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ON SOME CONNECTIONS ON LOCALLY PRODUCT RIEMANNIAN MANIFOLDS - PART I

Nevena Pušić[1]

Abstract. It is a well-known fact that a Riemannian metric on a differentible manifold induces a Riemannian metric on its submanifold and, hence, a Riemannian connection on the manifold induces a Riemannian connection on its submanifold. In this paper, we consider the problems in a converse direction and consider five connections from the family of "projective class" on global manifold which can induce a given Riemannian connection on the submanifold. Also, we consider the induced connection on the supplement of the submanifold, if the space is decomposable.

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1. Introduction

Subspaces of certain spaces have always attracted scientific interest. Especially Riemannian manifolds and their submanifolds were interesting for consideration. Also, other spaces which were built on the basis of Riemannian spaces, but carrying different structures besides Riemannian metrics and Levi-Civita (Riemannian) connection and objects induced by them on submanifolds have been a research subject over a centurye. Even contemporary generalizations of Riemannian space, which have very strong applications in physics and other sciences, like Lagrange and Hamiltonian spaces of any rank, still keep the same method: considering different kinds of subspaces (submanifolds) by a kind of coordinate transformation, then consider some kinds of connections and other geometric objects, and after that analyze which kinds of objects are induced ([1]). Even Riemannian spaces still can give some interesting and useful information about its subspaces carrying some special structures ([2, 4]).

Another topic in this paper are some special connections on a Riemannian manifold. Since the third decade of the last century, Riemannian spaces are mines of families of special connections and they still are ([3, 5, 6]). Here we consider some special connections. Shortly, we do not want to find out which kind of connection is induced on a submanifold by a connection given

[1]Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Serbia, e-mail: nevena@dmi.uns.ac.rs

on the ambiental space, but the converse, how could a connection given on the submanifold of a Riemannian space be naturally extended to the ambiental space.

Here we use the method which has been promoted by Kentaro Yano ([7]).

2. About locally product spaces

Let M_n be an $n-$dimensional **locally product space** such that M_n is a locally product space $M_p \times M_q$ of $p-$ and $q-$ dimensional spaces M_p and M_q $(p+q=n)$. Then, M_n is covered by such a system of coordinate neighborhoods $\{(U,x^h)\}$ that in any intersection of two coordinate neighborhoods (U,x^h) and $(U',x^{h'})$ we have

$$x^{a'} = x^{a'}(x^a) \text{ and } x^{x'} = x^{x'}(x^x) \tag{2.1}$$

with

$$|\partial_a x^{a'}| \neq 0 \text{ and } |\partial_x x^{x'}| \neq 0,$$

where ∂_h denotes $\partial/\partial x^h$, the indices a,b,c,d run over the range $1,2,...,p$, the indices x,y,z,w run over the range $p+1,...,p+q=n$ and the indices h,i,j,k,l run over the range $1,2,...,n$. Such a coordinate system will be called a **separating coordinate system** in M_n.

If we define F_i^h by

$$(F_i^h) = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_y^x \end{pmatrix} \tag{2.2}$$

in each separating coordinate neighborhood, F_i^h is a tensor field on M_n, which satisfies $F_j^i F_i^h = \delta_j^h$. If M_n is a Riemannian manifold, its metric tensor is of the form

$$(g_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{xy} \end{pmatrix} \tag{2.3}$$

and component-subspaces are mutually orthogonal. We call the tensor F the **product structure tensor.** Then, there exists the covariant structure tensor

$$F_{ji} = F_j^t g_{ti} \tag{2.4}$$

and there holds

$$F_j^t F_k^s g_{ts} = g_{jk}. \tag{2.5}$$

By this fact, it is obvious that the covariant structure tensor is symmetric and in fact

$$F_{ji} = \begin{bmatrix} g_{cb} & 0 \\ 0 & -g_{xy} \end{bmatrix} \tag{2.6}$$

If g_{cb} depends only on x^a and g_{xy} depends only on x^z, we call the space a *locally decomposable Riemannian space.* For a locally decomposable Riemannian space, we have

$$\left\{ {x \atop cb} \right\} = 0, \quad \left\{ {x \atop zb} \right\} = 0, \quad \left\{ {a \atop cy} \right\} = 0, \quad \left\{ {a \atop zy} \right\} = 0.$$

It is a very well-known fact ([7]) that the necessary and sufficient condition for a locally product Riemannian space to be a locally decomposable Riemannian space is that

$$\overset{\circ}{\nabla}_j \; F_i^h = 0, \tag{2.7}$$

or, equivalently

$$\overset{\circ}{\nabla}_j \; F_{ih} = 0 \tag{2.8}$$

where $\overset{\circ}{\nabla}$ denotes a covariant differentiation operator with respect to the Levi-Civita connection.

3. Almost product Riemannian spaces

Let it be given a differentiable manifold $\mathcal{M}_n$ and its submanifold $\mathcal{M}_p$. Then at any point of the submanifold the tangent space of the submanifold is a distribution $\mathcal{P}$ in the tangent space of the manifold; at any point, which means globally, there exists a complementary distribution $\mathcal{Q}$. If we introduce a properly chosen Riemannian metrics on this manifold, these two distributions will be mutually orthogonal. An alternative method of considering these two distributions is given in [1].

Under these conditions, for any vector with components (v^h) which is tangent to the manifold $\mathcal{M}_n$, there exist projection tensors P_i^h and Q_i^h such that there holds

$$v^h = \delta_i^h v^i = P_i^h v^i + Q_i^h v^i, \tag{3.1}$$

while $P_i^h v^i$ belongs to the distribution $\mathcal{P}$ and $Q_i^h v^i$ belongs to the distribution $\mathcal{Q}$. Then it is obvious that there holds

$$\delta_i^h = P_i^h + Q_i^h.$$

We can define the structure tensor F in the following way:

$$F_i^h = P_i^h - Q_i^h.$$

From the upper equality, there holds

$$P_i^h = \frac{1}{2}(\delta_i^h + F_i^h), \quad Q_i^h = \frac{1}{2}(\delta_i^h - F_i^h).$$

We can easily conclude that both projection tensors and structure tensor F are given globally.

If we take p linearly independent contravariant vectors B_b^h $(a, b, c, ... = 1, ..., p; x, y, z, ... = p+1, ..., p+q)$ which belong to the distribution $\mathcal{P}$ and q linearly independent contravariant vectors C_y^h which belong to the distribution $\mathcal{Q}$, then these n vectors are linearly independent (the lower index in both cases denotes the ordinal number of the vector). The inverse of the matrix (B_b^h, C_y^h) we denote by (B_i^a, C_i^z). There hold the identities

$$B_b^h B_h^a = \delta_b^a, \quad B_b^h C_h^y = 0, \quad C_y^h B_h^c = 0, \quad C_y^h C_h^z = \delta_y^z.$$

We call the set (B_b^h, C_y^h) a nonholomorphic frame. Then, there holds

$$B_a^h B_i^a = P_i^h, \quad C_x^h C_i^x = Q_i^h$$

for the projection tensors and also

$$P_i^h P_j^i = P_j^h, \quad P_i^h Q_j^i = 0, \quad P_j^i Q_i^h = 0, \quad Q_i^h Q_j^i = Q_j^h. \tag{3.2}$$

This obviously leads to the involutivity of the structure

$$F_i^h F_j^i = \delta_j^h. \tag{3.3}$$

The eigen values for the structure tensor F are ± 1. Let v^h be an eigen vector with eigenvalue 1. Then

$$F_i^h v^i = v^h, \quad Q_i^h v^i = \frac{1}{2}(\delta_i^h - F_i^h) v^i = 0,$$

which means that such a vector belongs to the distribution $\mathcal{P}$. The distribution $\mathcal{P}$ is an invariant subspace of the tangent space for the tensor F, with the eigen value 1, whose multiplicity is p. The distribution $\mathcal{Q}$ is the invariant subspace of the tangent space for the tensor F, with the eigen value -1, whose multiplicity is q.

An $n-$dimensional differentiable manifold $\mathcal{M}_n$ with a structure tensor F defined in such a way is called an **almost product space.**

A distribution is *integrable* (or it satisfies an integrability conditions) if and only if there exist a manifold of dimension p whose tangent space is the given distribution. As we have started from a submanifold of the given manifold which has a distribution $\mathcal{P}$ as the tangent space, the distribution $\mathcal{P}$ is integrable. In analytical form, the submanifold and its tangent space are expressed in the form of a system of (coordinate) functions at one side and a system of (formal) partial differential equations whose solution is that system of function. So we can give the conditions for such a system of partial differential equations to have a solution satisfying an initial problem; in fact, there are real integrability conditions expressed analytically

$$N^h_{ij} - N^l_{ij} F^h_l = 0,$$

where

$$NF(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] - [X, Y],$$

and in the case of locally product space, in local coordinates

$$N^h_{ij} = F^l_j(\partial_l F^h_i - \partial_i F^h_l) - F^l_i(\partial_l F^h_j - \partial_j F^h_l) \tag{3.4}$$

is the **Nijenhuis' tensor** for an almost product structure.

The complemetary distribution $\mathcal{Q}$ can also be tangent to a submanifold of a given manifold, if it is integrable. This integrability condition is

$$N^h_{ij} + N^l_{ij} F^h_l = 0.$$

As there exists the manifold $\mathcal{M}_p$, the first integrability condition is satisfied. The second one will be satisfied if and only if the Nijenhuis' tensor vanishes. If the complementary distribution is integrable, then "the rest " of the manifold is also a submanifold, and then we can consider it. Then the almost product structure on the manifold can became under some, not too strong, conditions, a locally product structure, which means that there will exist a system of coordinate neighborhoods of such kind that on the intersection of two of them there holds

$$x^{a'} = x^{a'}(x^a), \quad x^{x'} = x^{x'}(x^x).$$

Then, the locally product structure induced by such a system of coordinate neighborhoods will be equal to the initial almost product structure.

It is almost obvious that there holds the following proposition.

Proposition 1. *If on a submanifold of a differentiable manifold there is given a Riemannian metrics, there exists a globally given Riemannian metric on the whole manifold such that it induces on the submanifold the exactly given metric.*

It is easy to prove such a statement. One can consider in the tangent space of the manifold a distribution $\mathcal{P}$ which consists of all tangent vectors to the manifold that are also tangent to the submanifold. On that distribution $\mathcal{P}$ the function of scalar product is defined through the given metric tensor on the submanifold. There exists also a globally given comlementary distribution $\mathcal{Q}$

and its basic vectors at any point. We shall define a global metric taking into account two things:

- every basic vector from the distribution $\mathcal{Q}$ is orthogonal to any basic vector of the distribution $\mathcal{P}$.

- mutual scalar product of basic vectors from the distribution $\mathcal{Q}$ will be also defined.

In this way, on the whole manifold there will be given the scalar product function, that is metric coefficients. If the metric on the distribution $\mathcal{Q}$ is defined as symmetric and positively definite, then the complete metrics on the whole manifold is Riemannian.

The subsequence of such a statement is the following proposition.

Proposition 2. *If a Rimannian metric is given on a submanifold of a differentiable manifold, then there will exists a globally given Riemannian connection on the whole manifold such that the connection induced by it on the submanifold is the same which is deteminated by the given metric.*

Whenever there is a Riemannian metric given on a manifold, there will also be given a Riemannian connection.

If the other distribution of an almost product space is integrable (which happens if and only if the Nijenhuis' tensor vanihes), this Riemannian connection induces a Riemannian connection on the comlementary distribution. This is the reason for the well-known statement: **If a curve is a geodesic line on a submanifold, it is a geodesic line on the entire manifold.**

4. Some standard connections on Riemannian spaces

We shall consider a locally product Riemannian space and some standard connections which may be given on it; we are going to investigate which kinds of connections will be induced on each subspace by it. Also, we shall see what is going to happen in the case when the space is a locally decomposable Riemannian space.

First, we shall consider a **projective connection** on the Riemannian space, which is additionally a locally product space. It is a symmetric connection and its autoparallel lines are geodesic lines on the manifold (i. e. autoparallel lines of its Levi-Civita connection, given by the equation $\frac{d^2x^k}{dt^2} + \left\{ {k \atop ji} \right\} \frac{dx^i}{\partial t} \frac{dx^j}{\partial t} = \alpha(t) \frac{dx^k}{\partial t}$). Its coefficients are given by

$$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} + p_j \delta^h_i + p_i \delta^h_j, \tag{4.1}$$

where p_j are components of a covector field. Then, we can easily get for the coefficient of a projective connection in the separating coordinate system

$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} + p_b \delta^a_c + p_c \delta^a_b; \Gamma^a_{bz} = \left\{ {a \atop bz} \right\} + p_z \delta^a_b = \Gamma^a_{zb}; \Gamma^a_{xy} = \left\{ {a \atop xy} \right\};$

$\Gamma^x_{yz} = \left\{ {x \atop yz} \right\} + p_y \delta^x_z + p_z \delta^x_y; \Gamma^x_{yc} = \left\{ {x \atop yc} \right\} + p_c \delta^x_y = \Gamma^x_{cy}; \Gamma^x_{ab} = \left\{ {x \atop ab} \right\}.$

In the general case, a projective connection which is given on the whole space induces on every component of locally product Riemannian space also a projective connection. When the projection of the generator p_j on some of subspaces vanishes (although the vector itself does not vanish), the projective connection on global space induces a Riemannian connection on that subspace and a projective connection on the other one.

If the locally product Riemannian space is decomposable, then not all of mixed coefficients of the projective connection vanish; so, the projective connection is not a strongly $d-$connection. The covariant derivatives of the structure tensor are

$\nabla_a F_c^b = \overset{\circ}{\nabla}_a F_c^b; \nabla_a F_x^b = \overset{\circ}{\nabla}_a F_x^b - 2p_x \delta_a^b; \nabla_a F_b^y = \overset{\circ}{\nabla}_a F_b^y;$

$\nabla_a F_y^x = \overset{\circ}{\nabla}_a F_y^x; \nabla_x F_b^a = \overset{\circ}{\nabla}_x F_b^a; \nabla_x F_b^y = \overset{\circ}{\nabla}_x F_b^y + 2p_b \delta_x^y;$

$\nabla_x F_y^a = \overset{\circ}{\nabla}_x F_y^a; \nabla_x F_y^z = \overset{\circ}{\nabla}_x F_y^z.$

The symbol ∇ denotes the covariant differentiation operator with respect to the projective connection.

Secondly, we shall consider a **semi-symmetric metric connection** on the locally product space. It is a connection which is metric (i. e. $\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma_{ik}^h g_{hj} - \Gamma_{jk}^h g_{ih} = 0$), and its torsion tensor $T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i$ is expressed in the way

$$T_{jk}^i = p_j \delta_k^i - p_k \delta_j^i,$$

where p_j are components of a covector field. It is well known that the coefficients of such a connection can be expressed in the following way

$$\Gamma_{jk}^i = \left\{ {i \atop jk} \right\} + p_j \delta_k^i - p^i g_{jk}. \tag{4.2}$$

where p_j and p^i are covariant and contravariant components of the same vector (covector) field which is appearing in the expression for the torsion tensor components. Now we are going to calculate the coefficients of such a connection in a separating coordinate system.

$\Gamma_{bc}^a = \left\{ {a \atop bc} \right\} + p_b \delta_c^a - p^a g_{bc}; \Gamma_{bz}^a = \left\{ {a \atop bz} \right\} = \Gamma_{zb}^a; \Gamma_{xy}^a = \left\{ {a \atop xy} \right\} - p^a g_{xy} = \Gamma_{yx}^a;$

$\Gamma_{ab}^x = \left\{ {x \atop ab} \right\} - p^x g_{ab}; \Gamma_{yc}^x = \left\{ {x \atop yc} \right\} = \Gamma_{cy}^x; \Gamma_{yz}^x = \left\{ {x \atop yz} \right\} + p_y \delta_z^x - p^x g_{yz}.$

We can notice that all mixed components of metric semi-symmetric connection in a separating coordinate system are in fact symmetric. It is also obvious that the semi-symmetric metric component given globally on a locally product Riemannian space is inducing a semi-symmetric metric connection on every component of such a space. If the projection of a generating vector on some component of the space vanishes, then the induced connection on such a component will reduce to a Riemannian connection. The covariant derivatives of the structure tensor components in a such coordinate system are

$\nabla_a F_c^b = \overset{\circ}{\nabla}_a F_c^b; \nabla_a F_y^b = \overset{\circ}{\nabla}_a F_y^b; \nabla_a F_c^y = \overset{\circ}{\nabla}_a F_c^y + 2p^x g_{ac};$

$\nabla_a F_x^y = \overset{\circ}{\nabla}_a F_c^y; \nabla_x F_z^y = \overset{\circ}{\nabla}_x F_z^y; \nabla_x F_a^y = \overset{\circ}{\nabla}_x F_a^y;$

$\nabla_x F_z^a = \overset{\circ}{\nabla}_x F_z^a + 2p^a g_{xz}; \nabla_x F_b^a = \overset{\circ}{\nabla}_x F_b^a.$

The structure tensor cannot be parallel unless the connection reduces to a Riemannian connection trivially.

Then will hold the following theorem.

Theorem 1. *If on a submanifold of a Riemannian manifold there is given a Riemannian connection, then there exists a global uniquely determinated*

a) projective

b) semi-symmetric metric connection

with the generator whose projection to the submanifold vanishes, such that it induces the given Riemannian connection on the submanifold. If there exists a complementary manifold, the global connection induces

a) projective

b) semi-symmetric metric connection

on it.

These two connections are not $F-$connections as they are not taking account of its locally product nature and do not preserve it.

5. On holomorphically-projective, mirror and product semi-symmetric metric connections on locally product and decomposable Riemannian space

We consider a locally product Riemannian space and geodesic lines on it. Besides the property of minimizing the distance between two points, a geodesic line is autoparallel with respect to the Riemannian connection, which means that the direction of its tangent vector stays preserved during a parallel displacement along the same curve. Curves which are autoparallel for the projective connection are geodesic (autoparallel) lines for Riemannian connection. If we have a structure on the same space, for every vector v^h which is not an eigen vector for the structure, we have its image $F_i^h v^i$ made by the structure in the tangent space; in other words, we shall have a **holomorphic plane**. If, during the parallel displacement along a curve on the manifold holomorphic planes determined by a vector tangent to the curve and by the vector which is holomorphic to it stay parallel to itself, that curve is called *a* **holomorphically planar curve**.

If the space is decomposable, its geodesic lines are holomorphically planar, as the structure tensor is parallel.

Here is the differential equation of a holomorphically planar curve

$$\frac{d^2x^h}{dt^2} + \left\{ {h \atop ij} \right\} \frac{dx^i}{dt}\frac{dx^j}{dt} = \alpha(t)\frac{dx^h}{dt} + \beta(t)F_r^h\frac{dx^r}{dt} \tag{5.1}$$

for the Riemannian connection.
The connection which shares strictly holomorphically planar curves with Riemannian connection, if it is symmetric, has components of the form

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} + p_j\delta_i^h + p_i\delta_j^h + q_jF_i^h + q_iF_j^h. \tag{5.2}$$

We call the connection with components (5.2) a **holomorphically projective connection** ([3]).

As this connection should preserve the geometry of the space, then in the case of locally decomposable space, the structure tensor has also to be parallel to the holomorphically projective connection. We can calculate

$\nabla_k F^i_j = \overset{\circ}{\nabla}_k F^i_j + \delta^i_k (p_s F^s_j - q_j) + F^i_k (q_s F^s_j - p_j).$

The covariant derivative of the structure tensor is equal to its Levi-Civita covariant derivative if and only if there holds

$$p_s F^s_j = q_j, \quad q_s F^s_j = p_j. \tag{5.3}$$

The connection which displaces the tangent vector along the curve to a vector holomorphic to it at any close point we shall call *a* **mirror** connection. Its components are given by the expression

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + q_j F^i_k + q_k F^i_j. \tag{5.4}$$

The mirror connection is not an $F-$connection. It is a connection which has autoparallel lines in common with a holomorphically projective connection depending on the same generating vector.

There is one more connection which can be of interest in this context. This is **product semi-symmetric metric connection**. We considered a semi-symmetric metric connection on a Riemannian space. Its coefficients were uniquely determined by its torsion tensor and by the fact that it was metric. So, we have to broaden the definition of semi-symmetry and include the structure tensor in it. So,

$$\Gamma^i_{jk} - \Gamma^i_{kj} = T^i_{jk} = p_j \delta^i_k - p_k \delta^i_j + q_j F^i_k - q_k F^i_j. \tag{5.5}$$

From the form of torsion tensor and from the fact that the connection is metric we can obtain, by the standard calculation, the coefficients of the connection:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j \delta^i_k - p^i g_{jk} + q_j F^i_k - q^i F_{jk}. \tag{5.6}$$

Finally, we shall calculate the covariant derivarives for the structure tensor components:

$\nabla_k F^i_j = \overset{\circ}{\nabla}_k F^i_j + \delta^i_k (p_s F^s_j - q_j) + F_{jk} (q^s F^i_s - p^i) + F^i_k (q_s F^s_j - p_j)$
$+ g_{jk} (p^s F^i_s - q^i).$

It is obvious that the covariant derivative of the structure tensor equals to its Levi-Civita covariant derivative if and only if there holds

$$q_s F^s_j = p_j \tag{5.7}$$

and vice versa, as the structure is involutive.

Now we shall see the exact acting of such connections on locally product and decomposable Riemannian spaces.

1) Holomorphically projective connection

It the coefficients, in accordance with the structure tensor form in separating coordinate system, have the form

$$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} + p_b\delta^a_c + p_c\delta^a_b + q_bF^a_c + q_cF^a_b = \left\{ {a \atop bc} \right\} + 2p_b\delta^a_c + 2p_c\delta^a_b;$$
$$\Gamma^a_{bx} = \left\{ {a \atop bx} \right\} + 2p_x\delta^a_b = \Gamma^a_{xb}; \Gamma^a_{xy} = \left\{ {a \atop xy} \right\} = \Gamma^a_{yx};$$
$$\Gamma^x_{yc} = \left\{ {x \atop yc} \right\} + 2p_c\delta^x_y = \Gamma^x_{cy}; \Gamma^x_{ab} = \left\{ {x \atop ab} \right\}; \Gamma^{\ x}_{\ yz} = \left\{ {x \atop yz} \right\} + 2p_y\delta^x_z + 2p_z\delta^x_y.$$

So, a holomorphically projective connection given globally on the locally product Riemannian space induces on any its component a projective connection with double generator. If the projection of the generator on some component vanishes, then the holomorhically projective connection induces the Riemannian connection on that component and a projective connection on the other one.

We can notice that the holomorphically projective connection is not a strongly d-connection, which means that at least one of its mixed coefficients does not vanish identically; those mixed components which do not vanish have torsion tensors and look somehow "semi-symmetric", although the connection is symmetric.

2) Mirror connection

Mirror connection has the coefficients

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + q_jF^i_k + q_kF^i_j.$$

It is not a metric nor $F-$connection, although it is symmetric. We call it a "mirror" connection because during a parallel displacement along he curve which is satisfying the differential equation (3.1) the tangent vector direction turns into the holomorphic one (acts like a mirror). Let us denote $p_j = q_aF^a_j$. Then

$$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} + p_b\delta^a_c + p_c\delta^a_b; \Gamma^a_{bx} = \left\{ {a \atop bx} \right\} - p_x\delta^a_b = \Gamma^a_{xb}; \Gamma^a_{xy} = \left\{ {a \atop xy} \right\} = \Gamma^a_{yx};$$
$$\Gamma^x_{yc} = \left\{ {x \atop yc} \right\} - p_c\delta^x_y = \Gamma^x_{cy}; \Gamma^x_{ab} = \left\{ {x \atop ab} \right\}; \Gamma^x_{yz} = \left\{ {x \atop yz} \right\} + p_y\delta^x_z + p_z\delta^x_y.$$

This means that the mirror connection on both subspaces-components of locally product Riemannian space induces a projective connection whose generator is the image of the mirror-connection's generator by structure tensor, unless some of projections of the generator vanishes. Then, the induced connection on that subspace-component reduces to a Riemannian connection.

The mirror connection also cannot be a strongly d-connection, as at least two of its mixed coefficients do not vanish.

3) Product semi-symmetric metric connection

It has the coefficients

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j\delta^i_k - p^ig_{jk} + q_jF^i_k - q^iF_{jk},$$

where $q_j = F^a_jp_a$ and vice versa. In the separating coordinate system, it will be

$$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} + 2p_b\delta^a_c + 2p_c\delta^a_b; \Gamma^a_{bx} = \left\{ {a \atop bx} \right\} = \Gamma^a_{xb}; \Gamma^a_{xy} = \left\{ {a \atop xy} \right\} = \Gamma^a_{yx};$$
$$\Gamma^x_{ab} = \left\{ {x \atop ab} \right\} = \Gamma^x_{ba}; \Gamma^x_{ya} = \left\{ {x \atop ya} \right\} = \Gamma^x_{ay}; \Gamma^x_{yz} = \left\{ {x \atop yz} \right\} + 2p_y\delta^x_z + 2p_z\delta^x_y.$$

We can make the following conclusion: a metric semi-symmetric $F-$connection given globally on a locally product Riemannian space induces on any subspace-component a metric semi-symmetric connection with double generator. If the generator's projection on some subspace-components vanishes, the induced connection reduces to a Riemannian connection.

If the space is decomposable, the product semi-symmetric metric connection is a strongly $d-$connection, and this fact may be of some geometric meaning.

Meanwhile, similar statements hold for the other kinds of connections, on submanifolds and the entire manifold. The connections on submanifolds can have their extensions on entire manifold, which can be a connection of the same, but also of another kind. If the distributions $\mathcal{P}$ and $\mathcal{Q}$ are given (and if the submanifold is defined, they are given), there are known projections of an arbitrary vector tangent to the manifold on these distributions.

We have seen that the necessary and sufficient condition for the second distribution in almost product space to be integrable is that Nijenhuis' tensor should vanish. Let us suppose that this condition is satisfied and that, moreover, the almost product structure is of such kind that it is parallel to the given Riemannian connection. Then, the considered Riemannian space is decomposable and there exist some $F-$connections. Then there holds the following theorem.

Theorem 2. *If on one of subspace-components of a locally decomposable Riemannian space there is given a Riemannian connection, then there exists a uniquely determined global connection which is*

a) holomorphically projective

b) mirror

c) product semi-symmetric metric connection

with the generator whose projection on this subspace-component vanishes, which is a global extension of the given Riemannian connection. On the other subspace-component, this global connection will induce

a) projective

b) projective

c) semi-symmetric metric connection

respectively.

There also will hold the following theorem.

Theorem 3. *Let on a submanifold of a Riemannian manifold be given*

a) projective

b) semi-symmetric metric connection.

Then there will exist a globally given

a) projective

b) semi-symmetric metric connection

on the whole manifold which is an extension of a given connection on the submanifold.

For the decomposable Riemannian space, there will hold the following theorem.

Theorem 4. *If on one of subspace-components of a decomposable Riemannian space there is given*

a) projective

b) semi-symmetric metric connection,

then there exists a globally given uniquely determined

a) one holomorphically projective and one mirror

b) product semi-symmetric metric connection

respectively, which are natural extensions of the given connection on the subspace-component. On the other subspace-component, it induces

a) a projective or a Riemannian

b) a semi-symmetric metric or a Riemannian

connection, in dependance of the kind of generators of the globally given connection.

The proofs of these theorems are given in Sections 4 and 5.

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