a_i\|^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{d}\|y,x_2,\dots,x_{n-1},a_i\|^2\right)^{\frac{1}{2}}\\
&= \|x,x_2,\dots,x_{n-1}\|_{n-1}+\|y,x_2,\dots,x_{n-1}\|_{n-1}.
\end{aligned}$$

This inequality shows the triangle inequality in $(n-1)$-norm.

Theorem 2.5. *If the n-norm induced by generalized n-inner product satisfies the parallelogram law*

$$\begin{aligned}
&\|x+y,x_2,\dots,x_n\|^2+\|x-y,x_2,\dots,x_n\|^2\\
&\quad=2\|x,x_2,\dots,x_n\|^2+2\|y,x_2,\dots,x_n\|^2,
\end{aligned}$$

then the $(n-1)$-norm induced by generalized n-inner product given by (4) satisfies

$$\begin{aligned}
&\|x+y,x_2,\dots,x_{n-1}\|^2+\|x-y,x_2,\dots,x_{n-1}\|^2\\
&\quad=2\|x,x_2,\dots,x_{n-1}\|^2+2\|y,x_2,\dots,x_{n-1}\|^2
\end{aligned}$$

In particular, the derived norm satisfies

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.$$

Proof. Polarization Identity in a generalized n-inner product space is

$$\begin{aligned}
&\langle x,x_2,\dots,x_n|y,x_2,\dots,x_n\rangle\\
&\quad=\frac{1}{4}(\|x+y,x_2,\dots,x_n\|^2-\|x-y,x_2,\dots,x_n\|^2)
\end{aligned}$$

and a generalized $(n-1)$-inner product is derived from it with respect to $\{a_1,a_2,\dots,a_d\}$. One will then realize that the derived $(n-1)$-norm is the induced $(n-1)$-norm from the derived generalized $(n-1)$-inner product, and hence the parallelogram law follows. □

3. Examples

Example 3.1. Let $X=R^n$ be equipped with the standard generalized n-inner product space

$$\langle x_1,x_2,\dots,x_n|y_1,y_2,\dots,y_n\rangle=\begin{vmatrix}\langle x_1,y_1\rangle & \langle x_1,y_2\rangle & \dots & \langle x_1,y_n\rangle\\ \langle x_2,y_1\rangle & \langle x_2,y_2\rangle & \dots & \langle x_2,y_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle x_n,y_1\rangle & \langle x_n,y_2\rangle & \dots & \langle x_n,y_n\rangle\end{vmatrix}$$

where $\langle x, y\rangle$ is the usual inner product on R^n. Then the derived generalized $(n-k)$ inner product with respect to an orthonormal basis $\{b_1, b_2, \ldots, b_n\}$ coincides with the standard generalized $(n-k)$-inner product on R^n, that is,

$$\langle x_1, x_2, \ldots, x_{n-k} | y_1, y_2, \ldots, y_{n-k}\rangle = \begin{vmatrix} \langle x_1, y_1\rangle & \langle x_1, y_2\rangle & \ldots & \langle x_1, y_{n-k}\rangle \\ \langle x_2, y_1\rangle & \langle x_2, y_2\rangle & \ldots & \langle x_2, y_{n-k}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_{n-k}, y_1\rangle & \langle x_{n-k}, y_2\rangle & \ldots & \langle x_{n-k}, y_{n-k}\rangle \end{vmatrix}$$

In particular, the derived generalized inner product (inner product) $\langle x, y\rangle$ with respect to $\{b_1, b_2, \ldots, b_n\}$, which is given by

$$\begin{aligned} \langle x, y\rangle &= \langle x, b_2, b_3, \ldots, b_n | y, b_2, b_3, \ldots, b_n\rangle \\ &\quad + \langle x, b_1, b_3, \ldots, b_n | y, b_1, b_3, \ldots, b_n\rangle + \ldots \\ &\quad + \langle x, b_1, b_2, \ldots, b_{n-1} | y, b_1, b_2, \ldots, b_{n-1}\rangle \end{aligned}$$

is the usual inner product.

Example 3.2. Let $X = R^d$ be equipped with the standard n-inner product as in (5), with $\langle x, y\rangle$ being the usual inner product on R^d. Then one may particularly observe that the derived inner product with respect to an orthonormal basis $\{b_1, b_2, \ldots, b_d\}$ is given by

$$\begin{aligned} \langle x, y\rangle &= \sum_{\{i_1, i_2, \ldots, i_n\} \subseteq \{1,2,\ldots,d\}} \langle x, b_{i_2}, b_{i_3}, \ldots, b_{i_n} | y, b_{i_2}, b_{i_3}, \ldots, b_{i_n}\rangle \\ &= \binom{d-1}{n-1} \langle x, y\rangle, \end{aligned}$$

where $\binom{d-1}{n-1} = \frac{(d-1)!}{(d-n)!(n-1)!}$.

This derived inner product is better than the previous one in the sense that it is only a multiple of the usual inner product. This example may also be extended to any finite d-dimensional inner product space X.

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