Novi Sad J. Math.
Vol. 41, No. 2, 2011, 117-122

MULTIPLICATION AND COMULTIPLICATION MODULES

H. Ansari-Toroghy[1], **F. Farshadifar**[2]

Abstract. This paper deals with some results concerning multiplication and comultiplication modules over a commutative ring.

AMS Mathematics Subject Classification (2010): 13C13, 13C99

Key words and phrases: Multiplication and comultiplication modules

1. Introduction

Throughout this paper, R denotes a commutative ring with identity.

Let M be an R-module. M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$. Equivalently, M is a multiplication module if and only if for each submodule N of M, we have $N = (N :_R M)M$ [8].

The dual notion of multiplication modules was introduced by H. Ansari-Toroghy and F. Farshadifar in [1] and some properties of this class of modules have been considered. M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. Also, it is shown [1, 3.7] that M is a comultiplication module if and only if for each submodule N of M, we have $N = (0 :_M Ann_R(N))$. More information about this class of modules can be found in [2], [3], [4], and [5].

A proper submodule N of M is said to be *prime* if for each $a \in R$, the homomorphism $M/N \xrightarrow{a} M/N$ is either injective or zero. M is said to be a *prime module* if the zero submodule of M is prime [7].

A non-zero submodule N of M is said to be *second* if for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero [11].

A submodule N of M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [6].

M is said to be *co-Hopfian* if every injective endomorphism f of M is an isomorphism [10].

M is said to be a *domain* if $Zd(M) = 0$, where $Zd(M)$ is the set of all zero divisors of M [3].

[1]Department of Mathematics, Faculty of Science, University of Guilan, P. O. Box 1914, Rasht, Iran e-mail: ansari@guilan.ac.ir

[2]Department of Mathematics, Faculty of Science, University of Guilan, P. O. Box 1914, Rasht, Iran, e-mail: farshadifar@guilan.ac.ir

2. Main results

Theorem 2.1. *Let M be an R-module. Then we have the following.*

(a) *If M is a multiplication module, then for each endomorphism f of M, we have $Ker(f) = (0 :_M Ann_R(M/Im(f)))$.*

(b) *If M is a comultiplication module, then for each endomorphism f of M, we have $Im(f) = Ann_R(Ker(f))M$.*

(c) *If M is a multiplication module such that $Ann_R(M)$ is a prime ideal of R, then M is a prime module.*

(d) *If M is a comultiplication module such that $Ann_R(M)$ is a prime ideal of R, then M is a second module.*

(e) *If M is a comultiplication module, N is a minimal submodule of M and X and Y are submodules of M with $X \cap N = Y \cap N = 0$, then $N \cap (X+Y) = 0$.*

Proof. (a) Since M is a multiplication module, $Im(f) = (Im(f) :_R M)M$. Thus $M/(Ker(f)) \cong (Im(f) :_R M)M$. Now since

$$Ann_R((Im(f) :_R M)M) = Ann_R(M/(0 :_M (Im(f) :_R M))),$$

we have

$$Ann_R(M/Ker(f)) = Ann_R(M/(0 :_M (Im(f) :_R M))).$$

Hence $Ker(f) = (0 :_M Ann_R(M/Im(f))$ because M is a multiplication module.

(b) Since M is a comultiplication module, $Ker(f) = (0 :_M Ann_R(Ker(f))$. Thus $M/(0 :_M Ann_R(Ker(f)) \cong Im(f)$. Now since

$$Ann_R(M/(0 :_M Ann_R(Ker(f))) = Ann_R(Ann_R(Ker(f))M),$$

we have

$$Ann_R(Im(f)) = Ann_R(Ann_R(Ker(f))M).$$

Hence $Im(f) = Ann_R(Ker(f))M$ because M is a comultiplication module.

(c) Let $r \in R$. Consider the homomorphism $f_r : M \to M$ given by $f_r(m) = rm$ for all $m \in M$. Since M is a multiplication module, there exists an ideal I of R such that $Ker(f) = IM$. Thus $M/IM \cong rM$. Hence $rI \subseteq Ann_R(M)$. Since $Ann_R(M)$ is prime, $rM = 0$ or $IM = 0$ as required.

(d) Let $r \in R$. Consider the homomorphism $f_r : M \to M$ given by $f_r(m) = rm$ for all $m \in M$. Since M is a comultiplication module, there exists an ideal I of R such that $rM = (0 :_M I)$. Thus $rI \subseteq Ann_R(M)$. Since $Ann_R(M)$ is prime, $rM = 0$ or $M = (0 :_M I)$ as desired.

(e) Let N be a minimal submodule of M and let X, Y be two submodules of M such that $N \cap Y = N \cap X = 0$. Since M is a comultiplication module, $X = (0 :_M Ann_R(X))$ and $Y = (0 :_M Ann_R(Y))$. Now $(0 :_M Ann_R(X)Ann_R(Y)) \cap$

$N = N$ or $(0 :_M Ann_R(X)Ann_R(Y)) \cap N = 0$ because N is a minimal submodule of M. In the first case, we have

$$N = (0 :_N Ann_R(X)Ann_R(Y)) = (N \cap X :_N Ann_R(Y)) = N \cap Y = 0,$$

which is a contradiction. In the second case, $N \cap (0 :_M Ann_R(X)Ann_R(Y)) = 0$ implies that $N \cap (X + Y) = 0$ because

$$(0 :_M Ann_R(X)Ann_R(Y)) \supseteq (0 :_M Ann_R(X) \cap Ann_R(Y)) = X + Y.$$

□

Proposition 2.2. Let M be an R-module. Then the following hold.

(a) If for every non-zero submodule N of M, we have that M/N is a multiplication module and $(N :_R M) \neq Ann_R(M)$, then M is a multiplication module.

(b) If every proper submodule N of M is a comultiplication module and $Ann_R(N) \neq Ann_R(M)$, then M is a comultiplication module.

(c) If R is a principal ideal ring and M is a domain such that every submodule of M is a multiplication R-module, then every homomorphic image Q of M $(Q \neq M)$ is a comultiplication $R/Ann_R(Q)$-module.

Proof. (a) Let N be a non-zero submodule of M. Set $I = (N :_R M)$. If $IM = 0$, then $I = Ann_R(M)$, which is a contradiction. Hence $IM \neq 0$. Thus, by the assumption,

$$N/IM = (N/IM :_R M/IM)(M/IM) = 0,$$

as required.

(b) Let N be a proper submodule of M. If $(0 :_M Ann_R(N)) = M$, then $Ann_R(N) = Ann_R(M)$, which is a contradiction. Hence $(0 :_M Ann_R(N)) \neq M$. Thus by assumption,

$$N = (0 :_{(0:_M Ann_R(N))} Ann_R(N)) = (0 :_M Ann_R(N))$$

as desired.

(c) Let K be a submodule of M and let N/K be a submodule of M/K. Suppose that $(x + K)Ann_R(N/K) = 0$. Then $xAnn_R(N/K) \subseteq K$. By assumption, N is a multiplication R-module. Thus $xAnn_R(N/K) \subseteq Ann_R(N/K)N$. It follows that $x \in N$ because $Ann_R(N/K)$ is a principal ideal and M is a domain. Therefore, $(0 :_{M/K} Ann_R(N/K)) \subseteq N/K$. Clearly, $N/K \subseteq (0 :_{M/K} Ann_R(N/K))$ and the proof is completed. □

Example 2.3. Let R be a principal ideal domain and I a non-zero ideal of R. Then by Proposition 2.2 (c), R/I is a quasi-Frobenius ring [9, Exercise. 24.1].

Remark 2.4. It is well known that if M is a finitely generated multiplication R-module and I, J are ideals of R such that $IM \subseteq JM$, then $I \subseteq J + Ann_R(M)$. But the dual of this fact is not true in general. For example, the $\mathbb{Z}$-module (here $\mathbb{Z}$ denotes the ring of integers) $\mathbb{Z}_{p^\infty}$ is a faithful Artinian comultiplication $\mathbb{Z}$-module such that $(0 :_{\mathbb{Z}_{p^\infty}} q\mathbb{Z}) = (0 :_{\mathbb{Z}_{p^\infty}} \mathbb{Z})$ for each prime number $q \neq p$, while $q\mathbb{Z} \neq \mathbb{Z}$. Next proposition shows that this is true for comultiplication modules under some restrictive conditions.

Proposition 2.5. Let M be a comultiplication R-module and $(0 :_M I) \subseteq (0 :_M J)$ for some ideals I and J of R. Then we have the following.

(a) $J \subseteq I$ if there exists a finitely generated multiplication submodule N of M such that $Ann_R(N) \subseteq I$.

(b) $J \subseteq I$ if $I \in Supp_R(M)$.

Proof. (a) Let N be a finitely generated multiplication submodule of M. We have $(0 :_M I) \subseteq (0 :_M J)$ implying that $(0 :_N I) \subseteq (0 :_N J)$. By [1, 3.17], N is a comultiplication R-module. Therefore, $JN \subseteq IN$. Since N is a finitely generated multiplication module, $J \subseteq I + Ann_R(N) = I$ by [8, Theorem 9].

(b) Let $I \in Supp_R(M)$. Then there exists $m \in M$ such that $Ann_R(Rm) \subseteq I$. Now the result follows from part (a) and the proof is completed. □

Recall that an ideal I of R is a *pure ideal* if $IJ = I \cap J$ for each ideal J of R.

Proposition 2.6. Let M be an R-module. Then we have the following.

(a) If R is a Noetherian ring, I is a pure ideal of R, and N is a copure submodule of M, then $(N :_M I)$ is a copure submodule of M.

(b) If M is a multiplication R-module such that for each endomorphism f of M we have $Im(f)$ is a copure submodule of M, then M is co-Hopfian.

Proof. (a) Let J be an ideal of R. We show that

$$((N :_M I) :_M J) = (N :_M I) + (0 :_M J).$$

Since R is a Noetherian ring, it is enough to show this locally. Thus we may assume that R is a local ring. Since I is a pure ideal of R, we have $I = 0$ or $I = R$. If $I = 0$, then both sides of the equality is M. If $I = R$, then the copurity of N implies that

$$((N :_M I) :_M J) = (N :_M J) = N + (0 :_M J) = (N :_M I) + (0 :_M J).$$

(b) Let f be an endomorphism of M. Then $Im(f)$ is a copure submodule of M. Set $I = Ann_R(M/Im(f))$. Then by Theorem 2.1 (a), $Ker(f) = (0 :_M I)$. Thus

$$M/Im(f) = (0 :_{M/Im(f)} I) = (Im(f) :_M I)/Im(f) =$$
$$(Im(f) + (0 :_M I))/Im(f) = (Im(f) + Ker(f))/Im(f).$$

It follows that M is co-Hopfian. □

Theorem 2.7. *Let M be a comultiplication R-module. Then the following hold.*

(a) If M is a finitely generated faithful R-module and N is a direct summand of M, then $Ann_R(N)$ is a direct summand of R.

(b) If N is a copure submodule of M such that M/N is a finitely generated R-module, then N is a direct summand of M.

Proof. (a) Let K be a submodule of M such that $M = N \oplus K$. This implies that $Ann_R(N) \cap Ann_R(K) = 0$. Since $N \cap K = 0$,

$$(0 :_M Ann_R(N) + Ann_R(K)) = 0.$$

Thus by [4, 3.4], $Ann_R(N) + Ann_R(K) = R$, as required.

(b) Since $Ann_R(N)Ann_R(M/N) \subseteq Ann_R(M)$ and M is comultiplication,

$$M = (0 :_M Ann_R(N)Ann_R(M/N)) = (N :_M Ann_R(M/N)).$$

As N is copure, it follows that

$$M = N + (0 :_M Ann_R(M/N)).$$

Now we show that this sum is direct. Since M is a comultiplication module and N is a copure submodule of M,

$$\begin{aligned} M = (Ann_R(N)M :_M Ann_R(N)) &= ((0 :_M Ann_R(Ann_R(N)M)) :_M Ann_R(N)) \\ = (N :_M Ann_R(Ann_R(N)M)) &= N + (0 :_M Ann_R(Ann_R(N)M) \\ &= N + Ann_R(N)M. \end{aligned}$$

Hence $Ann_R(N)(M/N) = M/N$. Since M/N is finitely generated,

$$Ann_R(N) + Ann_R(M/N) = R$$

by Nakayama Lemma. Therefore, $N \cap (0 :_M Ann_R(M/N)) = 0$, as desired. □

References

[1] Ansari-Toroghy, H., Farshadifar, F., The dual notion of multiplication modules. Taiwanese J. Math. 11 (4) (2007), 1189–1201.

[2] Ansari-Toroghy, H., Farshadifar, F., On endomorphisms of multiplication and comultiplication modules. Arch. Math. (Brno) 44 (2008), 9–15.

[3] Ansari-Toroghy, H., Farshadifar, F., Comultiplication modules and related results. Honam Math. J. 30 (1) (2008), 91–99.

[4] Ansari-Toroghy, H., Farshadifar, F., On comultiplication modules. Korean Ann. Math. 25 (2) (2008), 57–66.

[5] Ansari-Toroghy, H., Farshadifar, F., On multiplication and comultiplication modules. Acta Math. Sci. 31B (2) (2011), 694-700.

[6] Ansari-Toroghy, H., Farshadifar, F., Strong comultiplication modules. CMU. J. Nat. Sci. 8 (1) (2009), 105–113.

[7] Dauns, J., Prime submodules. J. Reine Angew. Math. 298 (1978), 156–181.

[8] El-Bast, Z.A., Smith, P.F., Multiplication modules. Comm. Algebra 16 (1988), 755–779.

[9] Faith, C., Algebra II: Ring theory. New York-Heidelberg-Berlin: Springer-Verlag, 1976.

[10] Lam, T.Y., Lectures on Modules and Rings, Graduate texts in Math. New York-Heidelberg-Berlin: Springer-Verlag, 1999.

[11] Yassemi, S., The dual notion of prime submodules, Arch. Math (Brno) 37 (2001), 273–278.

Received by the editors February 22, 2010