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AN ITERATIVE PROCESS FOR A FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. We define an iterative process for a finite family of asymptotically quasi-nonexpansive mappings in CAT(0) space and obtain sufficient conditions for the convergence of this iterative scheme to a unique common fixed point of the family.

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1. Introduction and preliminaries

The notion of $\triangle-$ convergence in general metric spaces was introduced by Lim [13] in 1976. Kirk and Panyanak [12] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Afterwards, Chaoha and Phon-on [5], Dhompongsa and Panyanak [6], Shahzad [20] and Beg and Abbas [1] continued to work in this direction. Their results involve Mann and Ishikawa iteration schemes involving one mapping. In this paper, we initiate the study of convergence of an iterative sequence generated by a finite family of asymptotically quasi-nonexpansive mappings in CAT(0) spaces. Our result generalizes several comparable results in the existing literature.

Let us recall some basics: Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$, such

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that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles of an appropriate size satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$\text{(CN)} \qquad d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2$$

This is the (CN) inequality of Bruhat and Tits [3]. In fact (cf. [2, p. 163]), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality. A metric space X is called a CAT(0) space [10] if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. For a vigorous discussion, see [4, 2]. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space, see [9, 18].

The following are some elementary facts about CAT(0) spaces.

Lemma 1.1 ([6])**.** *Let (X, d) be a CAT(0) space.*

(i) (X, d) is uniquely geodesic.

(ii) Let p, x, y be points of X, let $\alpha \in [0, 1]$, and let m_1 and m_2 denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying $d(p, m_1) = \alpha d(p, x)$ and $d(p, m_2) = \alpha d(p, y)$. Then

$$d(m_1, m_2) \leq \alpha d(x, y). \tag{1.1}$$

(iii) Let $x, y \in X, x \neq y$ and $z, w \in [x, y]$, such that $d(x, z) = d(x, w)$. Then, $z = w$.

(iv) $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$ for all $x, y, z \in X$ and $t \in [0, 1]$.

(v) Let $x, y \in X$. For each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \tag{1.2}$$

For convenience, from now on we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (1.2).

Diaz and Metcalf [7] introduced the concept of quasi-nonexpansive mappings. Goebel and Kirk [8] gave the definition of asymptotically nonexpansive mappings.

Definition 1.2. A mapping $T : X \to X$ is called:

(1) Nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in X$.

(2) [8] Asymptotically nonexpansive if there exists $k_n \in [0, \infty)$ for all $n \in N$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, T^n y) \le (1 + k_n)d(x, y)$ for all $x, y \in X$.

(3) (cf. [14]) Quasi-nonexpansive if $d(Tx, p) \le d(x, p)$ for all $x \in X$, for all $p \in F(T)$ where $F(T)$ is the set of all fixed points of T.

(4) (cf.[16]) Asymptotically quasi-nonexpansive if there exists $k_n \in [0, \infty)$ for all $n \in N$ with $\lim_{n\to\infty} k_n = 0$ such that $d(T^n x, p) \le (1 + k_n)d(x, p)$ for all $p \in F(T)$.

Remark 1.3. From Definition 1.2, it is clear that the classes of quasi-nonexpansive mappings and asymptotically nonexpansive mappings include nonexpansive mappings, whereas the class of asymptotically quasi-nonexpansiveness is larger than that of quasi nonexpansive mappings, and asymptotically nonexpansive mappings. The reverse of these implications may not be true (see [8, 14]).

Dhompongsa and Panyanak [6, Theorems 3.1, 3.2 and 3.3] studied the $\triangle-$convergence of Picard, Mann and Ishikawa iterates.

Let C be a convex subset of a Banach space X, $S, T : C \to C$ be two mappings.

Schu [19] studied the following modified Mann iterative scheme on the pattern of the Mann scheme

$$x_0 \in C,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ which is bounded away from 0 and 1, i.e., $a \le \alpha_n \le b$ for all $n \in N \cup \{0\}$ and some $0 < a \le b < 1$.

Xu and Noor [22] introduced a three-step iterative scheme as follows:

$$\begin{aligned} x_0 &\in C,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n z_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n T^n x_n \end{aligned}$$

for all $n \in N \cup \{0\}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are some sequences in $[0, 1]$.

To approximate the common fixed points of two asymptotically nonexpansive mappings, S and T (say) the following Ishikawa type two-step iterative process is widely used (see, for example, [11, 15, 22] and references cited therein)

$$\begin{aligned} x_0 &\in C, \\ x_{n+1} &= (1 - a_n)x_n + a_n S^n y_n \\ y_n &= (1 - b_n)x_n + b_n T^n x_n \end{aligned}$$

for all $n \in N \cup \{0\}$, where $\{a_n\}$, $\{b_n\}$ are some appropriate sequences in $[0, 1]$.

In this paper, we study the convergence of the following iteration scheme in the framework of CAT(0) spaces:

Let C be a convex subset of a CAT(0) space X and $x_0 \in C$ $i = 1, 2, ..., k$. Let $\{T_i : i = 1, 2, ..., k\}$ be a finite family of self-mappings of C. Our process reads as follows:

$$\begin{aligned} x_{n+1} &= \alpha_{kn}x_n \oplus (1-\alpha_{kn})T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= \alpha_{(k-1)n}x_n \oplus (1-\alpha_{(k-1)n})T_{k-1}^n y_{(k-2)n}, \\ y_{(k-2)n} &= \alpha_{(k-2)n}x_n \oplus (1-\alpha_{(k-2)n})T_{k-2}^n y_{(k-3)n}, \\ &\ldots \\ y_{2n} &= \alpha_{2n}x_n \oplus (1-\alpha_{2n})T_2^n y_{1n}, \\ y_{1n} &= \alpha_{1n}x_n \oplus (1-\alpha_{1n})T_1^n y_{0n}, \end{aligned} \tag{1.3}$$

where $y_{0n} = x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Lemma 1.4 ([17]). *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three sequences of positive real numbers satisfying*

$$a_{n+1} \leq (1 + c_n)a_n + b_n \text{ for all } n \in N,$$

where $\sum\limits_{n=1}^{\infty} b_n < \infty$, $\sum\limits_{n=1}^{\infty} c_n < \infty$. Then $\lim\limits_{n\to\infty} a_n$ exists, and $\lim\inf\limits_{n\to\infty} a_n = 0$ implies that $\lim\limits_{n\to\infty} a_n = 0$.

Remark 1.5. It is noted that the above lemma holds under the hypothesis $\lim\sup\limits_{n\to\infty} a_n = 0$ as well.

2. Main results

In this section we study the necessary and sufficient condition for the convergence of an iterative sequence stated afore.

Theorem 2.1. *Let C be a nonempty convex subset C of a CAT(0) space X and $\{T_i : i = 1, 2, ..., k\}$ be a family of asymptotically quasi-nonexpansive self-mappings of C with $F \neq \phi$ and $y_{0n} \in C$. Suppose that $\{x_n\}$ is as in (1). Then, $\{x_n\}$ converges to a unique point in F if C is complete, and*

$$\lim\inf_{n\to\infty} d(x_n, F) = 0 \text{ or } \lim\sup_{n\to\infty} d(x_n, F) = 0.$$

Proof. First, note that there exist a points p in F and a sequence $\{k_n\} \subset [0, \infty)$ with $\lim\limits_{n\to\infty} k_n = 0$ such that

$$d(T_i^n x, p) \leq (1 + k_n)d(x, p),$$

for all $x \in C$ and for each $i = 1, 2, ..., k$. Since $\{T_i : i = 1, 2..., k\}$ is a finite family of asymptotically quasi-nonexpansive mappings, there exist $p_i \in F(T_i)$

and the sequences $\{k_i\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_{in} = 0$ for each $i = 1, 2, ..., k$ such that

$$d(T_i^n x, p_i) \leq (1 + k_{in}) d(x, p_i) \tag{2.1}$$

for each $x \in C$. Let $k_n = \max\{k_{1n}, k_{2n}, , ..., k_{kn}\}$. Hence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$. Also, $F \neq \phi$ implies that there exists $p \in F$. Hence, $p \in F(T_i)$ for each $i = 1, 2, ..., k$, and the inequality (2.1) holds for each $p \in F$. Thus, there exist $p \in F$ and a sequence $\{k_n\} \subset [0, \infty)$ such that

$$d(T_i^n x, p) \leq (1 + k_{in}) d(x, p) \leq (1 + k_n) d(x, p)$$

for all $x \in C$ and for each $i = 1, 2, ..., k$. Now for $p \in F$, consider

$$\begin{aligned} d(y_{1n}, p) &= d(\alpha_{1n} x_n \oplus (1 - \alpha_{1n}) T_1^n x_n, p) \\ &\leq \alpha_{1n} d(T_1^n x_n, p) + (1 - \alpha_{1n}) d(x_n, p) \\ &\leq \alpha_{1n}(1 + k_n) d(x_n, p) + (1 - \alpha_{1n}) d(x_n, p) \\ &= (1 + \alpha_{1n} k_n) d(x_n, p) \\ &\leq (1 + k_n) d(x_n, p). \end{aligned}$$

Assume that $d(y_{jn}, p) \leq (1 + k_n)^j d(x_n, p)$ holds for some $1 \leq j \leq k - 2$. Then

$$\begin{aligned} & d(y_{(j+1)n}, p) \\ &= d(\alpha_{(j+1)n} x_n \oplus (1 - \alpha_{(j+1)n} T_{(j+1)}^n y_{in}), p) \\ &\leq \alpha_{(j+1)n} d(T_j^n y_{in}, p) + (1 - \alpha_{(j+1)n}) d(x_n, p) \\ &\leq \alpha_{(j+1)n}(1 + k_n) d(y_{in}, p) + (1 - \alpha_{(j+1)n}) d(x_n, p) \\ &= \left[1 - \alpha_{(j+1)n} + \alpha_{(j+1)n} \left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j...(j+2-r)}{r!} k_n^r\right)\right] d(x_n, p) \\ &= \left[1 + \alpha_{(j+1)n} \sum_{r=1}^{j+1} \frac{(j+1)j...(j+2-r)}{r!} k_n^r\right] d(x_n, p) \\ &\leq (1 + k_n)^{j+1} d(x_n, p). \end{aligned}$$

By mathematical induction, we obtain that

$$d(y_{in}, p) \leq (1 + k_n)^i d(x_n, p)$$

for all $i = 1, 2, 3, ..., k-1$. Now,

$$\begin{aligned}
& d(x_{n+1}, p) \\
= \ & d(\alpha_{kn} x_n \oplus (1-\alpha_{kn}) T_k^{k-1} y_{(k-1)n}, p) \\
\leq \ & \alpha_{kn} d(T_k^n y_{(k-1)n}, p) + (1-\alpha_{kn}) d(x_n, p) \\
\leq \ & \alpha_{kn} (1+k_n)^k d(x_n, p) + (1-\alpha_{kn}) d(x_n, p) \\
= \ & \left[1 - \alpha_{kn} + \alpha_{kn}\left(1 + \sum_{r=1}^{k} \frac{k(k-1)...(k-r+1)}{r!} k_n^r\right)\right] d(x_n, p) \\
= \ & \left[1 + \alpha_{kn} \sum_{r=1}^{k} \frac{k(k-1)...(k-r+1)}{r!} k_n^r\right] d(x_n, p) \\
\leq \ & (1+k_n)^k d(x_n, p).
\end{aligned}$$

Following similar arguments to those given in [16], there exists a constant $M > 0$ such that, for all $n, m \in N$ and for every $p \in F$,

$$d(x_{n+m}, p) \leq M d(x_n, p)$$

holds. Now we show that $\{x_n\}$ is a Cauchy sequence in C. Since the given assumptions implies that $\lim_{n\to\infty} d(x_n, F) = 0$, so for each $\varepsilon > 0$, there exists $n_1 \in N$ such that

$$d(x_n, F) < \frac{\varepsilon}{M+1} \text{ for all } n \geq n_1.$$

Thus, there exists $q \in F$ such that

$$d(x_n, q) < \frac{\varepsilon}{M+1} \text{ for all } n \geq n_1,$$

and we obtain that

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, q) + d(x_n, q) \\
&\leq M d(x_n, q) + d(x_n, q) \\
&= (M+1) d(x_n, q) \\
&< (M+1)\left(\frac{\varepsilon}{M+1}\right) = \varepsilon,
\end{aligned}$$

for all $n, m \geq n_1$. Therefore, $\{x_n\}$ is a Cauchy sequence in C. From the completeness of C, we get that $\lim_{n\to\infty} x_n$ exists and equals $q \in C$, say. Therefore, for all $\varepsilon_1 > 0$, there exists an $n_1 \in N$ such that, for all$n \geq n_1$,

$$d(x_n, q) < \frac{\varepsilon}{2(2+u_1)}.$$

Now $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$ gives that $\lim_{n\to\infty} d(x_n, F) = 0$. So, there exists $n_2 \in N$ with $n_2 \geq n_1$ such that, for all $n \geq n_2$,

$$d(x_n, F) < \frac{\varepsilon_1}{2(4+3u_1)}.$$

Thus, there exists $r \in F$ such that

$$d(x_{n_2}, r) < \frac{\varepsilon_1}{2(4+3u_1)}$$

For any $T_i, i = 1, 2, ..., k$, we obtain

$$\begin{aligned} d.(T_i q, q) &\leq d(T_i q, r) + d(r, T_i x_{n_2}) + d(r, x_{n_2}) + d(x_{n_2}, q) \\ &= d(T_i q, r) + 2d(T_i x_{n_2}, r) + d(r, x_{n_2}) + d(x_{n_2}, q) \\ &\leq (2+u_1)d(q, r) + 2(1+u_1)d(x_{n_2}, r) + d(r, x_{n_2}) + d(x_{n_2}, q) \\ &\leq (2+u_1)\frac{\varepsilon_1}{2(2+u_1)} + (4+3u_1)\frac{\varepsilon_1}{2(4+3u_1)} = \varepsilon_1. \end{aligned}$$

Since ε_1 is arbitrary, so $d(T_i q, q) = 0$ for all $i = 1, 2, ..., k$; i.e., $T_i q = q$. Therefore, $q \in F$. □

Theorem 2.2. *Let C be a nonempty convex subset C of a CAT(0) space X and $\{T_i : i = 1, 2, ..., k\}$ be a family of asymptotically quasi-nonexpansive self-mappings of C with $F \neq \phi$ and $y_{0n} \in C$. Suppose that $\{x_n\}$ is as in (1). Then, $\lim\inf_{n\to\infty} d(x_n, F) = \lim\sup_{n\to\infty} d(x_n, F) = 0$ if $\{x_n\}$ converges to a unique point in F.*

Proof. *Let $p \in F$. Since $\{x_n\}$ converges to p, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that*

$$d(x_n, p) < \varepsilon \text{ for all } n \geq n_0.$$

Taking the infimum over $p \in F$, we find that

$$d(x_n, F) < \varepsilon \text{ for all } n \geq n_0.$$

This means that $\lim_{n\to\infty} d(x_n F) = 0$. We obtain that

$$\lim\inf_{n\to\infty} d(x_n, F) = \lim\sup_{n\to\infty} d(x_n, F) = 0.$$

Theorem 2.3. *Let $\{T_i : i = 1, 2, ..., k\}$ be a family of asymptotically quasi-nonexpansive mappings from a nonempty complete convex subset C of a CAT(0) space into C with $F \neq \phi$ ($T_i, i = 1, 2, ..., k$, need not to be continuous). Suppose that $\{x_n\}$ is as in (1). Assume that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.2}$$

if the sequence $\{z_n\}$ in C satisfies

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0, \tag{2.3}$$

then

$$\lim\inf_{n\to\infty} d(z_n, F) = 0 \text{ or } \lim\sup_{n\to\infty} d(z_n, F) = 0$$

Then, $\{x_n\}$ converges to a unique point in F.

Proof. From (2.2) and (2.3), we have that

$$\lim \inf_{n\to\infty} d(x_n, F) = 0 \text{ or } \lim \sup_{n\to\infty} d(x_n, F) = 0.$$

Therefore, we obtain from Theorem 2.1 that the sequence $\{x_n\}$ converges to a unique point in F. □

Theorem 2.4. *Let C, $\{T_i : i = 1, 2, ..., k\}$, F and $\{x_n\}$ be as in Theorem 2.3. Suppose that there exists a map T_j which satisfies the following conditions:*
(i) $\lim_{n\to\infty} d(x_n, T_j x_n) = 0$;
(ii) there exists a function $\phi : [0, \infty) \to [0, \infty)$ which is right-continuous at 0, $\phi(0) = 0$ and $\phi(d(x_n, T_j x_n)) \geq d(x_n, F)$ for all n. Then, the sequence $\{x_n\}$, defined by (1), converges to a unique point in F.

Proof. From (i) and (ii), we obtain

$$\begin{aligned} \lim_{n\to\infty} d(x_n, F) &\leq \lim_{n\to\infty} \phi(d(x_n, T_j x_n)) \\ &= \phi(\lim_{n\to\infty} d(x_n, T_j x_n)) \\ &= \phi(0) = 0; \end{aligned}$$

i.e., $\lim_{n\to\infty} d(x_n, F) = 0$. Thus, $\lim \inf_{n\to\infty} d(x_n, F) = \lim \sup_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.1, $\{x_n\}$ converges to a unique point in F. □

Remark 2.5. The results of Shahzad and Udomene [21, Theorem 3.2] and of Qihou [16, Theorem 1, together with its Corollaries 1 and 2] are special cases of our theorems.

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