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COMMON FIXED POINT RESULT IN SYMMETRIC SPACES

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Abstract. In this note, a common fixed point theorem for weakly compatible mappings satisfying a contractive condition of integral type and property (E.A) is established in symmetric spaces which generalizes recent results of Aamri, El Moutawakil and A. Aliouche.

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1. Introduction

Hicks and Rhoades [4] established some common fixed point theorems in symmetric spaces using the fact that some of the properties of metric are not required in the proofs of certain metric theorems. Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U, B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

The following two axioms appeared in [9].

Let (X, d) be a symmetric space.

(W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.

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Commuting, weakly commuting, compatible and weakly compatible mappings have been frequently used to prove existence theorems in common fixed point theory. Recall that Jungck and Rhoades [6] defined S and T to be weakly compatible if they commute at their coincidence points, i.e. if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

There are many examples in the literature verifying that for metric spaces, commuting implies weakly commuting implies compatible implies weakly compatible maps but the converse need not be true. (See [5] and [8]).

Aamri et al. [2] have established some common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying the property (E.A). Again recall that the pair (S, T) satisfies property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

Recently, Aliouche [3] proved a common fixed point theorem for self-mappings in a symmetric space under a contractive condition of integrals and satisfying a new property introduced recently in [2].

The main objective of this paper is to prove a common fixed point theorem for weakly compatible mappings in the setting of a symmetric space satisfying an integral type contractive condition and property (E.A). This Theorem generalizes the results of Aamri et al. [1, 2] and A. Aliouche [3].

2. Preliminaries

In the sequel we need a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions $0 < \phi(t) < t$ for each $t > 0$ and $\phi(0) = 0$. For example, $\phi(t) = t^2$ for each $t \in (0, 1)$. Note that, in these assumptions, ϕ is right continuous at $t = 0$.

Definition 2.1. [3, Def.4] Let (X, d) be a symmetric space. We say that (X, d) satisfies property (HE), if given $\{x_n\}$, $\{y_n\}$ and x in X,

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \lim_{n\to\infty} d(y_n, x) = 0 \text{ imply } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

For symmetric spaces, the following chain of implications is valid: commuting implies weakly commuting maps, compatible implies weakly compatible maps, but we must add hypothesis (HE) for every weakly commuting pair to be compatible. Indeed, suppose that (S, T) is weakly commuting, then $d(TSx, STx) \leq d(Sx, Tx)$, $\forall x \in X$. For a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \in X$, we get $d(TSx_n, STx_n) \leq d(Sx_n, Tx_n)$, $\forall n \in \mathbb{N}$. Since $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$ and (HE), then $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$, and, in consequence,

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0.$$

Note that condition (HE) is trivially valid in metric spaces.

3. Main Results

Now we state and prove our main result.

Theorem 3.1. *Let d be a symmetric for X which satisfies* (W.3), (W.4) *and (HE). Let A, B, S and T be self mappings of X such that*

$$A(X) \subseteq T(X) \quad \text{and} \quad B(X) \subseteq S(X), \tag{3.1}$$

$$\int_0^{d(Ax,By)} \varphi(t)dt \le \phi\Big(\int_0^{aL(x,y)+(1-a)M(x,y)} \varphi(t)dt\Big), \tag{3.2}$$

for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad \text{for all} \quad \epsilon > 0, \tag{3.3}$$

$$\begin{aligned}
L(x,y) &= \min\ \{L_1(x,y), L_2(x,y)\},\\
L_1(x,y) &= \max\{d(Sx,Ty), d(Sx,By), d(By,Ty)\},\\
L_2(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ax,Ty)\},\\
M(x,y) &= \min\ \{M_1(x,y), M_2(x,y)\},\\
M_1(x,y) &= \big[\max\{d^2(Sx,Ty), d(Sx,By)d(By,Ty), d(Sx,Ty)d(Sx,By),\\
&\qquad d(Sx,Ty)d(By,Ty), d^2(By,Ty)\}\big]^{1/2}\\
M_2(x,y) &= \big[\max\{d^2(Sx,Ty), d(Ax,Ty)d(Ax,Sx), d(Sx,Ty)d(Ax,Ty),\\
&\qquad d(Sx,Ty)d(Ax,Sx), d^2(Ax,Sx)\}\big]^{1/2}
\end{aligned}$$

and $0 \le a \le 1$. Suppose that (A,S) or (B,T) satisfies the property (E.A) and (A,S) and (B,T) are weakly compatible. If one of the subspaces $A(X)$, $B(X)$, $S(X)$ and $T(X)$ of X is complete, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that (B,T) satisfies the property (E.A), then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Bx_n, z) = \lim_{n\to\infty} d(Tx_n, z) = 0 \quad \text{for some} \quad z \in X.$$

Therefore, by the property (HE), we have

$$\lim_{n\to\infty} d(Bx_n, Tx_n) = 0.$$

Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence

$$\lim_{n\to\infty} d(Sy_n, z) = 0.$$

Now we show that $\lim_{n\to\infty} d(Ay_n, z) = 0$. For that, suppose

$$\limsup_{n\to\infty}\ d(Ay_n, Bx_n) = p\ \neq\ 0,$$

or

$$\limsup_{n\to\infty}\ d(Ay_n, Bx_n) = +\infty.$$

Using (3.2), we have

$$\int_0^{d(Ay_n,Bx_n)} \varphi(t)dt \le \phi\Big(\int_0^{aL(y_n,x_n)+(1-a)M(y_n,x_n)} \varphi(t)dt\Big), \tag{3.4}$$

where

$$\begin{aligned}
L(y_n,x_n) &= \min\ \{L_1(y_n,x_n), L_2(y_n,x_n)\},\\
L_1(y_n,x_n) &= \max\{d(Sy_n,Tx_n), d(Sy_n,Bx_n), d(Bx_n,Tx_n)\} = d(Bx_n,Tx_n),\\
L_2(y_n,x_n) &= \max\{d(Sy_n,Tx_n), d(Sy_n,Ay_n), d(Ay_n,Tx_n)\}\\
&= \max\{d(Bx_n,Tx_n), d(Bx_n,Ay_n), d(Ay_n,Tx_n)\},
\end{aligned}$$

which implies

$$\begin{aligned}
L(y_n,x_n) &= \min\ \{d(Bx_n,Tx_n),\\
&\quad \max\{d(Bx_n,Tx_n), d(Bx_n,Ay_n), d(Ay_n,Tx_n)\}\}\\
&= d(Bx_n,Tx_n),\\
M(y_n,x_n) &= \min\ \{M_1(y_n,x_n), M_2(y_n,x_n)\},\\
M_1(y_n,x_n) &= \big[\max\{d^2(Sy_n,Tx_n), d(Sy_n,Bx_n)d(Bx_n,Tx_n),\\
&\quad d(Sy_n,Tx_n)d(Sy_n,Bx_n),\\
&\quad d(Sy_n,Tx_n)d(Bx_n,Tx_n), d^2(Bx_n,Tx_n)\}\big]^{1/2}\\
&= d(Bx_n,Tx_n).\\
M_2(y_n,x_n) &= \big[\max\{d^2(Sy_n,Tx_n), d(Ay_n,Tx_n)d(Ay_n,Sy_n),\\
&\quad d(Sy_n,Tx_n)d(Ay_n,Tx_n),\\
&\quad d(Sy_n,Tx_n)d(Ay_n,Sy_n), d^2(Ay_n,Sy_n)\}\big]^{1/2}\\
&= \big[\max\{d^2(Bx_n,Tx_n), d(Ay_n,Tx_n)d(Ay_n,Bx_n),\\
&\quad d(Bx_n,Tx_n)d(Ay_n,Tx_n),\\
&\quad d(Bx_n,Tx_n)d(Ay_n,Bx_n), d^2(Ay_n,Bx_n)\}\big]^{1/2},
\end{aligned}$$

hence

$$M(y_n,x_n) = d(Bx_n,Tx_n).$$

Thus (3.4) becomes

$$\begin{aligned}\int_0^{d(Ay_n,Bx_n)}\varphi(t)dt &\le \phi\Big(\int_0^{ad(Bx_n,Tx_n)+(1-a)d(Bx_n,Tx_n)}\varphi(t)dt\Big)\\ &= \phi\Big(\int_0^{d(Bx_n,Tx_n)}\varphi(t)dt\Big)\\ &\le \int_0^{d(Bx_n,Tx_n)}\varphi(t)dt.\end{aligned}$$

Using (3.3), we get $\limsup_{n\to\infty}\int_0^{d(Ay_n,Bx_n)}\varphi(t)dt > 0$, and then

$$\limsup_{n\to\infty}\int_0^{d(Bx_n,Tx_n)}\varphi(t)dt > 0,$$

which is a contradiction. Therefore $\limsup_{n\to\infty} d(Ay_n,Bx_n)=0$ and

$$\lim_{n\to\infty} d(Ay_n,Bx_n)=0.$$

We deduce, by (W.4), that $\lim_{n\to\infty} d(Ay_n,z)=0$.

Now, suppose that $S(X)$ is a complete subspace of X. Then $z=Su$, for some $u\in X$. Consequently, we have

$$\lim_{n\to\infty} d(Ay_n,Su)=\lim_{n\to\infty} d(Bx_n,Su)=\lim_{n\to\infty} d(Tx_n,Su)=\lim_{n\to\infty} d(Sy_n,Su)=0.$$

If $Au\neq z$, using (3.2) we get

$$\begin{aligned}\int_0^{d(Au,Bx_n)}\varphi(t)dt &\le \phi\Big(\int_0^{aL(u,x_n)+(1-a)M(u,x_n)}\varphi(t)dt\Big)\\ &\le \int_0^{aL(u,x_n)+(1-a)M(u,x_n)}\varphi(t)dt,\end{aligned}\tag{3.5}$$

where

$$L(u,x_n)\le L_1(u,x_n)=\max\{d(Su,Tx_n),d(Su,Bx_n),d(Bx_n,Tx_n)\},$$

and

$$\begin{aligned}M(u,x_n)\le &M_1(u,x_n)=\big[\max\{d^2(Su,Tx_n),d(Su,Bx_n)d(Bx_n,Tx_n),\\ &d(Su,Tx_n)d(Su,Bx_n),d(Su,Tx_n)d(Bx_n,Tx_n),d^2(Bx_n,Tx_n)\}\big]^{1/2}.\end{aligned}$$

Taking $n\to\infty$, we get $\lim_{n\to\infty} L(u,x_n)=0$ and $\lim_{n\to\infty} M(u,x_n)=0$, respectively.
Hence (3.5) provides

$$\lim_{n\to\infty}\int_0^{d(Au,Bx_n)}\varphi(t)dt=0$$

and (3.3) implies that $\lim_{n\to\infty} d(Au, Bx_n) = 0$. By (W.3), we have $z = Au$, which is a contradiction. In consequence. $z = Au = Su$.

Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Au = Tv$.
If $Bv \neq z$, using (3.2), we have

$$\begin{aligned}
(3.6) \qquad \int_0^{d(z,Bv)} \varphi(t)dt =& \int_0^{d(Au,Bv)} \varphi(t)dt \\
\leq& \phi\Big(\int_0^{aL(u,v)+(1-a)M(u,v)} \varphi(t)dt\Big),
\end{aligned}$$

where $L(u,v) \leq d(z,Bv)$, and $M(u,v) \leq \left[d^2(z,Bv)\right]^{1/2}$, respectively.

In fact, $L(u,v) = 0$, and $M(u,v) = 0$. Indeed,

$$\begin{aligned}
L_1(u,v) =& \max\{d(Su,Tv), d(Su,Bv), d(Bv,Tv)\} = d(z,Bv), \\
L_2(u,v) =& \max\{d(Su,Tv), d(Su,Au), d(Au,Tv)\} = 0,
\end{aligned}$$

hence

$$L(u,v) = \min\ \{L_1(u,v), L_2(u,v)\} = 0,$$

and

$$\begin{aligned}
M_1(u,v) =& \Big[\max\{d^2(Su,Tv), d(Su,Bv)d(Bv,Tv), d(Su,Tv)d(Su,Bv), \\
& d(Su,Tv)d(Bv,Tv), d^2(Bv,Tv)\}\Big]^{1/2} \\
=& \Big[\max\{d^2(z,z), d(z,Bv)d(Bv,z), d(z,z)d(z,Bv), \\
& d(z,z)d(Bv,z), d^2(Bv,z)\}\Big]^{1/2}, \\
M_2(u,v) =& \Big[\max\{d^2(Su,Tv), d(Au,Tv)d(Au,Su), d(Su,Tv)d(Au,Tv), \\
& d(Su,Tv)d(Au,Su), d^2(Au,Su)\}\Big]^{1/2} = 0, \\
M(u,v) =& \min\ \{M_1(u,v), M_2(u,v)\} = 0.
\end{aligned}$$

Thus (3.6) becomes

$$\int_0^{d(z,Bv)} \varphi(t)dt \leq \phi\Big(\int_0^0 \varphi(t)dt\Big) = \phi(0) = 0,$$

which is a contradiction. Hence $z = Bv = Tv$.

The pair (A,S) is weakly compatible, so that $ASu = SAu$ since $Au = Su$, which implies $Az = Sz$.

Let us show that z is a common fixed point of A, B, S and T. If $z \neq Az$, using (3.2), we get

$$\begin{aligned}
(3.7) \qquad \int_0^{d(z,Az)} \varphi(t)dt =& \int_0^{d(Az,Bv)} \varphi(t)dt \\
\leq& \phi\Big(\int_0^{aL(z,v)+(1-a)M(z,v)} \varphi(t)dt\Big),
\end{aligned}$$

where

$$\begin{aligned}
L_1(z,v) =& \max\{d(Sz,Tv), d(Sz,Bv), d(Bv,Tv)\} = d(z,Az),\\
L_2(z,v) =& \max\{d(Sz,Tv), d(Sz,Az), d(Az,Tv)\} = d(z,Az),\\
L(z,v) =& d(z,Az),\\
&\text{and}\\
M_1(z,v) =& \big[\max\{d^2(Sz,Tv), d(Sz,Bv)d(Bv,Tv), d(Sz,Tv)d(Sz,Bv),\\
& d(Sz,Tv)d(Bv,Tv), d^2(Bv,Tv)\}\big]^{1/2}\\
=& \big[\max\{d^2(Az,z), d(Az,z)d(z,z), d(Az,z)d(Az,z),\\
& d(Az,z)d(z,z), d^2(z,z)\}\big]^{1/2}\\
=& d(Az,z),\\
M_2(z,v) =& \big[\max\{d^2(Sz,Tv), d(Az,Tv)d(Az,Sz), d(Sz,Tv)d(Az,Tv),\\
& d(Sz,Tv)d(Az,Sz), d^2(Az,Sz)\}\big]^{1/2}\\
=& \big[\max\{d^2(Az,z), d(Az,z)d(Az,Az), d(Az,z)d(Az,z),\\
& d(Az,z)d(Az,Az), d^2(Az,Az)\}\big]^{1/2}\\
=& d(Az,z),
\end{aligned}$$

hence $M(z,v) = \big[d^2(z,Az)\big]^{1/2}$.
Thus, using that $\int_0^{d(z,Az)} \varphi(t)dt > 0$ (by (3.3), then (3.7) becomes

$$\begin{aligned}
\int_0^{d(Az,z)} \varphi(t)dt \le \phi\Big(&\int_0^{ad(Az,z)+(1-a)d(Az,z)} \varphi(t)dt\Big)\\
&= \phi\Big(\int_0^{d(Az,z)} \varphi(t)dt\Big) < \int_0^{d(z,Az)} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Therefore $z = Az = Sz$.

Similarly, weak compatibility of B and T implies $BTv = TBv$, i.e., $Bz = Tz$.

If $z \neq Bz$, by using (3.2) and (3.3), a similar calculation as above yields $z = Bz = Tz$. Thus, z is a common fixed point of A, B, S and T.

For the uniqueness of the common fixed point z, let $w \neq z$ be another common fixed point of A, B, S and T. Then, using (3.2) and (3.3), and

$$\begin{aligned}
L_1(z,w) =& \max\{d(Sz,Tw), d(Sz,Bw), d(Bw,Tw)\} = d(z,w),\\
L_2(z,w) =& \max\{d(Sz,Tw), d(Sz,Az), d(Az,Tw)\}\} = d(z,w),\\
L(z,w) =& d(z,w),\\
M_1(z,w) =& \big[\max\{d^2(Sz,Tw), d(Sz,Bw)d(Bw,Tw), d(Sz,Tw)d(Sz,Bw),\\
& d(Sz,Tw)d(Bw,Tw), d^2(Bw,Tw)\}\big]^{1/2} = d(z,w),\\
M_2(z,w) =& \big[\max\{d^2(Sz,Tw), d(Az,Tw)d(Az,Sz), d(Sz,Tw)d(Az,Tw),\\
& d(Sz,Tw)d(Az,Sz), d^2(Az,Sz)\}\big]^{1/2} = d(z,w),\\
M(z,w) =& d(z,w),
\end{aligned}$$

we obtain $\int_0^{d(z,w)} \varphi(t)dt > 0$ and

$$\begin{aligned}
\int_0^{d(z,w)} \varphi(t)dt = & \int_0^{d(Az,Bw)} \varphi(t)dt\\
& \leq \phi\Big(\int_0^{aL(z,w)+(1-a)M(z,w)} \varphi(t)dt\Big)\\
& = \phi\Big(\int_0^{ad(z,w)+(1-a)d(z,w)} \varphi(t)dt\Big)\\
& = \phi\Big(\int_0^{d(z,w)} \varphi(t)dt\Big)\\
& < \int_0^{d(z,w)} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Therefore $z = w$.

When $T(X)$ is assumed to be a complete subspace of X, then $z = Tu$, for some $u \in X$. Consequently, we have

$$\lim_{n\to\infty} d(Ay_n, Tu) = \lim_{n\to\infty} d(Bx_n, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(Sy_n, Tu) = 0.$$

We check that $Bu = z$. If $Bu \neq z$, using (3.2) we get

$$\begin{aligned}
\int_0^{d(Ay_n,Bu)} \varphi(t)dt \leq & \phi\Big(\int_0^{aL(y_n,u)+(1-a)M(y_n,u)} \varphi(t)dt\Big)\\
\leq & \int_0^{aL(y_n,u)+(1-a)M(y_n,u)} \varphi(t)dt,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
L_1(y_n,u) =& \max\{d(Sy_n,Tu), d(Sy_n,Bu), d(Bu,Tu)\},\\
=& \max\{d(Bx_n,z), d(Bx_n,Bu), d(Bu,z)\},\\
L_2(y_n,u) =& \max\{d(Sy_n,Tu), d(Sy_n,Ay_n), d(Ay_n,Tu)\}\\
=& \max\{d(Bx_n,z), d(Bx_n,Ay_n), d(Ay_n,z)\},
\end{aligned}$$

hence $L(y_n, u) \leq L_2(y_n, u)$, yielding $\lim_{n\to\infty} L(y_n, u) = 0$, and

$$\begin{aligned} M_1(y_n, u) =& \big[\max\{d^2(Bx_n, z), d(Bx_n, Bu)d(Bu, z), d(Bx_n, z)d(Bx_n, Bu), \\ & d(Bx_n, z)d(Bu, z), d^2(Bu, z)\}\big]^{1/2}, \\ M_2(y_n, u) =& \big[\max\{d^2(Bx_n, z), d(Ay_n, z)d(Ay_n, Bx_n), d(Bx_n, z)d(Ay_n, z), \\ & d(Bx_n, z)d(Ay_n, Bx_n), d^2(Ay_n, Bx_n)\}\big]^{1/2}, \end{aligned}$$

hence $M(y_n, u) \leq M_2(y_n, u)$ and $\lim_{n\to\infty} M(y_n, u) = 0$.

Hence (3.8) provides

$$\lim_{n\to\infty} \int_0^{d(Ay_n, Bu)} \varphi(t)dt = 0$$

and (3.3) implies that $\lim_{n\to\infty} d(Ay_n, Bu) = 0$. By (W.3), we have $z = Bu$, which is a contradiction. In consequence. $z = Bu = Tu$.

The remaining part of the proof is similar to the case where $S(X)$ is complete.

On the other hand, by (3.1), the cases in which $A(X)$ or $B(X)$ is a complete subspace of X are similar to the cases in which $T(X)$ or $S(X)$ is complete, respectively.

The same procedure is valid for the case where (A, S) satisfies the property (E.A).

Note that the implicit relation can be relaxed if we restrict the hypotheses on the mappings A, B, T and S. For instance, if (B, T) satisfies (E.A) and $S(X)$ or $B(X)$ is a complete subspace of X, then we can use the implicit relation

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \phi\Big(\int_0^{aL_1(x,y)+(1-a)M_1(x,y)} \varphi(t)dt\Big),$$

for all $x, y \in X$, where $L_1(x, y)$ and $M_1(x, y)$ are given in the statement of Theorem 3.1, obtaining the following result.

Theorem 3.2. *Let d be a symmetric for X which satisfies* (W.3), (W.4) *and (HE). Let A, B, S and T be self-mappings of X such that*

$$A(X) \subseteq T(X) \quad \textit{and} \quad B(X) \subseteq S(X), \tag{3.9}$$

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \phi\Big(\int_0^{aL(x,y)+(1-a)M(x,y)} \varphi(t)dt\Big), \tag{3.10}$$

for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad \textit{for all} \quad \epsilon > 0, \tag{3.11}$$

$$\begin{aligned} L(x,y) =& \max\{d(Sx,Ty), d(Sx,By), d(By,Ty)\},\\ M(x,y) =& \big[\max\{d^2(Sx,Ty), d(Sx,By)d(By,Ty), d(Sx,Ty)d(Sx,By),\\ & d(Sx,Ty)d(By,Ty), d^2(By,Ty)\}\big]^{1/2} \end{aligned}$$

and $0 \le a \le 1$. *Suppose that* (B,T) *satisfies the property (E.A) and* (A,S) *and* (B,T) *are weakly compatible. If one of the subspaces* $B(X)$ *and* $S(X)$ *of* X *is complete, then* A, B, S *and* T *have a unique common fixed point in* X.

Remark 3.3. If (B,T) satisfies the property (E.A) and one of the subspaces $A(X)$, $T(X)$ of X is complete, we can use the same implicit as in Theorem 3.2, adding the following hypotheses: ϕ continuous, and

$$d(x_n,z) \to 0 \text{ as } n \to \infty \Longrightarrow d(x_n,v) \to d(z,v) \text{ as } n \to \infty.$$

Note that this last condition is not deduced from (W.3), (W.4) or (HE).

Remark 3.4. On the other hand, if (A,S) satisfies (E.A), and one of the subspaces $A(X)$ or $T(X)$ of X is complete, then we can assume the implicit relation

$$\int_0^{d(Ax,By)} \varphi(t)dt \le \phi\Big(\int_0^{aL(x,y)+(1-a)M(x,y)} \varphi(t)dt\Big), \tag{3.12}$$

for all $x, y \in X$, where $L(x,y) = \max\{d(Sx,Ty), d(Sx,Ax), d(Ax,Ty)\}$, and

$$\begin{aligned} M(x,y) =& \big[\max\{d^2(Sx,Ty), d(Ax,Ty)d(Ax,Sx), d(Sx,Ty)d(Ax,Ty),\\ & d(Sx,Ty)d(Ax,Sx), d^2(Ax,Sx)\}\big]^{1/2} \end{aligned}$$

In this case, if one of the subspaces $B(X)$ and $S(X)$ of X is complete, the implicit relation (3.12) can be assumed considering ϕ continuous, and

$$d(x_n,z) \to 0 \text{ as } n \to \infty \Longrightarrow d(x_n,v) \to d(z,v) \text{ as } n \to \infty.$$

Remark 3.5. If $a = 1$ in Theorem 3.1, we obtain

$$\int_0^{d(Ax,By)} \varphi(t)dt \le \phi\Big(\int_0^{L(x,y)} \varphi(t)dt\Big), \tag{3.13}$$

for all $x, y \in X$, where

$$L(x,y) = \min\ \{L_1(x,y), L_2(x,y)\},$$

$$L_1(x,y) = \max\{d(Sx,Ty), d(Sx,By), d(By,Ty)\},$$

$$L_2(x,y) = \max\{d(Sx,Ty), d(Sx,Ax), d(Ax,Ty)\}.$$

In case (B,T) satisfies (E.A) and $S(X)$ or $B(X)$ is a complete subspace of X, then we can take $L(x,y) = L_1(x,y)$ and get the implicit relation of Theorem

1 of Aliouche [3]. Note that, in Theorem 1 [3], the condition $\phi(0)=0$ is needed. This fact can be clearly deduced from the assertion

$$\limsup_{n\to\infty}\phi\Big(\int_0^{d(Bx_n,Tx_n)}\varphi(t)dt\Big)\le\limsup_{n\to\infty}\int_0^{d(Bx_n,Tx_n)}\varphi(t)dt,$$

to cover the case where $B=T$, or $Bx_n=Tx_n$, for n large enough.

Theorem 3.1 can be extended for the case where φ is summable on each compact interval of $\mathbb{R}_+$, making some minor changes in the proof.

Besides, the same remarks concerning Theorem 3.1 are applicable to the following Corollaries.

For $\varphi(t)=1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.6. *Let d be a symmetric for X which satisfies (W.3), (W.4) and (HE). Let A, B, S and T be self mappings of X such that*

$$A(X)\subseteq T(X)\quad\textit{and}\quad B(X)\subseteq S(X), \tag{3.14}$$

$$d(Ax,By)\le\phi\big(aL(x,y)+(1-a)M(x,y)\big) \tag{3.15}$$

for all $x,y\in X$, where

$$\begin{aligned}
L(x,y)&=\min\{L_1(x,y),L_2(x,y)\},\\
L_1(x,y)&=\max\{d(Sx,Ty),d(Sx,By),d(By,Ty)\},\\
L_2(x,y)&=\max\{d(Sx,Ty),d(Sx,Ax),d(Ax,Ty)\},\\
M(x,y)&=\min\{M_1(x,y),M_2(x,y)\},\\
M_1(x,y)&=\big[\max\{d^2(Sx,Ty),d(Sx,By)d(By,Ty),d(Sx,Ty)d(Sx,By),\\
&\qquad d(Sx,Ty)d(By,Ty),d^2(By,Ty)\}\big]^{1/2},\\
M_2(x,y)&=\big[\max\{d^2(Sx,Ty),d(Ax,Ty)d(Ax,Sx),d(Sx,Ty)d(Ax,Ty),\\
&\qquad d(Sx,Ty)d(Ax,Sx),d^2(Ax,Sx)\}\big]^{1/2}
\end{aligned}$$

and $0\le a\le 1$. Suppose that (A,S) or (B,T) satisfies the property (E.A) and (A,S) and (B,T) are weakly compatible. If one of the subspaces $A(X)$, $B(X)$, $S(X)$ and $T(X)$ of X is complete, then A, B, S and T have a unique common fixed point in X.

Remark 3.7. If $a=1$ in Corollary 3.6, we obtain the implicit relation

$$d(Ax,By)\le\phi\big(\min\ \{L_1(x,y),L_2(x,y)\}\big), \tag{3.16}$$

for all $x,y\in X$, where

$$\begin{aligned}
L_1(x,y)&=\max\{d(Sx,Ty),d(Sx,By),d(By,Ty)\},\\
L_2(x,y)&=\max\{d(Sx,Ty),d(Sx,Ax),d(Ax,Ty)\},
\end{aligned}$$

which is related to Theorem 2.2 of [1].

Again, for $B = A$ and $T = S$ in Theorem 3.1, we get the following corollary.

Corollary 3.8. *Let d be a symmetric for X which satisfies* (W.3) *and (HE). Let A and S be self mappings of X such that*

$$A(X) \subseteq S(X), \tag{3.17}$$

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \leq \phi\Big(\int_0^{aL(x,y)+(1-a)M(x,y)} \varphi(t)dt\Big) \tag{3.18}$$

for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable (or summable on each compact interval), non-negative and satisfying (3.3)*,*

$$\begin{aligned}
L(x,y) &= \min\ \{L_1(x,y), L_2(x,y)\},\\
L_1(x,y) &= \max\{d(Sx,Sy), d(Sx,Ay), d(Ay,Sy)\},\\
L_2(x,y) &= \max\{d(Sx,Sy), d(Sx,Ax), d(Ax,Sy)\},\\
M(x,y) &= \min\ \{M_1(x,y), M_2(x,y)\},\\
M_1(x,y) &= \big[\max\{d^2(Sx,Sy), d(Sx,Ay)d(Ay,Sy), d(Sx,Sy)d(Sx,Ay),\\
&\qquad d(Sx,Sy)d(Ay,Sy), d^2(Ay,Sy)\}\big]^{1/2},\\
M_2(x,y) &= \big[\max\{d^2(Sx,Sy), d(Ax,Sy)d(Ax,Sx), d(Sx,Sy)d(Ax,Sy),\\
&\qquad d(Sx,Sy)d(Ax,Sx), d^2(Ax,Sx)\}\big]^{1/2}
\end{aligned}$$

and $0 \leq a \leq 1$. Suppose that (A, S) satisfies property (E.A) and (A, S) is weakly compatible. If one of the subspaces $A(X)$ and $S(X)$ of X is complete, then A and S have a unique common fixed point in X.

Note that, following the lines of the proof of Theorem 3.1, it can be proved that, if $B = A$, $T = S$, hypothesis (W.4) can be avoided.

Again, for $\varphi(t) = 1$ in Corollary 3.8, we obtain the following corollary.

Corollary 3.9. *Let d be a symmetric for X which satisfies (W.3) and (HE). Let A and S be self-mappings of X such that*

$$A(X) \subseteq S(X), \tag{3.19}$$

$$d(Ax, Ay) \leq \phi\big(aL(x,y) + (1-a)M(x,y)\big) \tag{3.20}$$

for all $x, y \in X$, where $L(x, y)$ and $M(x, y)$ are given in Corollary 3.8 and $0 \leq a \leq 1$. Suppose that (A, S) satisfies the property (E.A) and (A, S) is weakly compatible. If one of the subspaces $A(X)$ and $S(X)$ of X is complete, then A and S have a unique common fixed point in X.

Remark 3.10. Compare the result obtained taking $a = 1$ in Corollary 3.8, with Theorem 2.1 of [1] and Corollary 1 [3].

Since noncompatibility of two mappings in a symmetric space implies that they satisfy the property (E.A), we get the following corollary.

Corollary 3.11. *Let d be a symmetric for X which satisfies* (W.3), (W.4) *and (HE). Let A, B, S and T be self-mappings of X satisfying* (3.1) *and* (3.2) *for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable (or summable on each compact interval), non-negative and satisfying* (3.3)*. Suppose that (A, S) or (B, T) is noncompatible and (A, S) and (B, T) are weakly compatible. If one of the subspaces $A(X)$, $B(X)$, $S(X)$ and $T(X)$ of X is complete, then A, B, S and T have a unique common fixed point in X.*

Remark 3.12. If $\varphi(t) = 1$ and $a = 1$ in Theorem 3.1, or $a = 1$ in Corollary 3.6, we obtain a result similar to Theorem 2 in [2] for symmetric spaces. Note that, in Theorem 2 [2], ϕ is assumed to satisfy $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, ϕ nondecreasing and $0 < \phi(t) < t$, for every $t > 0$. Under these assumptions, it is easy to check that $\phi(0) = 0$.

Recently, Y. Liu et al. [7] defined a common property (E.A) as follows:

Definition 3.13. Let $A, B, S, T : X \to X$. The pairs (A, S) and (B, T) satisfy a common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

If $B = A$ and $S = T$ in the definition above, we obtain the definition of property (E.A).

Example 3.14. Let A, B, S and T be self-maps on $X = [0, 1]$, with the usual metric $d(x, y) = |x - y|$, defined by:

$$Ax = \begin{cases} 1 - \frac{x}{2}, & \text{when } x \in [0, \frac{1}{2}), \\ 1, & \text{when } x \in [\frac{1}{2}, 1]. \end{cases}$$

$$Sx = \begin{cases} 1 - 2x, & \text{when } x \in [0, \frac{1}{2}), \\ 1, & \text{when } x \in [\frac{1}{2}, 1]. \end{cases}$$

$Bx = 1 - x$ and $Tx = 1 - \frac{x}{3}$, $\forall\, x \in X$.
Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by $x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{n^2+1}$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in X.$$

Thus, a common (E.A) property is satisfied by A, B, S and T.

Now, we prove a common fixed point Theorem using a common property (E.A).

Theorem 3.15. *Let d be a symmetric for X which satisfies (W.3) and (HE). Let A, B, S, and T be self mappings of (X, d) satisfying* (3.10) *and* (3.11)*. Suppose that (A, S) and (B, T) satisfy a common property (E.A), $S(X)$ and $T(X)$ are closed subspaces of X and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.*

Proof. Since (A, S) and (B, T) satisfy a common property (E.A), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$. Using that $S(X)$ and $T(X)$ are closed subspaces of X, then, $z = Su = Tv$ for some $u, v \in X$.
If $Au \neq z$, using (3.10), we get

$$\int_0^{d(Au,By_n)} \varphi(t)dt \leq \phi\Big(\int_0^{aL(u,y_n)+(1-a)M(u,y_n)} \varphi(t)dt\Big), \tag{3.21}$$

where

$$L(u, y_n) = \max\{d(Su, Ty_n), d(Su, By_n), d(By_n, Ty_n)\},$$

and

$$\begin{aligned} M(u, y_n) =\big[&\max\{d^2(Su, Ty_n), d(Su, By_n)d(By_n, Ty_n),\\ &d(Su, Ty_n)d(Su, By_n), d(Su, Ty_n)d(By_n, Ty_n), d^2(By_n, Ty_n)\}\big]^{\frac{1}{2}}. \end{aligned}$$

Taking the limit as $n \to \infty$ and using (HE), we get $\lim_{n\to\infty} L(u, y_n) = 0$ and $\lim_{n\to\infty} M(u, y_n) = 0$, respectively.

Hence (3.21) provides

$$\lim_{n\to\infty} \int_0^{d(Au,By_n)} \varphi(t)dt = 0$$

and (3.11) implies that

$$\lim_{n\to\infty} d(Au, By_n) = 0.$$

By (W.3), we have $z = Au$, which is a contradiction. In consequence, $z = Au = Su$.

On the other hand, if $Bv \neq z$, using (3.10), we get that

$$\begin{aligned} \int_0^{d(z,Bv)} \varphi(t)dt &= \int_0^{d(Au,Bv)} \varphi(t)dt \\ &\leq \phi\Big(\int_0^{aL(u,v)+(1-a)M(u,v)} \varphi(t)dt\Big), \end{aligned} \tag{3.22}$$

where $L(u, v) = d(z, Bv)$, and $M(u, v) = \big[d^2(z, Bv)\big]^{1/2}$, respectively.

Hence (3.22) becomes

$$\begin{aligned} \int_0^{d(z,Bv)} \varphi(t)dt &\leq \phi\Big(\int_0^{ad(z,Bv)+(1-a)d(z,Bv)} \varphi(t)dt\Big) \\ &= \phi\Big(\int_0^{d(z,Bv)} \varphi(t)dt\Big) \\ &< \int_0^{d(z,Bv)} \varphi(t)dt, \end{aligned}$$

which is a contradiction. This provides that $z = Bv = Tv$.

The rest of the proof follows as in Theorem 3.1 using L_1 and M_1.

Obviously, we can replace the hypothesis: $S(X)$ and $T(X)$ are closed subspaces of X by: $A(X)$ and $B(X)$ are closed subspaces of X.

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