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ON THE REAL PART OF A CLASS OF ANALYTIC FUNCTIONS

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Abstract. Let $\mathcal{T}(\beta, b)$, $\beta(\beta \geq 0)$ and $b \in \mathbb{C}$ denote the class of analytic functions $f(z)$ in the open unit disk which satisfy the condition $\text{Re}\{f'(z) + \beta z f''(z)\} > 1 - |b|$. Inclusion relations of functions in the class $\mathcal{T}(\beta, b)$ are given. Lower bounds are also obtained for the n-th partial sums $F_n(z)$ of the Libera integral operator $F(z)$ and the n-th partial sums of $f(z)$. Furthermore, some convolution properties of functions in $\mathcal{T}(\beta, b)$ are shown.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{T}(\beta, b)$ denote the class of functions $f(z) \in \mathcal{A}$ which satisfy the condition

$$\text{Re}\{f'(z) + \beta z f''(z)\} > 1 - |b| \tag{2}$$

for some $\beta(\beta \geq 0)$ and $b \in \mathbb{C}$, and for all $z \in \mathcal{U}$. The class $\mathcal{T}(\beta, b)$ for the function f of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geq 0) \tag{3}$$

was introduced and studied by Altintas and Ertekin [2]. For $\beta = 0$ and $b = 1-\alpha$, $0 \leq \alpha < 1$, the class $\mathcal{T}(0, 1-\alpha) = \mathcal{R}(\alpha)$, where the functions in $\mathcal{R}(\alpha)$ are called functions of bounded turning (see [5]).

In order to derive our main results, we have to recall here the following lemmas.

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Lemma 1.1 ([6]). *Let M be the positive root of the equation*

$$9t^7 + 55t^6 - 14t^5 - 948t^4 - 3247t^3 - 5013t^2 - 3780t - 1134 = 0.$$

If $-1 < t \leq M \approx 4.5678018$, then

$$Re\sum_{k=2}^{n} \frac{z^{k-1}}{k(k+t-1)} > -\frac{1}{1+t}, \quad n = 2, 3, \ldots .$$

Lemma 1.2 ([1]). *Let M be defined as in Lemma 1.1. If $-1 < t \leq M \approx 4.5678018$, then*

$$Re\sum_{k=2}^{n} \frac{z^{k-1}}{k+t-1} > -\frac{1}{1+t}, \quad n = 2, 3, \ldots .$$

A sequence $a_0, a_1, \ldots, a_n, \ldots$ of nonnegative numbers is called a convex null sequence if $a_n \to 0$ as $n \to \infty$ and

$$a_0 - a_1 \geq a_1 - a_2 \geq \ldots \geq a_n - a_{(n+1)} \geq \ldots \geq 0.$$

Lemma 1.3 ([4]). *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function $p(z) = c_0/2 + \sum\limits_{k=1}^{\infty} c_k z$, $z \in \mathcal{U}$, is analytic and $Rep(z) > 0$ in $\mathcal{U}$.*

Lemma 1.4. *Let $P(z)$ be analytic in $\mathcal{U}$, $P(0) = 1$, and $ReP(z) > 1/2$ in $\mathcal{U}$, then for any function Q, analytic in $\mathcal{U}$, the function $P * Q$ takes values in the convex hull of the image of $\mathcal{U}$ under Q.*

The above Lemma 1.4 can be derived from the Hergoltz representation for $P(z)$ in $\mathcal{U}$.(see ([5]).

The operator " $*$ " stands for the Hadamard product or convolution of two power series $f(z) = \sum\limits_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum\limits_{k=1}^{\infty} b_k z^k$ is defined as the power series $(f * g)(z) = \sum\limits_{k=1}^{\infty} a_k b_k z^k$.

2. Inclusion relations

Now we prove the following theorem.

Theorem 2.1. *Let $f(z) \in \mathcal{T}(\beta, b)$ and $b \neq 0$, then*

$$\text{Re}(f'(z)) > 1 - |b| . \tag{4}$$

that is,

$$\mathcal{T}(\beta, b) \subset \mathcal{T}(0, b)$$

Proof. For $c_0 = 1$ and

$$c_k = \frac{1}{1+\beta k}, \qquad k \geq 1,$$

we see that $\{c_k\}_{k=0}^{\infty}$ is a convex null sequence. Therefore, by Lemma 1.3, we have

$$\mathrm{Re}\left(1 + 2\,|b| \sum_{k=2}^{\infty} \frac{z^{k-1}}{1-\beta+\beta k}\right) > 1 - |b| \qquad (z \in \mathcal{U}).$$

Let $f(z) \in \mathcal{T}(\beta, b)$ be of the form (1). Then from (2), we have

$$\mathrm{Re}\left(1 + \sum_{k=2}^{\infty} k(1-\beta+\beta k) a_k z^{k-1}\right) > 1 - |b| \qquad (z \in \mathcal{U}),$$

or

$$\mathrm{Re}\left(1 + \frac{1}{2\,|b|} \sum_{k=2}^{\infty} k(1-\beta+\beta k) a_k z^{k-1}\right) > \frac{1}{2} \qquad (z \in \mathcal{U}).$$

Now

$$\begin{aligned} f'(z) &= 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \\ &= \left(1 + \frac{1}{2\,|b|} \sum_{k=2}^{\infty} k(1-\beta+\beta k) a_k z^{k-1}\right) \\ &\quad * \left(1 + 2\,|b| \sum_{k=2}^{\infty} \frac{z^{k-1}}{1-\beta+\beta k}\right) \\ &= P(z) * Q(z). \end{aligned}$$

Now on the application of Lemma 1.4 to $f'(z)$, we get the result. □

Letting $\beta = 1$ and $|b| = 1$, $b \in \mathbb{C}$ in Theorem 2.1, we have the following result obtained by Chichra [3]

Corollary 2.2. *If* $Re\{f'(z) + zf''(z)\} > 0$ *then* $Re(f'(z)) > 0$, $z \in \mathcal{U}$, *and hence* f *is univalent in* $\mathcal{U}$*.*

Letting $b = 1 - \alpha$, $0 \leq \alpha < 1$ in Theorem 2.1, we have

Corollary 2.3. *If* $Re\{f'(z) + \beta z f''(z)\} > \alpha$ *then* $f \in \mathcal{R}(\alpha)$.

We also have a better result than Theorem 2.1.

Theorem 2.4. *Let* $f(z) \in \mathcal{T}(\beta, b)$*, then*

(5) $$\mathrm{Re}(f'(z)) > 1 - \frac{(3\beta+1)\,|b|}{(1+\beta)(1+2\beta)} \geq 1 - |b|\,,$$

that is,

$$\mathcal{T}(\beta, b) \subset \mathcal{T}(0, \delta)$$

where

$$\delta = \frac{(3\beta+1)\,|b|}{(1+\beta)(1+2\beta)}$$

Proof. For $\beta \geq 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{z^k}{1-\beta+\beta k},$$

Zhonghu and Owa [12] proved that

$$\mathrm{Re}\frac{g(z)}{z} > \frac{4\beta^2+3\beta+1}{2(1+\beta)(1+2\beta)}.$$

Hence

$$\mathrm{Re}\left(1 + 2\,|b| \sum_{k=2}^{\infty} \frac{z^{k-1}}{1-\beta+\beta k}\right) > 1 - \frac{(3\beta+1)\,|b|}{(1+\beta)(1+2\beta)}.$$

The application of Lemma 1.4 to $f'(z)$ in Theorem 2.4 completes the proof. □

Letting $b = 1-\alpha$ in Theorem 2.4, we have the following result obtained by Al-Oboudi [8].

Corollary 2.5. *Let* $f \in \mathcal{A}$ *and* $0 \leq \alpha < 1$. *If*

(6) $$\mathrm{Re}\left\{f'(z) + \beta z f''(z)\right\} > \alpha, \qquad (z \in \mathcal{U})$$

then

(7) $$\mathrm{Re}(f'(z)) > \frac{2\beta^2 + (1+3\beta)\alpha}{(1+\beta)(1+2\beta)}.$$

Letting $\beta = 1$ and $b = 1-\alpha$ in Theorem 2.4, we have

Corollary 2.6. *Let* $f \in \mathcal{A}$ *and* $0 \leq \alpha < 1$. *If*

(8) $$Re\left\{f'(z) + z f''(z)\right\} > \alpha, \qquad (z \in \mathcal{U})$$

then

(9) $$Re(f'(z)) > \frac{1+2\alpha}{3}.$$

Remark 2.7. It is shown by Saitoh [10] that for $\beta > 0$ and $0 \leq \alpha < 1$, $\text{Re}\{f'(z) + \beta z f''(z)\} > \alpha$ implies $\text{Re}(f'(z)) > (2\alpha + \beta)/(2 + \beta)$, so if we put $\beta = 1$, we have Corollary 2.6.

Letting $\alpha = 0$ in Corollary 2.6, we have

Corollary 2.8. *Let $f \in \mathcal{A}$. If*

$$\text{Re}\{f'(z) + zf''(z)\} > 0, \qquad (z \in \mathcal{U}) \tag{10}$$

then

$$\text{Re}(f'(z)) > \frac{1}{3}. \tag{11}$$

Remark 2.9. The result in Corollary 2.8 is an improvement of the result of Singh and Singh [11], where they show that $\text{Re}\{f'(z) + zf''(z)\} > 0$ implies $\text{Re}(f'(z)) > 2\log 2 - 1 \approx -0.39$.

3. Partial sum

For f of the form (1), the Libera integral operator F is given by

$$F(z) = \frac{2}{z}\int f(\zeta)d\zeta = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k$$

then the n-th partial sums $F_n(z)$ of the Libera integral operator $F(z)$ are given by

$$F_n(z) = z + \sum_{k=2}^{n} \frac{2}{k+1} a_k z^k. \tag{12}$$

Furthermore, let $f_n(z)$ be the n-th partial sums of $f(z)$ defined by

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k. \tag{13}$$

In this section, we determine lower bounds for $\text{Re}\{F_n(z)/z\}$ and $\text{Re}F_n'(z)$ when $F(z) \in \mathcal{T}(\beta, b)$ and for $\text{Re}\{f_n(z)/z\}$ and $\text{Re}f_n'(z)$ when $f(z) \in \mathcal{T}(\beta, b)$.

Theorem 3.1. *Let $0 < 1/\beta \leq M$, where M is defined as in Lemma 1.1. If $F(z) \in \mathcal{T}(\beta, b)$, then*

$$Re\left(\frac{F_n(z)}{z}\right) > 1 - \frac{2\,|b|}{\beta + 1} \tag{14}$$

and

$$Re(F_n'(z)) > 1 - \frac{2\,|b|}{\beta + 1}. \tag{15}$$

Proof. Let $F(z) \in \mathcal{T}(\beta, b)$ be of the form (1). Then we have

$$\text{Re}\left(1 + \sum_{k=2}^{\infty} \frac{2k}{k+1}(1 - \beta + \beta k)a_k z^{k-1}\right) > 1 - |b| \quad (z \in \mathcal{U}).$$

or

$$\text{Re}\left(1 + \frac{1}{2\,|b|}\sum_{k=2}^{\infty} \frac{2k}{k+1}(1 - \beta + \beta k)a_k z^{k-1}\right) > \frac{1}{2} \quad (z \in \mathcal{U}).$$

Now

$$\begin{aligned} \frac{F_n(z)}{z} &= 1 + \sum_{k=2}^{n} \frac{2}{k+1} a_k z^{k-1} \\ &= \left(1 + \frac{1}{2\,|b|}\sum_{k=2}^{\infty} \frac{2k}{k+1}(1 - \beta + \beta k)a_k z^{k-1}\right) \\ &\quad * \left(1 + 2\,|b| \sum_{k=2}^{n} \frac{z^{k-1}}{k(1 - \beta + \beta k)}\right). \end{aligned}$$

From Lemma 1.1, we see that, for $t = 1/\beta$

$$\text{Re}\left(1 + 2\,|b| \sum_{k=2}^{n} \frac{z^{k-1}}{k(1 - \beta + \beta k)}\right) > 1 - \frac{2\,|b|}{\beta + 1}$$

and the result follows by application of Lemma 1.4.

Using a similar argument and applying Lemma 1.2 instead of Lemma 1.1, we can prove (15). □

Theorem 3.2. *Let* $0 < 1/\beta \le M$, *where* M *is defined as in Lemma 1.1. If* $f(z) \in \mathcal{T}(\beta, b)$, *then*

$$Re\left(\frac{f_n(z)}{z}\right) > 1 - \frac{2\,|b|}{\beta + 1} \tag{16}$$

and

$$Re(f_n'(z)) > 1 - \frac{2\,|b|}{\beta + 1} \tag{17}$$

Proof. Let $f \in \mathcal{T}(\beta, b)$ be of the form (12). Then we have

$$\operatorname{Re}\left(1+\sum_{k=2}^{\infty} k(1-\beta+\beta k)a_k z^{k-1}\right) > 1-|b|$$

or

$$\operatorname{Re}\left(1+\frac{2}{\beta+1}\sum_{k=2}^{\infty} k(1-\beta+\beta k)a_k z^{k-1}\right) > 1-\frac{2\,|b|}{\beta+1}.$$

Now

$$\begin{aligned}\frac{f_n(z)}{z} &= 1+\sum_{k=2}^{n} a_k z^{k-1}\\ &= \left(1+\frac{2}{\beta+1}\sum_{k=2}^{\infty} k(1-\beta+\beta k)a_k z^{k-1}\right)\\ &\quad * \left(1+\frac{\beta+1}{2}\sum_{k=2}^{n}\frac{z^{k-1}}{k(1-\beta+\beta k)}\right)\end{aligned}$$

From Lemma 1.1, we see that, for $t = 1/\beta$

$$\operatorname{Re}\left(1+\frac{\beta+1}{2}\sum_{k=2}^{n}\frac{z^{k-1}}{k(1-\beta+\beta k)}\right) > \frac{1}{2}$$

and the result follows by application of Lemma 1.4.

Using a similar argument and applying Lemma 1.2 instead of Lemma 1.1, we can prove (17). □

4. Convolution properties

Pólya and Schoenberg [7] conjectured that if $f \in \mathcal{C}$ and $g \in \mathcal{C}$, then $f * g \in \mathcal{C}$ and this conjecture was proved by Ruscheweyh and Sheil-Small [9]. Also, they proved that if $f \in \mathcal{C}$ and $g \in \mathcal{K}$, then $f * g \in \mathcal{K}$ and if $f \in \mathcal{S}^\star$ and $g \in \mathcal{S}^\star$, then $f * g \in \mathcal{S}^\star$, where $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{S}^\star$ denote the classes of convex, close-to-convex and starlike functions, respectively.

In the next theorems, we prove the analogue of Pólya-Schoenberg conjecture for the classes $\mathcal{T}(\beta, b)$ and $\mathcal{R}(\alpha)$.

Theorem 4.1. *Let $f \in \mathcal{T}(\beta, b)$ and $g \in \mathcal{C}$. Then $f * g \in \mathcal{T}(\beta, b)$.*

Proof. Let $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, then it is sufficient to show that

$$\operatorname{Re}\left(1+\sum_{k=2}^{\infty} k(1-\beta+\beta k)a_k b_k z^{k-1}\right) > 1-|b|\,.$$

It is known that if $g \in \mathcal{C}$ then

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) = \operatorname{Re}\left(1+\sum_{k=2}^{\infty} b_k z^{k-1}\right) > \frac{1}{2}.$$

Now

$$\begin{aligned} &1+\sum_{k=2}^{\infty} k(1-\beta+\beta k)a_k b_k z^{k-1} \\ = &\left(1+\sum_{k=2}^{\infty} k(1-\beta+\beta k)a_k z^{k-1}\right) * \left(1+\sum_{k=2}^{\infty} b_k z^{k-1}\right). \end{aligned}$$

Since $f \in \mathcal{T}(\beta, b)$ the result follows by application of Lemma 1.4. □

Letting $\beta = 0$ and $b = 1-\alpha$, $0 \le \alpha < 1$, in Theorem 4.1, we have

Corollary 4.2. *Let $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{C}$. Then $f * g \in \mathcal{R}(\alpha)$.*

Theorem 4.3. *Let $f \in \mathcal{T}(0, b)$ and $g \in \mathcal{T}(\beta, b)$. Then $f * g \in \mathcal{T}(0, \gamma)$ where*

$$\gamma = \frac{|b|\,(2\beta+3) - (\beta+1)}{2(\beta+1)} \tag{18}$$

Proof. Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{T}(\beta, b)$, then

$$\operatorname{Re}\left(1+\sum_{k=2}^{\infty} k(1-\beta+\beta k) b_k z^{k-1}\right) > 1-|b|\,. \tag{19}$$

Let $c_0 = 1$ and

$$c_k = \frac{\beta+1}{1+\beta k}, \qquad k \ge 1,$$

we see that $\{c_k\}_{k=0}^{\infty}$ is a convex null sequence. Therefore, by Lemma 1.3, we have

$$\operatorname{Re}\left(1+\sum_{k=2}^{\infty} \frac{\beta+1}{1-\beta+\beta k} z^{k-1}\right) > \frac{1}{2}. \tag{20}$$

Take the convolution of (19) and (20) and apply Lemma 1.4 to obtain

$$\operatorname{Re}\left(1+(\beta+1)\sum_{k=2}^{\infty} b_k z^{k-1}\right) > 1-|b|$$

or

$$\mathrm{Re}\left(\frac{g(z)}{z}\right) = \mathrm{Re}\left(1+\sum_{k=2}^{\infty} b_k z^{k-1}\right) > \frac{\beta+1-|b|}{\beta+1}$$

or

$$\mathrm{Re}\left(\frac{g(z)}{z} - \frac{\beta+1-2\,|b|}{2(\beta+1)}\right) > \frac{1}{2}.$$

Since $f \in \mathcal{T}(0,b)$, by applying Lemma 1.4, we obtain

$$\mathrm{Re}\left(f'(z) * \left(\frac{g(z)}{z} - \frac{\beta+1-2\,|b|}{2(\beta+1)}\right)\right) > 1-|b|$$

or

$$\begin{aligned}\mathrm{Re}(f(z)*g(z))' &= \mathrm{Re}\left(f'(z) * \left(\frac{g(z)}{z}\right)\right)\\ &= 1-\left(\frac{|b|\,(2\beta+3)-(\beta+1)}{2(\beta+1)}\right).\end{aligned}$$

□

Letting $\beta = 0$ and $b = 1-\alpha$, in Theorem 4.3, we have

Corollary 4.4. *Let f and g be in $\mathcal{R}(\alpha); 0 \le \alpha < 2/3$. Then $f*g \in \mathcal{R}(\mu)$ where*

$$\mu = \frac{3\alpha}{2} \tag{21}$$

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