Novi Sad J. Math.
Vol. 41, No. 1, 2011, 85-121

GENERALIZED SOLUTIONS TO A SINGULAR NONLINEAR CAUCHY PROBLEM

Victor Dévoué[1]

Abstract. Using regularization techniques, we give a meaning to a singular, strong non-linear Cauchy problem by replacing it by a three-parameter family of Lipschitz, non-characteristic, regular problems in an appropriate algebra of genaralized functions. We prove existence and uniqueness of the solution and we specify how it depends on the choices made.

AMS Mathematics Subject Classification (2000): Primary 35D99, 35L05, 35L70; Secondary 46F30

Key words and phrases: Nonlinear partial differential equations, characteristic Cauchy problem, algebra of genaralized functions; regularization of problems.

1. Introduction

The main purpose of this paper is to establish the existence of a global solution to the non-Lipschitz Cauchy problem formally written as

$$(P) \qquad \frac{\partial^2 u}{\partial x \partial t} = F(.\ ,\ ., u')$$

where $u' = \dfrac{\partial u}{\partial t}$, with a non-Lipschitz nonlinear function F on the right-hand side and irregular data as distributions or generalized functions. For example: $\dfrac{\partial^2 u}{\partial x \partial t} = -\left|u'\right|^p u' + f$ with $p > 0$. So we tackle the example of a hyperbolic nonlinear equation given by J. L. Lions in [18], pages 38 to 43, in another way: we take the equation in canonical form, on the right-hand side the function is non-Lipschitz nonlinear but not specified, the data are irregular and the problem may be characteristic.

To give a meaning to this problem we use the recent theories of generalized functions ([2], [3], [4],[5], [15], [17], [25], [25], [26], [27]) and particularly the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras of J.-A. Marti (see [19]- [21], [22], [24]). The $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras give an efficient algebraic framework which permits a precise study of solutions as in [8], [13], [14], [22],[23], [24]. We investigate solutions with distributions or other generalized functions as initial data; thus we must search for solutions

[1]Equipe Analyse Algébrique Non Linéaire, Université des Antilles et de la Guyane, Laboratoire Ceregmia, Campus de Schoelcher. BP 7209, 97275 Schoelcher Cedex, Martinique. F.W.I., e-mail: devoue-vi@orange.fr

in the algebras which are invariant under nonlinear functions and contain the space of distributions.

To do this, we study the nonlinear problem formally written

$$(P_{form})\left\{\begin{array}{l}\dfrac{\partial^2 u}{\partial x \partial t}=F(\ .\ ,\ .\ ,u'),\\ u|_{\gamma}=\varphi,\\ u'|_{\gamma}=\psi.\end{array}\right.$$

The notation $F(\ .\ ,\ .\ ,u')$ extends, with a meaning to be defined later, the expression $(x,t)\mapsto F(x,t,u'(x,t))$ in the case where u is a generalized function of two variables x and t. Here φ and ψ are one-variable generalized functions. The data are given along a characteristic curve γ of the equation $t=f(x)$.

This ill-posed problem remains unsolvable in classical function spaces. To overcome this difficulty, by means of regularizations, we associate to the problem (P_{form}) a generalized one (P_{gen}) well formulated in a convenient algebra $\mathcal{A}\left(\mathbb{R}^2\right)$. We search for a generalized solution u in $\mathcal{A}\left(\mathbb{R}^2\right)$.

For equation (P), the characteristic Cauchy problem is ill-posed in Hadamard sense. This characteristic irregular Cauchy problem has no smooth solution (not even C^2) even if the data φ and ψ are smooth.

So we must use all the results of [1], [8], [9], [11], [12], [13] to solve this problem.

The general idea goes as follows. The characteristic problem is approached by a three-parameter family of classical smooth problems $(P_{\varepsilon,\eta,\rho})$. We then get a three-parameter family of classical solutions. A generalized solution is defined as the class of this family of smooth functions satisfying some asymptotical growth restrictions [26]. We obtain a solution which has no classical counterpart.

The plan of this article is as follows. This section is followed by section 2 which introduces the algebras of generalized functions.

In Section 3 we define a well formulated generalized differential problem (P_{gen}) associated to the ill-posed classical one. It is constructed by means of a family $(P_{\varepsilon,\eta,\rho})$ of regularized problems, where $(\varepsilon,\eta,\rho)\in(0,1]^3$. We give estimates needed in the sequel. We replace F with a family of Lipschitz functions (F_ε) given by suitable cutoff techniques which gives rise to a family of regularized Lipschitz problems. We use a family mollifiers $(\theta_\rho)_\rho$ to regularize the data in singular case. Then the parameter ε is used to render the problem Lipschitz, ρ makes it regular. Moreover, by deforming the characteristic curve $t=f(x)$ into a family of non-characteristic ones $t=f_\eta(x)$ we obtain a family of classical problems. Then we can build a $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra, $\mathcal{A}\left(\mathbb{R}^2\right)$, stable under the family (F_ε), adapted to the generalized Cauchy problem in which the irregular problem can be solved.

Then we proceed in Section 4 with the proof of the existence of the generalized solution in the case where the irregular data are given along the characteristic curve γ. To prove the existence of solution a three parametric representative $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)}$ is constructed from the existence of smooth solutions $u_{\varepsilon,\eta,\rho}$ for each regularized Lipschitz problem $\left(P_{(\varepsilon,\eta,\rho)}\right)$. The class of $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)}$ is the expected generalized solution. Thus, we obtain a global generalized solution, when the classical smooth solutions often break down in finite time [30]. We show that this solution is unique in the constructed algebra. However, the generalized problem (P_{gen}), and obviously its solution, depend on the choice of the cutoff functions and, in the case of irregular data, on the family of mollifiers. With regard to the regularization, we show that this solution depends solely on the class of cutoff functions as a generalized function, not on the particular representative. In the case of irregular data, the solution of the problem depends on the family of mollifiers but not on a class of that family. We take $(f_\eta)_\eta$ to be equivalent to f for some sense in an appropriate algebra of generalized functions. Furthermore, by imposing some restrictions on the asymptotical growth of the f_η, we are able to prove that the generalized solution depends solely on the class of $(f_\eta)_\eta$ as a generalized function, not on the particular representative.

So the theory of generalized functions appears as the natural continuation of the classical theory of functions and distributions.

In section 5, we indicate how to treat the case of non-characteristic Cauchy problem and irregular data. Moreover, we show that if the initial problem admits a smooth solution v satisfying appropriate growth estimates on some open subset Ω of $\mathbb{R}^2$, then this solution and the generalized one are equal in a meaning given in Theorem 15. In the last Section we compute an example of a characteristic equation. In Appendix we specify the results and estimates obtained in classical problem.

2. Algebras of generalized functions

2.1. The presheaves of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

2.1.1. Definitions

We refer the reader to [19], [20], [21], [22] for more details. Take

- Λ a set of indices;
- A a solid subring of the ring $\mathbb{K}^\Lambda$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), that is A has the following stability property: whenever $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$ (i.e. for any λ, $|s_\lambda| \leq r_\lambda$) for any pair $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times |A|$, it follows that $(s_\lambda)_\lambda \in A$, with $|A| = \{(|r_\lambda|)_\lambda : (r_\lambda)_\lambda \in A\}$;
- I_A a solid ideal of A with the same property;
- $\mathcal{E}$ a sheaf of $\mathbb{K}$-topological algebras on a topological space X, such that for any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (p_i)_{i \in I(\Omega)}$ of seminorms satisfying

  $$\forall i \in I(\Omega), \exists (j,k,C) \in I(\Omega) \times I(\Omega) \times \mathbb{R}_+^*, \forall f, g \in \mathcal{E}(\Omega) : p_i(fg) \leq C p_j(f) p_k(g).$$

  Assume that

- For any two open subsets Ω_1, Ω_2 of X such that $\Omega_1 \subset \Omega_2$, we have $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$, then, for each $p_i \in \mathcal{P}(\Omega_1)$, the seminorm $\widetilde{p_i} = p_i \circ \rho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$;

- For any family $\mathcal{F} = (\Omega_h)_{h \in H}$ of open subsets of X if $\Omega = \cup_{h \in H} \Omega_h$, then, for each $p_i \in \mathcal{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily $\Omega_1, ..., \Omega_{n(i)}$ of $\mathcal{F}$ and corresponding seminorms $p_1 \in \mathcal{P}(\Omega_1), ..., p_{n(i)} \in \mathcal{P}(\Omega_{n(i)})$, such that, for each $u \in \mathcal{E}(\Omega)$,

$$p_i\left(u\right) \leq p_1\left(u_{|\Omega_1}\right) + ... + p_{n(i)}(u_{|\Omega_{n(i)}}).$$

Set

$$\begin{aligned} \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ ((p_i(u_\lambda))_\lambda \in |A|\}, \\ \mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ (p_i(u_\lambda))_\lambda \in |I_A|\}, \\ \mathcal{C} &= A/I_A. \end{aligned}$$

One can prove that $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ is a sheaf of subalgebras of the sheaf $\mathcal{E}^\Lambda$ and $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ is a sheaf of ideals of $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ [20]. Moreover, the constant sheaf $\mathcal{X}_{(A,\mathbb{K},|.|)}/\mathcal{N}_{(I_A,\mathbb{K},|.|)}$ is exactly the sheaf $\mathcal{C} = A/I_A$.

Definition 1. *We call presheaf of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra the factor presheaf of algebras over the ring $\mathcal{C} = A/I_A$*

$$\mathcal{A} = \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}.$$

We denote by $[u_\lambda]$ the class in $\mathcal{A}(\Omega)$ defined by the representative $(u_\lambda)_{\lambda \in \Lambda} \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$.

2.1.2. Overgenerated rings

See [9]. Let $B_p = \left\{(r_{n,\lambda})_\lambda \in (\mathbb{R}_+^*)^\Lambda : n = 1, ..., p\right\}$ and B be the subset of $(\mathbb{R}_+^*)^\Lambda$ obtained as rational functions with coefficients in $\mathbb{R}_+^*$ of elements in B_p as variables. Define

$$A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists\, (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\}.$$

Definition 2. *In the above situation, we say that A is* overgenerated *by B_p (and it is easy to see that A is a solid subring of $\mathbb{K}^\Lambda$). If I_A is some solid ideal of A, we also say that $\mathcal{C} = A/I_A$ is* overgenerated *by B_p.*

Example 1. *For example, as a "canonical" ideal of A, we can take*

$$I_A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall\, (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\}.$$

Remark 1. *We can see that with this definition B is stable by inverse.*

2.1.3. Relationship with distribution theory

Let Ω be an open subset of $\mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ can be embedded into $\mathcal{A}(\Omega)$. If $(\theta_\rho)_{\rho\in(0,1]}$ is a family of mollifiers $\theta_\rho(x) = \frac{1}{\rho^n}\theta\left(\frac{x}{\rho}\right)$, $x \in \mathbb{R}^n$, $\int \theta(x)\,dx = 1$ and if $T \in \mathcal{D}'(\mathbb{R}^n)$, the convolution product family $(T * \theta_\rho)_\rho$ is a family of smooth functions slowly increasing in $\frac{1}{\rho}$. Then, taking ρ as a component of the multi-index $\lambda \in \Lambda$, we shall choose the subring A overgenerated by some B_p of $(\mathbb{R}_+^*)^\Lambda$ containing the family $(\rho)_\lambda$ [6], [27].

2.1.4. The association process

We assume that Λ is left-filtering for a given partial order relation $\prec$. We denote by Ω an open subset of X, E a given sheaf of topological $\mathbb{K}$-vector spaces containing $\mathcal{E}$ as a subsheaf, a a given map from Λ to $\mathbb{K}$ such that $(a(\lambda))_\lambda = (a_\lambda)_\lambda$ is an element of A. We also assume that

$$\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \left\{ (u_\lambda)_\lambda \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) : \lim_{E(\Omega),\Lambda} u_\lambda = 0 \right\}.$$

Definition 3. *We say that $u = [u_\lambda]$ and $v = [v_\lambda] \in \mathcal{E}(\Omega)$ are a-E associated if*

$$\lim_{E(\Omega),\Lambda} a_\lambda(u_\lambda - v_\lambda) = 0.$$

That is to say, for each neighborhood V of 0 for the E-topology, there exists $\lambda_0 \in \Lambda$ such that $\lambda \prec \lambda_0 \Longrightarrow a_\lambda(u_\lambda - v_\lambda) \in V$. We write

$$u \underset{E(\Omega)}{\overset{a}{\sim}} v.$$

Remark 2. *We can also define an association process between $u = [u_\lambda]$ and $T \in \mathcal{E}(\Omega)$ by writing simply*

$$u \sim T \iff \lim_{E(\Omega),\Lambda} u_\lambda = T.$$

Taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda = (0,1]$, we recover the association process defined in the literature (J.-F. Colombeau [4]).

2.2. *$\mathcal{D}'$-singular support*

Assume that

$$\mathcal{N}_{\mathcal{D}'}^{\mathcal{A}}(\Omega) = \left\{ (u_\lambda)_\lambda \in \mathcal{X}(\Omega) : \lim_{\lambda\to 0} u_\lambda = 0 \text{ in } \mathcal{D}'(\Omega) \right\} \supset \mathcal{N}(\Omega).$$

Set

$$\mathcal{D}'_{\mathcal{A}}(\Omega) = \left\{ [u_\lambda] \in \mathcal{A}(\Omega) : \exists T \in \mathcal{D}'(\Omega),\ \lim_{\lambda\to 0} (u_\lambda) = T \text{ in } \mathcal{D}'(\Omega) \right\}.$$

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is clearly well defined because the limit is independent of the chosen representative; indeed, if $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$ we have $\lim\limits_{\substack{\lambda\to 0\\ \mathcal{D}'(\mathbb{R})}} i_\lambda = 0$.

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is an $\mathbb{R}$-vector subspace of $\mathcal{A}(\Omega)$. Therefore we can consider the set $\mathcal{O}_{D'_{\mathcal{A}}}$ of all x having a neighborhood V on which u is associated to a distribution:

$$\mathcal{O}_{D'_{\mathcal{A}}}(u) = \{x \in \Omega : \exists V \in \mathcal{V}(x),\ \ u|_V \in \mathcal{D}'_{\mathcal{A}}(V)\},$$

$\mathcal{V}(x)$ being the set of all neighborhoods of x.

Definition 4. *The $\mathcal{D}'$-singular support of $u \in \mathcal{A}(\Omega)$, denoted* $\operatorname{singsupp}_{\mathcal{D}'}(u) = S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$*, is the set*

$$S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) = \Omega \backslash \mathcal{O}_{D'_{\mathcal{A}}}(u).$$

2.3. Algebraic framework for our problem

Set $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and Λ a set of indices, $\lambda \in \Lambda$. For any open set Ω, in $\mathbb{R}^d$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$ topology of uniform convergence of all derivatives on compact subsets of Ω. This topology may be defined by the family of the seminorms

$$P_{K,l}(u_\lambda) = \sup_{|\alpha|\le l} P_{K,\alpha}(u_\lambda) \ \text{ with } P_{K,\alpha}(u_\lambda) = \sup_{x\in K} |D^\alpha u_\lambda(x)|\,,\ \ K \Subset \Omega$$

and $D^\alpha = \dfrac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial z_1^{\alpha_1}\ldots\partial z_d^{\alpha_d}}$ for $z = (z_1, \ldots, z_d) \in \Omega$, $l \in \mathbb{N}$, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$. Let A be a subring of the ring $\mathbb{R}^\Lambda$ of the family of reals with the usual laws. We consider a solid ideal I_A of A. Then we have

$$\begin{aligned}
&\mathcal{X}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |A|\},\\
&\mathcal{N}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |I_A|\},\\
&\mathcal{A}(\Omega) = \mathcal{X}(\Omega)/\mathcal{N}(\Omega).
\end{aligned}$$

The generalized derivation $D^\alpha : u(= [u_\lambda]) \mapsto D^\alpha u = [D^\alpha u_\lambda]$ provides $\mathcal{A}(\Omega)$ with a differential algebraic structure.

Example 2. *Set* $\Lambda = (0, 1]$. *Consider*

$$\begin{aligned}
A &= \mathbb{R}^\Lambda_M\\
&= \big\{(m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}^*_+,\ \exists C \in \mathbb{R}^*_+,\ \exists \mu \in\ (0, 1],\ \forall \lambda \in\ (0, \mu],\\
&\qquad |m_\lambda| \le C\lambda^{-p}\big\}
\end{aligned}$$

and the ideal

$$\begin{aligned}
I_A = &\big\{(m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}^*_+,\ \exists D \in \mathbb{R}^*_+,\ \exists \mu \in\ (0, 1],\ \forall \lambda \in\ (0, \mu],\\
&\ |m_\lambda| \le D\lambda^q\big\}\,.
\end{aligned}$$

In this case we denote $\mathcal{X}^s(\Omega) = \mathcal{X}(\Omega)$ and $\mathcal{N}^s(\Omega) = \mathcal{N}(\Omega)$. The sheaf of factor algebras $\mathcal{G}(\cdot) = \mathcal{X}^s(\cdot)/\mathcal{N}^s(\cdot)$ is called the sheaf of simplified Colombeau algebras *[4].*

We have an analogue of Theorem 1.2.3. of [17] for $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras. We suppose here that Λ is left filtering and give this proposition for $\mathcal{A}\left(\mathbb{R}^2\right)$, although it is valid in more general situations.

Proposition 1. *Assume that there exists* $(a_\lambda)_\lambda \in B$ *with* $\lim_\Lambda a_\lambda = 0$. *Consider* $(u_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$ *such that*

$$\forall K \Subset \mathbb{R}^2, \quad (P_{K,0}(u_\lambda))_\lambda \in |I_A| .$$

Then $(u_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$.

For a detailed proof we refer the reader to [7], [9].

Definition 5. *Tempered generalized functions, [10], [17], [28], [29]. For* $f \in \mathrm{C}^\infty(\mathbb{R}^n)$, $r \in \mathbb{Z}$ *and* $m \in \mathbb{N}$, *we put*

$$\mu_{r,m}(f) = \sup_{x \in \mathbb{R}^n, |\alpha| \leq m} (1+|x|)^r \left|\mathcal{D}^\alpha f(x)\right| .$$

The space of functions with slow growth is

$$\mathcal{O}_M(\mathbb{R}^n) = \left\{ f \in \mathrm{C}^\infty(\mathbb{R}^n) : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \mu_{-q,m}(f) < +\infty \right\} .$$

Definition 6. *We put*

$$\begin{aligned}\mathcal{X}_\tau\left(\mathbb{R}^n\right) =& \{ (f_\eta)_{(\varepsilon,\eta,\rho)} \in \mathcal{O}_M(\mathbb{R}^n)^\Lambda : \forall m \in \mathbb{N},\ \exists q \in \mathbb{N}, \exists N \in \mathbb{N}, \\ & \mu_{-q,m}(f_\eta) = O(\eta^{-N}) \quad (\eta \to 0) \},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_\tau\left(\mathbb{R}^n\right) =& \{ (f_\eta)_{(\varepsilon,\eta,\rho)} \in \mathcal{O}_M(\mathbb{R}^n)^\Lambda : \forall m \in \mathbb{N},\ \exists q \in \mathbb{N}, \forall p \in \mathbb{N}, \\ & \mu_{-q,m}(f_\eta) = O(\eta^{p}) \quad (\eta \to 0) \}.\end{aligned}$$

$\mathcal{X}_\tau\left(\mathbb{R}^n\right)$ *is a subalgebra of* $\mathcal{O}_M(\mathbb{R}^n)^\Lambda$ *and* $\mathcal{N}_\tau\left(\mathbb{R}^n\right)$ *an ideal of* $\mathcal{X}_\tau\left(\mathbb{R}^n\right)$. *The algebra* $\mathcal{G}_\tau\left(\mathbb{R}^n\right) = \mathcal{X}_\tau\left(\mathbb{R}^n\right)/\mathcal{N}_\tau\left(\mathbb{R}^n\right)$ *is called the algebra of tempered generalized functions. The generalized derivation* $\mathcal{D}^\alpha : u = [u_\eta] \mapsto \mathcal{D}^\alpha u = [\mathcal{D}^\alpha u_\eta]$ *provides* $\mathcal{G}_\tau\left(\mathbb{R}^n\right)$ *with a differential algebraic structure.*

2.4. Some regularizing conditions

2.4.1. Generalized operator associated to a stability property

Set $\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3$, denote by $\lambda = (\varepsilon, \eta, \rho)$ an element of Λ.

Definition 7. *Let* Ω *be an open subset of* $\mathbb{R}^2$, $\Omega' = \Omega \times \mathbb{R} \subset \mathbb{R}^3$. *Let* $F_\varepsilon \in \mathrm{C}^\infty(\Omega', \mathbb{R})$. *We say that the algebra* $\mathcal{A}(\Omega)$ *is* stable under the family $(F_\varepsilon)_\lambda$ *if the following two conditions are satisfied:*

- *For each* $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$ *and* $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$, *there is a positive finite sequence* C_0,..., C_l, *such that*

$$P_{K,l}(F_\varepsilon(\cdot,\cdot,u_\lambda)) \leq \sum_{i=0}^{l} C_i P_{K,l}^i(u_\lambda).$$

- *For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, $(v_\lambda)_\lambda$ and $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$, there is a positive finite sequence D_1,..., D_l such that*

$$P_{K,l}(F_\varepsilon(\cdot,\cdot,v_\lambda) - F_\varepsilon(\cdot,\cdot,u_\lambda)) \leq \sum_{j=1}^{l} D_j P_{K,l}^{j}(v_\lambda - u_\lambda).$$

Remark 3. *If $\mathcal{A}(\Omega)$ is stable under $(F_\varepsilon)_\lambda$ then, for all $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$ and $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$, we have $(F_\varepsilon(\cdot,\cdot,u_\lambda))_\lambda \in \mathcal{X}(\Omega)$; $(F_\varepsilon(\cdot,\cdot,u_\lambda + i_\lambda) - F_\varepsilon(\cdot,\cdot,u_\lambda))_\lambda \in \mathcal{N}(\Omega)$.*

Set $f \in \mathrm{C}^\infty(\mathbb{R}^2)$, we define

$$\begin{aligned} \mathrm{C}^\infty\left(\mathbb{R}^2\right) &\to \mathrm{C}^\infty\left(\mathbb{R}^2\right) \\ f &\mapsto H_\lambda(f) = F_\varepsilon(\cdot,\cdot,f)\,. \end{aligned}$$

$$H_\lambda(f) = F_\varepsilon(\cdot,\cdot,f) : (x,t) \mapsto F_\varepsilon(x,t,f(x,t))$$

Clearly, the family $(H_\lambda)_\lambda$ maps $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ into $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ and allows to define a map from $\mathcal{A}\left(\mathbb{R}^2\right)$ into $\mathcal{A}\left(\mathbb{R}^2\right)$. For $u = [u_\lambda] \in \mathcal{A}\left(\mathbb{R}^2\right)$, $([F_\varepsilon(.,.,u_\lambda)])$ is a well defined element of $\mathcal{A}(\mathbb{R}^2)$ (i.e. not depending on the representative $(u_\lambda)_\lambda$ of u). This leads to the following definition [9]:

Definition 8. *If $\mathcal{A}\left(\mathbb{R}^2\right)$ if stable under $(F_\varepsilon)_\lambda$, the operator*

$$\mathcal{F} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}^2\right), \quad u = [u_\lambda] \mapsto [F_\varepsilon(.,.,u_\lambda)] = [H_\lambda(u_\lambda)]$$

is called the generalized operator associated to the family $(F_\varepsilon)_\lambda$. *See [9].*

Definition 9. *Let $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$ and $(g_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$, we define $F_\varepsilon(x,t,z) = F(x,t,zg_\varepsilon(z))$. The family $(F_\varepsilon)_\lambda$ is called the* family associated to F *via* the family $(g_\varepsilon)_\varepsilon$. *If $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under $(F_\varepsilon)_\lambda$, the operator*

$$\mathcal{F} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}^2\right), \quad u = [u_\lambda] \mapsto [F_\varepsilon(.,.,u_\lambda)] = [H_\lambda(u_\lambda)]$$

is called the generalized operator associated to F *via* the family $(g_\varepsilon)_\varepsilon$.

2.4.2. Generalized restriction mappings

Set $(f_\eta)_\lambda$ be a family of functions in $\mathrm{C}^\infty(\mathbb{R})$. For each $g \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$ set

$$R_\eta(g) : \mathrm{C}^\infty(\mathbb{R}) \to \mathrm{C}^\infty(\mathbb{R}), \; f_\eta \mapsto (x \mapsto g(x, f_\eta(x)))\,.$$

The family $(R_\eta)_\lambda$ map $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ into $(\mathrm{C}^\infty(\mathbb{R}))^\Lambda$.

Definition 10. *The family of smooth function $(f_\eta)_\lambda$ is* compatible with second side restriction *if*

$$\begin{aligned} &\forall\,(u_\lambda)_\lambda \in \mathcal{X}((\mathbb{R}^2), \quad (u_\lambda(\cdot, f_\eta(\cdot)))_\lambda \in \mathcal{X}(\mathbb{R})\;; \\ &\forall\,(i_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2), \quad (i_\lambda(\cdot, f_\eta(\cdot)))_\lambda \in \mathcal{N}(\mathbb{R}). \end{aligned}$$

Clearly, if $u = [u_\lambda] \in \mathcal{A}(\mathbb{R}^2)$ then $[u_\lambda(\cdot, f_\eta(\cdot))]$ is a well defined element of $\mathcal{A}(\mathbb{R})$ (i.e. not depending on the representative of u.) This leads to the following:

Definition 11. *If the family of smooth function $(f_\eta)_\lambda$ is compatible with second side restriction, the mapping*

$$\mathcal{R} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}(\mathbb{R}), \quad u = [u_\lambda] \mapsto [u_\lambda(\cdot, f_\eta(\cdot))] = [R_\eta(u_\lambda)]$$

is called the generalized second side restriction mapping *associated to the family* $(f_\eta)_\lambda$.

Remark 4. *The previous process generalizes the standard one defining the restriction of the generalized function $u = [u_\lambda] \in \mathcal{A}\left(\mathbb{R}^2\right)$ to the manifold $\{t = f(x)\}$ obtained when taking $f_\eta = f$ for each $\eta \in \Lambda_2$.*

First let us state a useful definition used throughout this article:

Definition 12. *[17] Let $(f_\eta)_\lambda$ be a family of $C^\infty(\mathbb{R}^n)$ functions. This family is c-bounded if for all compact sets $K \subset \mathbb{R}^n$ it exists another compact set $L \subset \mathbb{R}^n$ such that $f_\eta(K) \subset L$ for all η (L is independent of η).*

Proposition 2. *Assume that:*
(i) For each $K \Subset \mathbb{R}$, it exists $K' \Subset \mathbb{R}$ such that, for all $\eta \in \Lambda_2$, $f_\eta(K) \subset K'$,
(ii) $(f_\eta)_\lambda$ belongs to $\mathcal{X}(\mathbb{R})$.
Then the family $(f_\eta)_\lambda$ is compatible with restriction.

For a detailed proof we refer the reader to [1].

3. A Cauchy problem

We study the differential Cauchy problem formally written as

$$(P_{form}) \begin{cases} \dfrac{\partial^2}{\partial x \partial t} u = F(\cdot, \cdot, u'), \\ u|_\gamma = r, \\ u'|_\gamma = s \end{cases}$$

where F, a nonlinear function of its arguments, may be non-Lipschitz (in u'), γ is the monotonic curve of equation $t = f(x)$, the data r, s may be as irregular as distributions. We don't have a classical surrounding in which we can pose (and a fortiori solve) the problem.

We treat in details the case of irregular data given along the characteristic curve γ, we add some remarks for the non-characteristic cases.

3.1. Estimates for a parametrized regular problem

First, we are going to prove that $\left(P_{(\varepsilon,\eta,\rho)}\right)$ has a unique smooth solution under the following assumption

$$(H1) \qquad (H_{\varepsilon,\eta,\rho}) \begin{cases} \text{a)} & f_\eta \in \mathrm{C}^\infty(\mathbb{R}),\ f_\eta' > 0,\ f_\eta(\mathbb{R}) = \mathbb{R}; \\ \text{b)} & F_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R}),\ \forall K \Subset \mathbb{R}^2, \\ & \quad \sup_{(x,t)\in K; z \in \mathbb{R}} |\partial_z F_\varepsilon(x,t,z)| = m_{K,\varepsilon} < +\infty; \\ \text{c)} & \varphi_\rho \text{ and } \psi_\rho \in \mathrm{C}^\infty(\mathbb{R}), \end{cases}$$

where the notation $K \Subset \mathbb{R}^2$ means that K is a compact subset of $\mathbb{R}^2$. According to Appendix 7, we can say that $\left(P_{(\varepsilon,\eta,\rho)}\right)$ is equivalent to the integral formulation

$$(1)\quad \left(I_{(\varepsilon,\eta,\rho)}\right) : u_{\varepsilon,\eta,\rho}(x,t) = u_{0,\varepsilon,\eta,\rho}(x,t) - \iint_{D(x,t,f_\eta)} F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\xi d\zeta,$$

where $u_{0,\varepsilon,\eta,\rho}(x,t) = \Upsilon_{\eta,\rho}(t) - \Upsilon_{\eta,\rho}(f_\eta(x)) + \varphi_\rho(x)$ and Υ_ρ denotes a primitive of $\psi_\rho \circ f_\eta^{-1}$, with

$$D(x,t,f_\eta) = \begin{cases} \left\{(\xi,\zeta) :\ f_\eta^{-1}(t) \leq \xi \leq x, t \leq \zeta \leq f_\eta(\xi)\right\} & \text{if } t \leq f_\eta(x) \\ \left\{(\xi,\zeta) : x \leq \xi \leq f_\eta^{-1}(t), f_\eta(\xi) \leq \zeta \leq t\right\} & \text{if } t \geq f_\eta(x). \end{cases}$$

Theorem 3. *Under Assumption* $(H_{\varepsilon,\eta,\rho})$*, Problem* $\left(P_{(\varepsilon,\eta,\rho)}\right)$ *has a unique solution in* $C^\infty(\mathbb{R}^2)$*.*

Corollary 4. *Set* $K_{a,\eta} = \left[f_\eta^{-1}(-a), f_\eta^{-1}(a)\right] \times [-a,a]$,

$$m_{a,\varepsilon,\eta} = \sup_{(x,t)\in K_{a,\eta};\ t\in\mathbb{R}} \left|\frac{\partial F_\varepsilon}{\partial z}(x,t,t)\right|$$

and $\Phi_{a,\varepsilon,\eta,\rho} = \left\|F_\varepsilon(.,.,0)\right\|_{\infty,K_{a,\eta}} + m_{a,\varepsilon,\eta}\left\|u'_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K_{a,\eta}}$. *We have the estimate*
(2)

$$\left\|u_{\varepsilon,\eta,\rho}\right\|_{\infty,K_{a,\eta}} \leq \left\|u_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K_{a,\eta}} + \frac{\Phi_{a,\varepsilon,\eta,\rho}}{m_{a,\varepsilon,\eta}} \exp[m_{a,\varepsilon,\eta}(2a)\left(f_\eta^{-1}(a) - f_\eta^{-1}(-a)\right)].$$

These results are proved in Appendix 7.

3.2. Cut off procedure

Let ε be a parameter belonging to the interval $(0,1]$. Let $(r_\varepsilon)_\varepsilon$ be in $\mathbb{R}_*^{(0,1]}$ such that $r_\varepsilon > 0$ and $\lim\limits_{\varepsilon\to 0} r_\varepsilon = +\infty$. Set $L_\varepsilon = [-r_\varepsilon, r_\varepsilon]$. Consider a family of smooth one-variable functions $(g_\varepsilon)_\varepsilon$ such that

$$(A1)\qquad \sup_{z\in L_\varepsilon} |g_\varepsilon(z)| = 1,\ g_\varepsilon(z) = \begin{cases} 0, \text{ if } & |z| \geq r_\varepsilon \\ 1, \text{ if } & -r_\varepsilon + 1 \leq z \leq r_\varepsilon - 1 \end{cases}$$

and $\dfrac{\partial^n g_\varepsilon}{\partial z^n}$ is bounded on L_ε for any integer n, $n > 0$. Set

$$\sup_{z\in L_\varepsilon} \left|\frac{\partial^n g_\varepsilon}{\partial z^n}(z)\right| = M_n.$$

Let $\phi_\varepsilon(z) = z g_\varepsilon(z)$. We approximate the function F by the family of functions $(F_\varepsilon)_\varepsilon = (F_\varepsilon)_\varepsilon$ defined by

$$F_\varepsilon(x,t,z) = F(x,t,\phi_\varepsilon(z)).$$

3.3. Construction of $\mathcal{A}\left(\mathbb{R}^2\right)$

We recall that $\lambda = (\varepsilon, \eta, \rho) \in \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \Lambda$, $\Lambda_1 = \Lambda_2 = \Lambda_3 = (0,1]$ where the parameter ρ is used to regularize the distributions s and t, the more general case. Consider the previous family $(r_\varepsilon)_\varepsilon$.

If γ is a characteristic curve, we consider a family of smooth functions $(f_\eta)_{\eta\in(0,1]}$ such that

$$(A2)\qquad \begin{cases} f_\eta \in \mathrm{C}^\infty(\mathbb{R}),\ f_\eta \text{ strictly increasing, } f_\eta(\mathbb{R}) = \mathbb{R}, \\ \forall x \in \mathbb{R}, f'_\eta(x) \neq 0, \\ (f_\eta)_\eta, (f_\eta^{-1})_\eta \in \mathcal{X}_\tau(\mathbb{R}),\ (f_\eta)_\eta \text{ is c-bounded and } \lim\limits_{\eta \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} 0} f_\eta = f. \end{cases}$$

Then we consider the family of smooth non-characteristic curves γ_η whose equation is $y = f_\eta(x)$, such that γ_η is diffeomorphic to γ which is a consequence of the previous assumption.

The idea is then to approach the Cauchy problem (P_{form}) by a family of non-characteristic ones by replacing the characteristic curve γ by a family of smooth non-characteristic curves γ_η.

Each compact $K \Subset \mathbb{R}^2$ is contained in some product $\left[f_\eta^{-1}(-a), f_\eta^{-1}(a)\right] \times [-a,a]$. Set

$$(3)\qquad \begin{cases} a_{K,\eta} = 2\max(f_\eta^{-1}(a), \left|f_\eta^{-1}(-a)\right|), \\ K_{a,\eta} = K_{1,\eta} \times K_2 \text{ with } K_{1,\eta} = [-a_{K,\eta}/2, a_{K,\eta}/2] \text{ and } K_2 = [-a,a]\,. \end{cases}$$

By construction we have $K \subset K_{a,\eta}$.

We make the following assumptions to generate a convenient $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra adapted to our problem

$$(A3)\qquad \forall \eta \in (0,1], \forall K \Subset \mathbb{R}^2, \exists \nu_K > 0, \exists a_\eta > 0, a_{K,\eta} \leq \nu_K a_\eta.$$

$$(A4)\qquad \begin{cases} \exists\, (l_{\eta,\rho})_{(\eta,\rho)} \in \mathbb{R}_*^{(0,1]\times(0,1]} \text{ such that} \\ \qquad \forall K_2 \Subset \mathbb{R}, \forall \alpha_2 \in \mathbb{N}, \exists D_2 = D_{K_2,\alpha_2,\eta,\rho} \in \mathbb{R}_+^*, \exists q \in \mathbb{N}, \\ \max\left[\sup\limits_{K_2}\left|D^{\alpha_2}\psi_\rho(f_\eta^{-1}(t))\right|, \sup\limits_{K_2}\left|D^{\alpha_2}\Upsilon_{\eta,\rho}(t)\right|\right] \leq D_2\,(l_{\eta,\rho})^{-q}\,. \end{cases}$$

$$(A5)\qquad \begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated} \\ \qquad \text{by the following elements of } \mathbb{R}_*^{(0,1]\times(0,1]\times(0,1]} \\ (\varepsilon)_{(\varepsilon,\eta,\rho)}, (\eta)_{(\varepsilon,\eta,\rho)}, (\rho)_{(\varepsilon,\eta,\rho)}, (r_\varepsilon)_{(\varepsilon,\eta,\rho)}, (l_{\eta,\rho})_{(\varepsilon,\eta,\rho)}, (e^{r_\varepsilon a_\eta})_{(\varepsilon,\eta,\rho)}\,. \end{cases}$$

Then $\mathcal{A}\left(\mathbb{R}^2\right) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}^2), (P_{K,l})_{K\Subset\mathbb{R}^2, l\in\mathbb{N}}\right)$. In the same way $\mathcal{A}(\mathbb{R}) = \mathcal{X}(\mathbb{R})/\mathcal{N}(\mathbb{R})$ is built on $\mathcal{C}$ with $(\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}), (P_{K,l})_{K\Subset\mathbb{R}, l\in\mathbb{N}}\right)$. We take ϕ, $\psi \in O_M(\mathbb{R})$.

Proposition 5. *We have the following relations*

$$\left(f_\eta^{-1}(-a)\right)_{(\varepsilon,\eta,\rho)}, \left(f_\eta^{-1}(a)\right)_{(\varepsilon,\eta,\rho)}, \left(a_{K,\eta}\right)_{(\varepsilon,\eta,\rho)} \in |A|, \tag{4}$$

$$\forall\eta, \forall(x,y) \in K_\eta, D(x,y,f_\eta) \subset K_\eta. \tag{5}$$

Proof. First $(f_\eta^{-1})_{(\varepsilon,\eta,\rho)} \in \mathcal{X}_\tau(\mathbb{R})$ so $\left(f_\eta^{-1}(-a)\right)_{(\varepsilon,\eta,\rho)}, \left(f_\eta^{-1}(a)\right)_{(\varepsilon,\eta,\rho)} \in |A|$ and then obviously $\left(a_{K,\eta}\right)_{(\varepsilon,\eta,\rho)} \in |A|$. Next, as $(f_\eta)_{(\varepsilon,\eta,\rho)} \in \mathcal{X}_\tau(\mathbb{R})$, we can find $p \in \mathbb{N}$ such that $\forall x, \eta, |f_\eta(x)| \leq \eta^{-p}(1+|x|)^p$ so we have

$$|\mu_\eta| = |f_\eta(a_{K,\eta})| \leq \eta^{-p}(1+|a_{K,\eta}|)^p$$

then $(|\mu_\eta|)_{(\varepsilon,\eta,\rho)} \in |A|$. □

3.4. Stability of $\mathcal{A}\left(\mathbb{R}^2\right)$

Proposition 6. *Set* $S_n = \left\{\alpha \in \mathbb{N}^3 : |\alpha| = n\right\}$ *when* $n \in \mathbb{N}^*$. *Let* $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$, F_ε *defined as above in Section 3.2. Assume that*

$$\forall\varepsilon \in (0,1], \forall(x,t) \in \mathbb{R}^2, F_\varepsilon(x,t,0) = 0, \tag{H3}$$

$$\exists p > 0, \forall n \in \mathbb{N}, \exists c_n > 0, \forall\varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \sup_{(x,t)\in K;\ z\in\mathbb{R};\alpha\in Sn} |D^\alpha F_\varepsilon(x,t,z)| \leq c_n r_\varepsilon^p, \tag{H4}$$

then $\mathcal{A}\left(\mathbb{R}^2\right)$ *is stable under the family* $\left(F_\varepsilon\right)_\varepsilon$.

Corollary 7. *Set* $F(x,t,z) = G(z) = z^p$, $G_\varepsilon(z) = F_\varepsilon(x,t,z)$, *then* $\mathcal{A}\left(\mathbb{R}^2\right)$ *is stable under* $\left(G_\varepsilon\right)_\varepsilon$.

For a detailed proof we refer the reader to [13].

3.5. A generalized differential problem associated to the formal one

Our goal is to give a meaning to the differential Cauchy problem formally written as (P_{form}).

As the data r and s are as irregular as distributions, we set

$$\varphi_\rho = r * \theta_\rho \text{ and } \varphi = [\varphi_\rho], \tag{A8}$$

$$\psi_\rho = s * \theta_\rho \text{ and } \psi = [\psi_\rho] \tag{A9}$$

where $\left(\theta_\rho\right)_\rho$ is a chosen family of mollifiers. Then the data φ, ψ belong to $\mathcal{A}(\mathbb{R})$ and u is sought in the algebra $\mathcal{A}\left(\mathbb{R}^2\right)$.

Let $\left(g_\varepsilon\right)_\varepsilon \in \left(\mathrm{C}^\infty(\mathbb{R})\right)^{\Lambda_1}$ and $\mathcal{F}$ the generalized operator associated to F *via* the family $\left(g_\varepsilon\right)_\varepsilon$ in Definition 9. Let $f = \left(f_\eta\right)_\eta$ and $\mathcal{R}_f$ given by Definition 11.

The problem associated to problem (P_{form}) can be written as the well formulated one

$$(P_{gen}) \begin{cases} \dfrac{\partial^2 u}{\partial x \partial t} = \mathcal{F}(u'), \\ \mathcal{R}_f(u) = \varphi, \\ \mathcal{R}_f\left(\dfrac{\partial u}{\partial t}\right) = \psi \end{cases}$$

where u is searched in the algebra $\mathcal{A}\left(\mathbb{R}^2\right)$ and $\mathcal{F}$, $\mathcal{R}_f$ are defined as previously by taking into account the family $(g_\varepsilon)_\varepsilon$and f.

In terms of representatives, and thanks to the stability and restriction hypothesis, solving (P_{gen}) amounts to find a family $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{X}(\mathbb{R}^2)$ such that

$$\begin{cases} \dfrac{\partial^2 u_{\varepsilon,\eta,\rho}}{\partial x \partial t}(x,t) - F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}(x,t)) = i_{\varepsilon,\eta,\rho}(x,t)\,, \\ u_{\varepsilon,\eta,\rho}(x, f_\eta(x)) - \varphi_\rho(x) = j_\rho(x)\,, \\ \dfrac{\partial u_{\varepsilon,\eta,\rho}}{\partial t}(x, f_\eta(x)) - \psi_\rho(x) = l_\rho(x)\,, \end{cases}$$

where $(i_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}\left(\mathbb{R}^2\right)$, $(j_\rho)_{(\varepsilon,\eta,\rho)}$, $(l_\rho)_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R})$.

Suppose we can find $u_{\varepsilon,\eta,\rho} \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$ verifying

$$\left(P_{(\varepsilon,\eta,\rho)}\right) \begin{cases} \dfrac{\partial^2 u_{\varepsilon,\eta,\rho}}{\partial x \partial t}(x,t) = F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}(x,t)), \\ u_{\varepsilon,\eta,\rho}(x, f_\eta(x)) = \varphi_\rho(x)\,, \\ \dfrac{\partial u_{\varepsilon,\eta,\rho}}{\partial t}(x, f_\eta(x)) = \psi_\rho(x)\,, \end{cases}$$

then, if we can prove that $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{X}(\mathbb{R}^2)$, $u = [u_{\varepsilon,\eta,\rho}]$ is a solution of (P_{gen}).

Remark 5. *Uniqueness in the algebra* $\mathcal{A}\left(\mathbb{R}^2\right)$. *Let* $v = [v_{\varepsilon,\eta,\rho}]$ *another solution to* (P_{gen}). *There are* $(k_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}\left(\mathbb{R}^2\right)$, $(\alpha_\rho)_{(\varepsilon,\eta,\rho)}$, $(\beta_\rho)_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R})$, *such that*

$$\begin{cases} \dfrac{\partial^2 v_{\varepsilon,\eta,\rho}}{\partial x \partial t}(x,t) - F_\varepsilon(x,t,v'_{\varepsilon,\eta,\rho}(x,t)) = k_{\varepsilon,\eta,\rho}(x,t)\,, \\ v_{\varepsilon,\eta,\rho}\left(x, f_{\eta(\lambda)}(x)\right) = \varphi_\rho(x) + \alpha_\rho(x), \\ \dfrac{\partial v_{\varepsilon,\eta,\rho}}{\partial t}(x, f_\eta(x)) = \psi_\rho(x) + \beta_\rho(t). \end{cases}$$

The uniqueness of the solution to (P_{gen}) *will be a consequence of*

$$(w_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} = (v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2).$$

Remark 6. *Dependence on some regularizing family. The problem* (P_{gen}) *itself, so a solution of it, a priori depends on the family of cutoff functions and, in the case of irregular data, on the family of mollifiers.*
If $(\theta_\rho)_{\rho\in\Lambda_3}$ *and* $(\tau_\rho)_{\rho\in\Lambda_3}$ *are families of mollifiers in* $\mathcal{D}(\mathbb{R})$ *and* $T \in \mathcal{D}'(\mathbb{R})$, *it is well known that generally* $[T * \theta_\rho] \neq [T * \tau_\rho]$ *in the Colombeau simplified algebra even if* $[\theta_\rho] = [\tau_\rho]$ *in these algebras. Therefore, in the case of irregular data the solution of Problem* (P_{gen}) *in some Colombeau algebra depends on the family of mollifiers* $(\theta_\rho)_\rho$ *but not on a class of that family.*
We have associated the generalized operator $\mathcal{F}$ *to* F *via the family* $(g_\varepsilon)_\varepsilon$. *Let* $(h_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$ *another family representative of the class* $[g_\varepsilon] = g$ *in a meaning specified in (4.2) and leading to another generalized operator* $\mathcal{H}$ *associated to* F. *We can prove that in fact* $\mathcal{H} = \mathcal{F}$, *that is to say Problem* (P_{gen}) *only depends on the class* g *of cutoff functions.*

Remark 7. *Independence of the solution from the class* $[f_\eta]$. *If* $v = [v_{\varepsilon,\eta,\rho}]$ *is another solution of* (P_{gen}) *obtained by replacing* γ *by another family of smooth non-characteristic curves* γ'_η *whose equation is* $t = l_\eta(x)$. *We have to prove that*

$$(v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2)$$

if we intend to prove that the solution of (P_{gen}) *in the algebra* $\mathcal{A}\left(\mathbb{R}^2\right)$ *does not depend on the representative of the class* $[f_\eta]$ *in* $\mathcal{G}_\tau\left(\mathbb{R}\right)$.

4. Non Lipschitz characteristic problem with irregular data

4.1. Solution to (P_{gen})

Theorem 8. *With the previous assumptions, if* $u_{\varepsilon,\eta,\rho}$ *is the solution to problem* $\left(P_{(\varepsilon,\eta,\rho)}\right)$, *then problem* (P_{gen}) *admits* $[u_{\varepsilon,\eta,\rho}]_{\mathcal{A}(\mathbb{R}^2)}$ *as solution.*

Proof. We have

$$u_{\varepsilon,\eta,\rho}(x,t) = u_{0,\varepsilon,\eta,\rho}(x,t) - \iint_{D(x,t,f_\eta)} F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\xi d\zeta,$$

where $u_{0,\varepsilon,\eta,\rho}(x,t) = \Upsilon_{\eta,\rho}\left(t\right) - \Upsilon_{\eta,\rho}(f_\eta(x)) + \varphi_\rho(x)$ and $\Upsilon'_{\eta,\rho} = \psi_\rho \circ f_\eta^{-1}$. Then $u'_{0,\varepsilon,\eta,\rho}(x,t) = \psi_\rho \circ f_\eta^{-1}\left(t\right) - f'_\eta\left(x\right)\psi_\rho(x) + \varphi'_\rho(x)$. We will actually prove that $(P_{K_\varepsilon,n}(u_\varepsilon))_\varepsilon \in |A|$.
We have $f_\eta^{-1}(K_2) = K_{1\eta}$ and $\psi_\rho \in \mathcal{O}_M(\mathbb{R})$, $\left(f_\eta^{-1}(-a)\right)_{(\varepsilon,\eta,\rho)}, \left(f_\eta^{-1}(a)\right)_{(\varepsilon,\eta,\rho)} \in |A|$,

$$\forall\alpha_2 \in \mathbb{N}, \exists D_2 \in \mathbb{R}_+^*, \exists q \in \mathbb{N} : \sup_{K_2}\left|D^{\alpha_2}\Upsilon_{\eta,\rho}\left(t\right)\right| \le D_2\left(l_{\eta,\rho}\right)^{-q}$$

and

$$\sup_{K_2}\left|D^{\alpha_2}\psi_\rho(f_\eta^{-1}\left(t\right))\right| \le D_2\left(l_{\eta,\rho}\right)^{-q},$$

then

$$\forall l \in \mathbb{N}, (P_{K_2,l}(\Upsilon_{\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in |A|\,, \left(P_{K_2,l}(\psi_\rho \circ f_\eta^{-1}\right)_{(\varepsilon,\eta,\rho)} \in |A|\,.$$

Moreover as $\varphi_\rho \in \mathcal{O}_M(\mathbb{R})$ we also have that

$$\forall l \in \mathbb{N}, \left(P_{K_{1,\eta}l}(\varphi_\rho)\right)_{(\varepsilon,\eta,\rho)} \in |A|$$

and as $(\Upsilon_{\eta,\rho} \circ f_\eta)' = f'_\eta\psi_\rho$ and $(f'_\eta)_\eta \in \mathcal{X}_\tau(\mathbb{R})$ we can conclude that

$$\forall l \in \mathbb{N}, \left(P_{K_\eta,l}\left(u_{0,\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in |A|\,, \left(P_{K_\eta,l}\left(u'_{0,\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in |A|\,.$$

We have $\forall K \Subset \mathbb{R}^2$, $\exists K_{a,\eta} = K_{1\eta} \times K_2 \Subset \mathbb{R}^2$, $K \subset K_{a,\eta}$,

$$\|u_{\varepsilon,\eta,\rho}\|_{\infty,K} \le \|u_{\varepsilon,\eta,\rho}\|_{\infty,K_{a,\eta}} \le \|u_{0,\varepsilon,\eta,\rho}\|_{\infty,K_{a,\eta}} + \frac{\Phi_{a,\varepsilon,\eta,\rho}}{m_{a,\varepsilon,\eta}}\exp\left(m_{a,\varepsilon,\eta}2aa_{K,\eta}\right)$$

where

$$m_{a,\varepsilon,\eta}=\sup_{(x,t)\in K_{a,\eta};\ z\in\mathbb{R}}\left|\frac{\partial F_{\varepsilon}}{\partial z}(x,t,z)\right|\leq c_{1}r_{\varepsilon}^{p}$$

and

$$\Phi_{a,\varepsilon,\eta,\rho}=m_{a,\varepsilon,\eta}\left\|u_{0,\varepsilon,\eta,\rho}^{\prime}\right\|_{\infty,K_{a,\eta}}.$$

Then

$$\left\|u_{\varepsilon,\eta,\rho}\right\|_{\infty,K}\leq\left\|u_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K_{a,\eta}}+\left\|u_{0,\varepsilon,\eta,\rho}^{\prime}\right\|_{\infty,K_{a,\eta}}\exp(2ac_{1}r_{\varepsilon}^{p}\nu_{K}a_{\eta}).$$

We have $(\left\|u_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K_{a,\eta}})_{(\varepsilon,\eta,\rho)}\in A$ and $(\left\|u_{0,\varepsilon,\eta,\rho}^{\prime}\right\|_{\infty,K_{a,\eta}})_{(\varepsilon,\eta,\rho)}\in A$, thus

$$\left\|u_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K_{\eta}}+\left\|u_{0,\varepsilon,\eta,\rho}^{\prime}\right\|_{\infty,K_{\eta}}\exp(2ac_{1}r_{\varepsilon}^{p}\nu_{K}a_{\eta})\in\left|A\right|.$$

A being stable, we have $(\left\|u_{\varepsilon,\eta,\rho}\right\|_{\infty,K_{\eta}})_{(\varepsilon,\eta,\rho)}\in|A|$ and then $(\left\|u_{\varepsilon,\eta,\rho}\right\|_{\infty,K})_{(\varepsilon,\eta,\rho)}\in|A|$, that is

$$(P_{K,0}\left(u_{\varepsilon,\eta,\rho}\right))_{(\varepsilon,\eta,\rho)}\in\left|A\right|.$$

Let us show that $(P_{K,1}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)}\in|A|$. We have

$$\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(x,t)=\frac{\partial u_{0,\varepsilon,\eta,\rho}}{\partial x}(x,t)+\int_{f(x)}^{t}F_{\varepsilon}(x,\zeta,u_{\varepsilon,\eta,\rho}^{\prime}(x,\zeta))\,\mathrm{d}\zeta,$$

thus

$$P_{K,(1,0)}(u_{\varepsilon,\eta,\rho})\leq\sup\nolimits_{K}\left|\frac{\partial u_{0,\varepsilon,\eta,\rho}}{\partial x}(x,t)\right|+2a\sup\nolimits_{K_{a,\eta}}\left|F_{\varepsilon}(x,t,u_{\varepsilon,\eta,\rho}^{\prime}(x,t))\right|.$$

We have

$$P_{K_{a,\eta},(0,0)}(F_{\varepsilon}(\ .\ ,\ .\ ,u_{\varepsilon,\eta,\rho}^{\prime}))\leq P_{K_{a,\eta},0}(F_{\varepsilon}(\ .\ ,\ .\ ,u_{\varepsilon,\eta,\rho}^{\prime}))\leq c_{0}r_{\varepsilon}^{p}.$$

Then

$$P_{K,(1,0)}(u_{\varepsilon,\eta,\rho})\leq\left\|\partial/\partial x\ u_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K}+c_{0}r_{\varepsilon}^{p}2a.$$

Moreover $(\left\|\partial/\partial x\ u_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K})_{(\varepsilon,\eta,\rho)}\in|A|$, then we get $(P_{K,(1,0)}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)}\in|A|$. We have

$$\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial t}(x,t)=\frac{\partial u_{0,\varepsilon,\eta,\rho}}{\partial t}(x,t)-\int_{x}^{f^{-1}(t)}F_{\varepsilon}(\xi,t,u_{\varepsilon,\eta,\rho}^{\prime}(\xi,t))\,\mathrm{d}\xi,$$

thus

$$P_{K,(0,1)}(u_{\varepsilon,\eta,\rho})\leq\sup\nolimits_{K}\left|\frac{\partial u_{0,\varepsilon,\eta,\rho}}{\partial t}(x,t)\right|+a_{K,\eta}\sup\nolimits_{K_{a,\eta}}\left|F_{\varepsilon}(x,t,u_{\varepsilon,\eta,\rho}^{\prime}(x,t))\right|.$$

We obtain
$$P_{K,(0,1)}(u_{\varepsilon,\eta,\rho}) \leq \|\partial/\partial t\ u_{0,\varepsilon,\eta,\rho}\|_{\infty,K} + \nu_K a_\eta c_0 r_\varepsilon^p$$
and then
$$(\|\partial/\partial t\ u_{\varepsilon,\eta,\rho}\|_{\infty,K})_{(\varepsilon,\eta,\rho)} \in |A|\,.$$
Now we proceed by induction. Suppose that $(P_{K,l}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in |A|$ for every $l \leq n$, and let us show that it implies $(P_{K,n+1}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in |A|$. We have $P_{K,n+1} = \max\left(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n}\right)$ with

$$P_{1,n} = P_{K,(n+1,0)},\ \ P_{2,n} = P_{K,(0,n+1)},$$
$$P_{3,n} = \sup\nolimits_{\alpha+\beta=n;\ \beta\geq 1} P_{K,(\alpha+1,\beta)},\ \ P_{4,n} = \sup\nolimits_{\alpha+\beta=n;\ \alpha\geq 1} P_{K,(\alpha,\beta+1)}.$$

First, let us show that $(P_{1,n}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)}, (P_{2,n}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in |A|$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\begin{aligned}\frac{\partial^{n+1} u_{\varepsilon,\eta,\rho}}{\partial x^{n+1}}(x,t) =& \frac{\partial^{n+1} u_{0,\varepsilon,\eta,\rho}}{\partial x^{n+1}}(x,t) \\ &- \sum_{j=0}^{n-1} C_n^j f_\eta^{(n-j)}(x)\ \frac{\partial^j}{\partial x^j} F_\varepsilon(x, f(x), \psi_\rho(x)) \\ &+ \int_{f(x)}^{t} \frac{\partial^n}{\partial x^n} F_\varepsilon(x,\zeta,u'_{\varepsilon,\eta,\rho}(x,\zeta))\,\mathrm{d}\zeta.\end{aligned}$$

As $K \subset K_{a,\eta}$, we can write

$$\begin{aligned}\sup\nolimits_{(x,t)\in K}\left|\frac{\partial^{n+1} u_{\varepsilon,\eta,\rho}}{\partial x^{n+1}}(x,t)\right| \leq& \left\|\frac{\partial^{n+1} u_{0,\varepsilon,\eta,\rho}}{\partial x^{n+1}}\right\|_{\infty,K} \\ &+ \sup\nolimits_{x\in K_{1\eta}} \sum_{j=0}^{n-1} C_n^j \left|f_\eta^{(n-j)}(x)\right| \left|\frac{\partial^j}{\partial x^j} F_\varepsilon(x,f(x),\psi_\rho(x))\right| \\ &+ a_{K,\eta} \sup\nolimits_{(x,t)\in K}\left|\frac{\partial^n}{\partial x^n} F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}(x,t))\right|.\end{aligned}$$

We have $f_\eta \in \mathcal{X}_\tau$ then for all k, we can find $p \in \mathbb{N}$ such that

$$\forall\eta, \sup_{\mathbb{R}}(1+|x|)^{-p}\left|f_\eta^{(k)}(x)\right| \leq \eta^{-p},$$

but then we have

$$\left\|f_\eta^{(k)}\right\|_{K_{1,\eta}} \leq \max\left\{(1+\left|f_\eta^{-1}(a)\right|)^p, (1+\left|f_\eta^{-1}(-a)\right|)^p\right\}\eta^{-p} \in |A|\,.$$

Moreover

$$\sup\nolimits_{(x,t)\in K}\left|\frac{\partial^n}{\partial x^n} F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}(x,t))\right| \leq P_{K,n}(F_\varepsilon(\ .\ ,\ .\ , u'_{\varepsilon,\eta,\rho})) \leq c_n r_\varepsilon^p,$$

$$\sup_{x\in[-a,a]}\left|\frac{\partial^j}{\partial x^j}F_\varepsilon(x,f(x),\psi_\rho(x))\right|\le P_{K,n}(F_\varepsilon(\ .\ ,\ .\ ,u'_{\varepsilon,\eta,\rho}))\le c_n r_\varepsilon^p,$$

and $(\left\|\partial^{n+1}/\partial x^{n+1}\ u_{0,\varepsilon,\eta,\rho}\right\|_{\infty,K})_{(\varepsilon,\eta,\rho)}\in|A|$. According to the stability hypothesis, a simple calculation shows that, for every $K\Subset\mathbb{R}^2$,

$$(P_{K,(n+1,0)}\,(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)}\in|A|\,.$$

Let us show that $(P_{2,n}(u_{\varepsilon,\eta,\rho}))_\varepsilon\in|A|$, for every $n\in\mathbb{N}$. We have by successive derivations, for $n\ge 1$

$$\begin{aligned}\frac{\partial^{n+1}u_{\varepsilon,\eta,\rho}}{\partial t^{n+1}}(x,t)=\frac{\partial^{n+1}u_{0,\varepsilon,\eta,\rho}}{\partial t^{n+1}}(x,t)-\int_x^{f^{-1}(t)}\frac{\partial^n}{\partial t^n}F_\varepsilon(\xi,t,u'_{\varepsilon,\eta,\rho}(\xi,t))\,\mathrm{d}\xi\\ -\sum_{j=0}^{n-1}C_n^j\left(f_\eta^{-1}\right)^{(n-j)}(t)\frac{\partial^j}{\partial t^j}F_\varepsilon(f_\eta^{-1}(t),t,\psi_\varepsilon(f^{-1}(t)))\,.\end{aligned}$$

As $K\subset K_{a,\eta}$, we can write

$$\begin{aligned}&\sup_{(x,t)\in K}\left|\frac{\partial^{n+1}u_{\varepsilon,\eta,\rho}}{\partial t^{n+1}}(x,t)\right|\\ &\quad\le\left\|\frac{\partial^{n+1}u_{0,\varepsilon,\eta,\rho}}{\partial t^{n+1}}\right\|_{\infty,K}+a_{K,\eta}\sup_{(x,t)\in K}\left|\frac{\partial^n}{\partial t^n}F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}(x,t))\right|\\ &\qquad+\sup_{t\in[-a,a]}\sum_{j=0}^{n-1}C_n^j\left|\left(f_\eta^{-1}\right)^{(n-j)}(t)\right|\left|\frac{\partial^j}{\partial t^j}F_\varepsilon(f_\eta^{-1}(t),t,\psi_\varepsilon(f^{-1}(t)))\right|\,.\end{aligned}$$

We have

$$\sup_{(x,t)\in K}\left|\frac{\partial^n}{\partial t^n}F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}(x,t))\right|\le P_{K,n}(F_\varepsilon(.,.,u'_{\varepsilon,\eta,\rho}))\le c_n r_\varepsilon^p,$$

$$\sup_{t\in[f(-a),f(a)]}\left|\frac{\partial^j}{\partial t^j}F_\varepsilon(f_\eta^{-1}(t),t,\psi_\varepsilon(f_\eta^{-1}(t)))\right|\le P_{K,n}(F_\varepsilon(\ .\ ,\ .\ ,u'_{\varepsilon,\eta,\rho}))\le c_n r_\varepsilon^p.$$

For all k, we can find $p\in\mathbb{N}$ such that

$$\left|\left|f_\eta^{(k)}\right|\right|_{K_{1,\eta}}\le\max\left\{(1+\left|f_\eta^{-1}(a)\right|)^p,(1+\left|f_\eta^{-1}(-a)\right|)^p\right\}\eta^{-p}\in|A|\,.$$

According to the stability hypothesis, a simple calculation shows that, for every $K\Subset\mathbb{R}^2$ and $n\in\mathbb{N}$, $(P_{K,(0,n+1)}\,(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)}\in|A|$. For $\alpha+\beta=n$ and $\beta\ge 1$, we now have

$$\begin{aligned}P_{K,(\alpha+1,\beta)}(u_{\varepsilon,\eta,\rho})&=\sup_{(x,t)\in K}\left|D^{(\alpha,\beta-1)}D^{(1,1)}u_{\varepsilon,\eta,\rho}\,(x,t)\right|\\ &=\sup_{(x,t)\in K}\left|D^{(\alpha,\beta-1)}F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}\,(x,t))\right|\\ &=P_{K,(\alpha,\beta-1)}(F_\varepsilon(\ .\ ,\ .\ ,u'_{\varepsilon,\eta,\rho}))\\ &\le P_{K,n}(F_\varepsilon(\ .\ ,\ .\ ,u'_{\varepsilon,\eta,\rho}))\le c_n r_\varepsilon^p.\end{aligned}$$

Then we finally have

$$P_{3,n}(u_{\varepsilon,\eta,\rho}) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_{\varepsilon,\eta,\rho}) \leq c_n r_\varepsilon^p$$

and the stability hypothesis ensures that $(P_{3,n}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in |A|$. In the same way, for $\alpha + \beta = n$ and $\alpha \geq 1$, we have

$$\begin{aligned} P_{K,(\alpha,\beta+1)}(u_{\varepsilon,\eta,\rho}) &= \sup_{(x,t)\in K} \left| D^{(\alpha-1,\beta)} D^{(1,1)} u_{\varepsilon,\eta,\rho}(x,t) \right| \\ &= \sup_{(x,t)\in K} \left| D^{(\alpha-1,\beta)} F_\varepsilon(x,t,u'_{\varepsilon,\eta,\rho}(x,t)) \right| \\ &= P_{K,(\alpha-1,\beta)}(F_\varepsilon(\,.\,,\,.\,,u'_{\varepsilon,\eta,\rho})) \\ &\leq P_{K,n}(F_\varepsilon(\,.\,,\,.\,,u'_{\varepsilon,\eta,\rho})) \leq c_n r_\varepsilon^p. \end{aligned}$$

Thus we have $P_{4,n}(u_{\varepsilon,\eta,\rho}) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_{\varepsilon,\eta,\rho}) \leq c_n r_\varepsilon^p$ and the stability hypothesis ensures that $(P_{4,n}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in |A|$. Finally, we clearly have $(P_{K,n+1}(u_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in |A|$, consequently $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{X}(\mathbb{R}^2)$. □

Theorem 9. *Problem (P_{gen}) has a unique solution in the algebra $\mathcal{A}\left(\mathbb{R}^2\right)$.*

Proof. Let $[u_{\varepsilon,\eta,\rho}]_{\mathcal{A}(\mathbb{R}^2)}$ be the solution to (P_{gen}) obtained in Theorem 8. Let $v = [v_\varepsilon]$ be another solution to (P_{gen}). There are $(i_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}\left(\mathbb{R}^2\right)$ and $(\alpha_\rho)_\rho$, $(\beta_\rho)_\rho \in \mathcal{N}(\mathbb{R})$, such that

$$\left\{ \begin{array}{l} \dfrac{\partial^2 v_{\varepsilon,\eta,\rho}}{\partial x \partial t}(x,t) = F_\varepsilon(x,t,v'_{\varepsilon,\eta,\rho}(x,t)) + i_{\varepsilon,\eta,\rho}(x,t), \\ v_{\varepsilon,\eta,\rho}(x,f(x)) = \varphi_\rho(x) + \alpha_\rho(x), \\ \dfrac{\partial v_{\varepsilon,\eta,\rho}}{\partial t}(x,f(x)) = \psi_\rho(x) + \beta_\rho(x). \end{array} \right.$$

The uniqueness of the solution to (P_G) will be a consequence of

$$(v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2).$$

It is easy to see that

$$\left((x,t) \mapsto \iint\limits_{D(x,t,f)} i_{\varepsilon,\eta,\rho}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \right)_{(\varepsilon,\eta,\rho)} \in \mathcal{N}\left(\mathbb{R}^2\right).$$

So, there is $(j_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}\left(\mathbb{R}^2\right)$ such that

$$v_{\varepsilon,\eta,\rho}(x,t) = v_{0,\varepsilon,\rho}(x,t) - \iint\limits_{D(x,t,f)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\,\mathrm{d}\xi\,\mathrm{d}\zeta + j_{\varepsilon,\eta,\rho}(x,t),$$

with $v_{0,\varepsilon,\eta,\rho}(x,t) = u_{0,\varepsilon,\eta,\rho}(x,t) + \theta_\rho(x,t)$, where $\theta_\rho(x,t) = B_\rho(t) - B_\rho(f(x)) + \alpha_\rho(x)$ and B_ρ is a primitive of $\beta_\rho \circ f_\eta^{-1}$. So $(\theta_\rho)_{\varepsilon,\rho}$ belongs to $\mathcal{N}(\mathbb{R}^2)$. Hence there is $(\sigma_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2)$ such that

$$v_{\varepsilon,\eta,\rho}(x,t) = u_{0,\varepsilon,\eta,\rho}(x,t) + \sigma_{\varepsilon,\eta,\rho}(x,t) - \iint\limits_{D(x,t,f)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\,\mathrm{d}\xi\,\mathrm{d}\zeta.$$

Let us put $w_{\varepsilon,\eta,\rho} = v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho}$ and show that $(w_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2)$. We have to prove that

$$\forall K \Subset \mathbb{R}^2, \forall n \in \mathbb{N},\ (P_{K,n}(w_{\varepsilon,\eta,\rho}))_{(\varepsilon,\eta,\rho)} \in I_A.$$

We have

$$w_{\varepsilon,\eta,\rho}(x,t) = \iint\limits_{D(x,t,f)} \left(-F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) + F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta))\right) \mathrm{d}\xi\,\mathrm{d}\zeta + \sigma_{\varepsilon,\eta,\rho}(x,t),$$

but

$$F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) - F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta)) = w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\int_0^1 \frac{\partial F_\varepsilon}{\partial z}(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta) + \theta w'_{\varepsilon,\eta,\rho}(\xi,\zeta))\,\mathrm{d}\theta,$$

then

$$w_{\varepsilon,\eta,\rho}(x,t) = -\iint\limits_{D(x,t,f)} w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\left(\int_0^1 \frac{\partial F_\varepsilon}{\partial z}(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta) + \theta(w'_{\varepsilon,\eta,\rho}(\xi,\zeta)))\,\mathrm{d}\theta\right)\mathrm{d}\xi\,\mathrm{d}\zeta + \sigma_{\varepsilon,\eta,\rho}(x,t).$$

Let $(x,t) \in K_{a,\eta}$. Since $D(x,t,f) \subset K_{a,\eta}$, if $t \geq f(x)$, we have

$$\text{(6)} \qquad \begin{aligned} |w_{\varepsilon,\eta,\rho}(x,t)| &\leq m_{a,\varepsilon,\eta} \int_x^{f_\eta^{-1}(t)} \int_{f_\eta(\xi)}^{t} \left|w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\right| \mathrm{d}\xi\,\mathrm{d}\zeta + \|\sigma_{\varepsilon,\eta,\rho}\|_{\infty,K_a,\eta} \\ &\leq c_1 r_\varepsilon^p \int_{f_\eta^{-1}(-a)}^{f_\eta^{-1}(a)} \int_{f_\eta(x)}^{t} \left|w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\right| \mathrm{d}\xi\,\mathrm{d}\zeta + \|\sigma_{\varepsilon,\eta,\rho}\|_{\infty,K_a,\eta}. \end{aligned}$$

As

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial u_0}{\partial t}(x,t) - \int_x^{f_\eta^{-1}(t)} F_\varepsilon(\xi,t,u'_{\varepsilon,\eta,\rho}(\xi,t))\,\mathrm{d}\xi,$$

we have

$$w'_{\varepsilon,\eta,\rho}(\xi,t) = \int_x^{f_\eta^{-1}(t)} \left(F_\varepsilon(\xi,t,u'_{\varepsilon,\eta,\rho}(\xi,t)) - F_\varepsilon(\xi,t,v'_{\varepsilon,\eta,\rho}(\xi,t))\right) \mathrm{d}\xi$$
$$= \int_x^{f_\eta^{-1}(t)} \left(w'_{\varepsilon,\eta,\rho}(\xi,t) \int_0^1 \frac{\partial F_\varepsilon}{\partial z}(\xi,t,u'_{\varepsilon,\eta,\rho}(\xi,\eta) + \theta w''_{\varepsilon,\rho}(\xi,\eta))\,\mathrm{d}\theta \right) \mathrm{d}\xi$$

so

$$\left|w'_{\varepsilon,\eta,\rho}(\xi,t)\right| \leq c_1 r_\varepsilon^p \int_x^{f_\eta^{-1}(t)} \left|w'_{\varepsilon,\eta,\rho}(\xi,t)\right| \mathrm{d}\xi.$$

Put $e_{\varepsilon,\eta,\rho}(\xi) = \sup_{t\in[-a,a]} \left|w'_{\varepsilon,\eta,\rho}(\xi,t)\right|$. Then

$$\left|w'_{\varepsilon,\eta,\rho}(\xi,t)\right| \leq c_1 r_\varepsilon^p \int_x^{f_\eta^{-1}(t)} e_{\varepsilon,\eta,\rho}(\xi)\,\mathrm{d}\xi.$$

We deduce that

$$\forall \xi \in \left[f_\eta^{-1}(-a), f_\eta^{-1}(a)\right], e_{\varepsilon,\eta,\rho}(\xi) \leq c_1 r_\varepsilon^p \int_x^{f^{-1}(t)} e_{\varepsilon,\eta,\rho}(\xi)\,\mathrm{d}\xi.$$

Thus, according to Gronwall's lemma, $e_{\varepsilon,\eta,\rho}(\xi) = 0$ and consequently

$$\forall\,(\xi,t) \in \left[f_\eta^{-1}(-a), f_\eta^{-1}(a)\right] \times [-a,a], w'_{\varepsilon,\eta,\rho}(\xi,t) = 0.$$

We obtain the same result for $t \leq f_\eta(x)$. Then, according to (6), we have

$$|w_{\varepsilon,\eta,\rho}(x,t)| \leq \|\sigma_{\varepsilon,\eta,\rho}\|_{\infty,K_{a,\eta}}.$$

Since $(\sigma_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2)$ we have $(\|\sigma_{\varepsilon,\eta,\rho}\|_{\infty,K_{a,\eta}})_{(\varepsilon,\eta,\rho)} \in I_A$ then

$$(\|w_{\varepsilon,\eta,\rho}\|_{\infty,K_{a,\eta}})_{(\varepsilon,\eta,\rho)} \in I_A$$

This implies the 0th order estimate. According to (1) $(w_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2)$, and consequently u is the unique solution to (P_G). □

Remark 8. *Construction of $\mathcal{A}\left(\mathbb{R}^2\right)$ in the case of regular data. If the data s and t are smooth, we take $(\varepsilon,\eta) \in \Lambda = \Lambda_1 \times \Lambda_2 = (0,1] \times (0,1]$. Let $(r_\varepsilon)_\varepsilon$ be in $(\mathbb{R}_*^+)^{(0,1]}$ such that $\lim_{\varepsilon\to 0} r_\varepsilon = +\infty$. We take $\mathcal{C} = A/I_A$ the ring overgenerated by $(\varepsilon)_{(\varepsilon,\eta)}$, $(\eta)_{(\varepsilon,\eta)}$, $(r_\lambda)_{(\varepsilon,\eta)}$, $(l_\eta)_{(\varepsilon,\eta)}$, $(e^{a_\eta r_\varepsilon})_{(\varepsilon,\eta)}$, elements of $(\mathbb{R}_*^+)^{(0,1]\times(0,1]}$. Then $\mathcal{A}\left(\mathbb{R}^2\right) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E},\mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}^2), (P_{K,l})_{K\Subset\mathbb{R}^2, l\in\mathbb{N}}\right)$ and, in the same*

way, $\mathcal{A}(\mathbb{R}) = \mathcal{X}(\mathbb{R})/\mathcal{N}(\mathbb{R})$ *is built on* $\mathcal{C}$ *with* $(\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}), (P_{K,l})_{K \Subset \mathbb{R}, l \in \mathbb{N}}\right)$*. Nonetheless, the algebra* $\mathcal{A}\left(\mathbb{R}^2\right)$ *is not the same in the two cases, regular data and irregular data.*
We set $\varphi = s$ *and* $\psi = t$*, elements of* $\mathrm{C}^\infty(\mathbb{R})$ *canonically embedded in* $\mathcal{A}(\mathbb{R})$*. If* $\alpha \in \mathcal{A}(\mathbb{R})$ *we take* $\alpha_\rho = \alpha$*, if* $\alpha \in \mathcal{N}(\mathbb{R})$ *we take* $\alpha_\rho = 0$*. Then we can rewrite this section and get similar results. We have the same definitions as previously and we obtain the same theorems, the same proofs replacing* φ_ρ *by* φ *and* ψ_ρ *by* ψ*. As previously, we can prove that Problem* (P_{gen}) *has a generalized solution* $u = [u_{\varepsilon,\eta}]$ *in the algebra* $\mathcal{A}\left(\mathbb{R}^2\right)$*.*

4.2. Independence of the generalized solution from the class of cutoff functions

See [14]. Recall that $\Lambda_1 = (0, 1]$, set

$$\mathcal{X}_1(\mathbb{R}) = \{(g_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N},\ (P_{K,l}(g_\varepsilon))_\varepsilon \in |A|\},$$
$$\mathcal{N}_1(\mathbb{R}) = \{(g_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N},\ (P_{K,l}(g_\varepsilon))_\varepsilon \in |I_A|\},$$
$$\mathcal{A}_1(\mathbb{R}) = \mathcal{X}_1(\mathbb{R})/\mathcal{N}_1(\mathbb{R}).$$

Consider $\mathcal{T}(\mathbb{R})$ the set of families of smooth one-variable functions $(h_\varepsilon)_{\varepsilon \in \Lambda_1} \in \mathcal{X}_1(\mathbb{R})$, verifying the following assumptions

$$\exists\, (s_\varepsilon)_\varepsilon \in \mathbb{R}_*^{(0,1]} : \sup_{z \in [-s_\varepsilon, s_\varepsilon]} |h_\varepsilon(z)| = 1, \tag{7}$$
$$h_\varepsilon(z) = \begin{cases} 0, \text{ if } \ |z| \geq s_\varepsilon \\ 1, \text{ if } \ -s_\varepsilon + 1 \leq z \leq s_\varepsilon - 1 \end{cases}.$$

$$\exists q \in \mathbb{N}^*, \forall\, (h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R}), \forall \varepsilon, s_\varepsilon \leq r_\varepsilon^q. \tag{8}$$

Moreover, assume that $\dfrac{\partial^n h_\varepsilon}{\partial z^n}$ is bounded on $J_\varepsilon = [-s_\varepsilon, s_\varepsilon]$ for any integer n, $n > 0$.

We have $(g_\varepsilon)_{\varepsilon \in \Lambda_1} \in \mathcal{T}(\mathbb{R})$. Recall that $\phi_\varepsilon(z) = z g_\varepsilon(z)$ for $z \in \mathbb{R}$, $F_\varepsilon(x, y, z) = F(x, y, \phi_\varepsilon(z))$ for $(x, y, z) \in \mathbb{R}^3$ and

$$\sup_{z \in [-r_\varepsilon, r_\varepsilon]} \left| \frac{\partial^n g_\varepsilon}{\partial z^n}(z) \right| = M_n.$$

Let $g \in \mathcal{T}(\mathbb{R})/\mathcal{N}_1(\mathbb{R})$ be the class of $(g_\varepsilon)_\varepsilon$. Take $(h_\varepsilon)_\varepsilon$ another representative of g, that is to say $(h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R})$ and

$$(g_\varepsilon - h_\varepsilon)_\varepsilon \in \mathcal{N}_1(\mathbb{R}). \tag{9}$$

Set $\sigma_\varepsilon(z) = z h_\varepsilon(z)$ for $z \in \mathbb{R}$, $H_\varepsilon(x, y, z) = F(x, y, \sigma_\varepsilon(z))$ for $(x, y, z) \in \mathbb{R}^3$ and

$$\sup_{z \in [-s_\varepsilon, s_\varepsilon]} \left| \frac{\partial^n h_\varepsilon}{\partial z}(z) \right| = M'_n.$$

Our choice is made such that $(\mathrm{supp}\,(h_\varepsilon))_\varepsilon$ have the same growth as $(\mathrm{supp}\,(f_\varepsilon))_\varepsilon$ with respect to the scale $(r_\varepsilon^q)_\varepsilon$, in this way the corresponding solutions are lying in the same algebra $\mathcal{A}\left(\mathbb{R}^2\right)$.

Proposition 10. *Set* $S_n = \left\{\alpha \in \mathbb{N}^3 : |\alpha| = n\right\}$ *when* $n \in \mathbb{N}^*$. *Let* $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$, H_ε *defined by*

$$H_\varepsilon(x, y, z) = F(x, y, \sigma_\varepsilon(z)).$$

Assume that

$$\forall\,(x, y) \in \mathbb{R}^2, F(x, y, 0) = 0\ ,$$

$$\exists p_0 > 0, \forall \alpha \in \mathbb{N}^3, |\alpha| = n > p_0, D^\alpha F(x, y, z) = 0,$$

$$\begin{gathered}\forall n \in \mathbb{N}, n \leq p_0, \exists d_n > 0, \forall \varepsilon \in (0, 1]\,, \forall K \Subset \mathbb{R}^2, \\ \sup_{(x,y)\in K;\ z\in J_\varepsilon;\alpha\in Sn} |D^\alpha F(x, y, z)| \leq d_n r_\varepsilon^{p_0},\end{gathered} \tag{10}$$

then

$$\begin{gathered}\forall n \in \mathbb{N}, n \leq p_0, \exists c_n > 0, \forall \varepsilon \in (0, 1]\,, \forall K \Subset \mathbb{R}^2, \\ \sup_{(x,y)\in K;\ z\in\mathbb{R};\alpha\in Sn} |D^\alpha H_\varepsilon(x, y, z)| \leq c_n r_\varepsilon^{p_0(1+q)}\end{gathered}$$

and $\mathcal{A}\left(\mathbb{R}^2\right)$ *is stable under the family* $(H_\varepsilon)_{(\varepsilon,\rho)}$.

We refer the reader to [14] for a detailed proof.

Theorem 11. *Assume that* $p = p_0(1 + q)$ *and the hypotheses of Proposition 10 are verified. Let* $\mathcal{F}$ *be the generalized operator associated to* F *via the family* $(g_\varepsilon)_\varepsilon$. *Let* $(h_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$ *be another family representative of the class* $[g_\varepsilon] = g$ *and leading to another generalized operator* $\mathcal{H}$ *associated to* F. *Then we have* $\mathcal{H} = \mathcal{F}$, *that is to say* $\mathcal{H}\,(u) = \mathcal{F}\,(u)$ *for any* $u \in \mathcal{A}(\mathbb{R}^2)$. *Then, in terms of representatives, that is to say, if* $(u_\lambda)_\lambda$, $(v_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$ *and* $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$, *then*

$$\left(F(\cdot, \cdot, \sigma_{\mu(\lambda)}\,(v_\lambda)) - F(\cdot, \cdot, \phi_{\mu(\lambda)}\,(u_\lambda))\right)_\lambda \in \mathcal{N}(\mathbb{R}^2).$$

We refer the reader to [14] for a detailed proof.

Corollary 12. *Problem* (P_{gen}), *a fortiori its solution, does not depend of the choice of the representative* $(f_\varepsilon)_\varepsilon$ *of the class* $f \in \mathcal{T}(\mathbb{R})/\mathcal{N}_1(\mathbb{R})$.

Proof. $(w_\lambda)_{(\varepsilon,\eta,\rho)} = (v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2)$ then

$$\left(w'_{\varepsilon,\eta,\rho}\right)_{(\varepsilon,\eta,\rho)} = \left(v'_{\varepsilon,\eta,\rho} - u'_{\varepsilon,\eta,\rho}\right)_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2).$$

We deduce that

$$\left(F(\cdot, \cdot, \sigma_\varepsilon\left(v'_{\varepsilon,\eta,\rho}\right)) - F(\cdot, \cdot, \phi_\varepsilon\left(u'_{\varepsilon,\eta,\rho}\right))\right)_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2),$$

that is to say $\mathcal{H}\,(u') = \mathcal{F}\,(u')$ for any $u \in \mathcal{A}(\mathbb{R}^2)$. □

4.3. Independence of the generalized solution from the class $[f_\eta]$

Lemma 13. *Let $(f_\eta)_\eta, (h_\eta)_\eta \in \mathcal{X}_\tau(\mathbb{R})$ such that for every η, f_η, h_η are bijective and*

$$\left(f_\eta^{-1}\right)_\eta, \left(h_\eta^{-1}\right)_\eta \in \mathcal{X}_\tau(\mathbb{R}).$$

If moreover $(h_\eta - f_\eta)_\eta \in \mathcal{N}_\tau(\mathbb{R})$ we have that

$$\left(f_\eta^{-1} - h_\eta^{-1}\right)_\eta \in \mathcal{N}_\tau(\mathbb{R})$$

We refer the reader to [1] for a detailed proof.

Theorem 14. *Under the same hypotheses as Theorem 8, the generalized function u represented by the family $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)}$ of solutions to Problems $(P_{\varepsilon,\eta,\rho})$, does not depend on the choice of the representative $(f_\eta)_\eta$ of the class $f = [f_\eta] \in \mathcal{G}_\tau(\mathbb{R})$.*

Proof. We have

$$u_{\varepsilon,\eta,\rho}(x,t) = u_{0,\varepsilon,\eta,\rho}(x,t) - \iint_{D(x,t,f_\eta)} F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\xi d\zeta,$$

where $u_{0,\varepsilon,\eta,\rho}(x,t) = \Upsilon^f_{\eta,\rho}(t) - \Upsilon^f_{\eta,\rho}(f_\eta(x)) + \varphi_\rho(x)$ and $\left(\Upsilon^f_{\eta,\rho}\right)' = \psi_\rho \circ f_\eta^{-1}$. So we take $(h_\eta)_\eta \in \mathcal{X}_\tau(\mathbb{R})$, such that $(f_\eta - h_\eta)_\eta \in \mathcal{N}_\tau(\mathbb{R})$; let $v = [v_{\varepsilon,\eta,\rho}]$ be the corresponding generalized solution. Let us prove that $u = v$. We will in fact prove a slightly stronger statement for reasons that will be clear in the course of the proof

$$\forall K \Subset \mathbb{R}^2, \forall \alpha \in \mathbb{N}^2, \left(P_{K_\eta,\alpha}(u_{\varepsilon,\eta,\rho} - v_{\varepsilon,\eta,\rho})\right)_{(\varepsilon,\eta,\rho)} \in I_A.$$

Let us now fix $K \Subset \mathbb{R}^2$. We have

$$v_{\varepsilon,\eta,\rho}(x,t) = v_{0,\varepsilon,\eta,\rho}(x,t) - \iint_{D(x,t,h_\eta)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\xi d\zeta,$$

where $v_{0,\varepsilon,\eta,\rho}(x,t) = \Upsilon^h_{\eta,\rho}(t) - \Upsilon^h_{\eta,\rho}(h_\eta(x)) + \varphi_\rho(x)$ and $\left(\Upsilon^h_{\eta,\rho}\right)' = \psi_\rho \circ h_\eta^{-1}$. Then we get

$$u_{0,\varepsilon,\eta,\rho}(x,t) - v_{0,\varepsilon,\eta,\rho}(x,t) = \Upsilon^f_{\eta,\rho}(t) - \Upsilon^h_{\eta,\rho}(t) - \Upsilon^f_{\eta,\rho}(f_\eta(x) + \Upsilon^h_{\eta,\rho}(h_\eta(x)).$$

We compute

$$\frac{\partial}{\partial x}\left(-\Upsilon^f_{\eta,\rho}(f_\eta(x) + \Upsilon^h_{\eta,\rho}(h_\eta(x))\right) = \psi_\rho(x)\left(f'_\eta(x) - h'_\eta(x)\right)$$

and $(f_\eta - h_\eta)_\eta \in \mathcal{N}_\tau(\mathbb{R})$, but we can find $p \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ we have

$$\forall x \in \mathbb{R},\ \frac{\partial}{\partial x}\left(-\Upsilon^f_{\eta,\rho}(f_\eta(x) + \Upsilon^h_{\eta,\rho}(h_\eta(x))\right) \le \eta^m(1+|x|)^p,$$

so

$$\left\| \frac{\partial}{\partial x} \left(-\Upsilon^f_{\eta,\rho}(f_\eta(x) + \Upsilon^h_{\eta,\rho}(h_\eta(x)) \right) \right\|_{K_{a,\eta}} \leq \eta^m \max\left\{ (1 + |a_{K,\eta}/2|)^p \right\},$$

but $((1 + |a_{K,\eta}/2|)^p)_{(\varepsilon,\eta,\rho)} \in |A|$ thus we have obtained that

$$\left(\left\| \frac{\partial}{\partial x} \left(-\Upsilon^f_{\eta,\rho}(f_\eta(x) + \Upsilon^h_{\eta,\rho}(h_\eta(x)) \right) \right\|_{K_\eta} \right)_{(\varepsilon,\eta,\rho)} \in I_A.$$

Then we obtain that $\left(P_{K_\eta,1} \left(-\Upsilon^f_{\eta,\rho}(f_\eta(x) + \Upsilon^h_{\eta,\rho}(h_\eta(x)) \right) \right)_{(\varepsilon,\eta,\rho)} \in I_A$. The proof is similar for higher derivatives. Now as $\left(f_\eta^{-1} - h_\eta^{-1} \right)_\eta \in \mathcal{N}(\mathbb{R})$ and $\psi_\rho \in \mathcal{O}_M$ then $\left(\psi_\rho \circ f_\eta^{-1} - \psi_\rho \circ h_\eta^{-1} \right)_\eta \in \mathcal{N}(\mathbb{R})$ and then we finally obtain that

$$\forall \alpha, \left(P_{K_\eta,\alpha}(u_{0,\varepsilon,\eta,\rho} - v_{0,\varepsilon,\eta,\rho}) \right)_{(\varepsilon,\eta,\rho)} \in I_A.$$

We compute

$$\begin{aligned}
& u_{1,\varepsilon,\eta,\rho}(x,t) - v_{1,\varepsilon,\eta,\rho}(x,t) \\
= & \int_{f_\eta^{-1}(t)}^{x} \int_t^{f_\eta(\xi)} F_\varepsilon(\xi, \zeta, u'_{\varepsilon,\eta,\rho}(\xi,\zeta)) d\zeta d\xi \\
& - \int_{h_\eta^{-1}(t)}^{x} \int_t^{h_\eta(\xi)} F_\varepsilon(\xi, \zeta, v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) d\zeta d\xi \\
= & \int_{f_\eta^{-1}(t)}^{x} \int_t^{f_\eta(\xi)} \left[F_\varepsilon(\xi, \zeta, u'_{\varepsilon,\eta,\rho}(\xi,\zeta)) - F_\varepsilon(\xi, \zeta, v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) \right] d\zeta d\xi \\
& - \int_{h_\eta^{-1}(t)}^{f_\eta^{-1}(t)} \int_t^{h_\eta(\xi)} F_\varepsilon(\xi, \zeta, v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) d\zeta d\xi \\
& - \int_{f_\eta^{-1}(t)}^{x} \int_{f_\eta(\xi)}^{h_\eta(\xi)} F_\varepsilon(\xi, \zeta, v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) d\zeta d\xi.
\end{aligned}$$

As $f_\eta \circ h_\eta^{-1} \equiv id \mod \mathcal{N}^s(\mathbb{R})$, we have

$$\begin{aligned}
& \sup_{t \in [-a,a]} \left| \int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)} \int_t^{h_\eta(\xi)} F_\varepsilon(\xi, \zeta, v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) d\zeta d\xi \right| \\
& \qquad \leq 2b \int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)} \sup_{\zeta \in [-a,a]} \left| F_\varepsilon(\xi, \zeta, v'_{\varepsilon,\eta,\rho}(\xi,\zeta)) \right| d\xi \\
& \qquad \leq 2b \left\| f_\eta^{-1} - h_\eta^{-1} \right\|_{[-a,a]} \left\| F_\varepsilon \right\|_{[\lambda_\eta,\mu_\eta] \times [-a,a] \times \mathbb{R}}
\end{aligned}$$

where

$$\begin{cases} \lambda_\eta = \min\{f_\eta^{-1}(-a), h_\eta^{-1}(-a)\} \\ \mu_\eta = \max\{f_\eta^{-1}(a), h_\eta^{-1}(a)\}. \end{cases}$$

As $(f_\eta - h_\eta)_\eta \in \mathcal{N}_\tau(\mathbb{R})$ and $\|F_\varepsilon\|_{[\lambda_\eta,\mu_\eta]\times[-a,a]\times\mathbb{R}} \in |A|$, then

$$\sup_{t\in[-a,a]} \left| \int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)} \int_t^{h_\eta(\xi)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\zeta d\xi \right| \in I_A.$$

For the first derivative we have

$$\frac{d}{dt}\left(\int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)} \int_t^{h_\eta(\xi)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\zeta d\xi \right)$$
$$= -\int_t^{h_\eta(f_\eta^{-1}(t))} F_\varepsilon(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta))d\zeta$$
$$+ \int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)} F_\varepsilon(\xi,t,v'_{\varepsilon,\eta,\rho}(\xi,t))d\xi.$$

And the same kind of arguments take care of those two terms. Now for the higher derivatives

$$\frac{d^2}{dt^2}\left(\int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)} \int_t^{h_\eta(\xi)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\zeta d\xi \right) \tag{11}$$
$$= -F_\varepsilon(f_\eta^{-1}(t),h_\eta(f_\eta^{-1}(t)),v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),h_\eta(f_\eta^{-1}(t)))) + F_\varepsilon(f_\eta^{-1}(t),t,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),t)) \tag{12}$$
$$-\int_t^{h_\eta(f_\eta^{-1}(t))} \frac{d}{dt}\left(F_\varepsilon(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta))\right)d\zeta \tag{13}$$
$$+F_\varepsilon(h_\eta^{-1}(t),t,v'_{\varepsilon,\eta,\rho}(h_\eta^{-1}(t),t)) - F_\varepsilon(f_\eta^{-1}(t),t,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),t)) \tag{14}$$
$$+\int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)} \frac{d}{dt}\left(F_\varepsilon(\xi,t,v'_{\varepsilon,\eta,\rho}(\xi,t))\right)d\xi. \tag{15}$$

The hypotheses on F_ε and the fact that $(h_\eta^{-1} - f_\eta^{-1})_\eta \in \mathcal{N}_\tau(\mathbb{R})$ takes care of the terms of lines (12) and (14). Let us now turn our attention to line (13). We have

$$\int_t^{h_\eta(f_\eta^{-1}(t))} \frac{d}{dt}\left(F_\varepsilon(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta))\right)d\zeta$$
$$= \int_t^{h_\eta(f_\eta^{-1}(t))} [(f_\eta^{-1})'(t)\frac{\partial F_\varepsilon}{\partial \xi}(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta))$$
$$+ (f_\eta^{-1})'(t)\frac{\partial^2 v_{\varepsilon,\eta,\rho}}{\partial x \partial t}(f_\eta^{-1}(t),\zeta)\frac{\partial F_\varepsilon}{\partial z}(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta))]d\zeta.$$

As $(h_\eta \circ f_\eta^{-1} - id)_\eta \in \mathcal{N}_\tau(\mathbb{R})$ we can find a compact $L \subset \mathbb{R}$ such that

$$\forall \eta,\ \{h_\eta \circ f_\eta^{-1}(t) : t \in [-b,b]\} \cup [-b,b] \subset L,$$

and moreover it is sufficient to prove that

$$\Big(\sup_{t\in[-b,b],\zeta\in L} \mid (f_\eta^{-1})'(t)\frac{\partial F_\varepsilon}{\partial \xi}(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta)) + (f_\eta^{-1})'(t)\frac{\partial^2 v_\eta}{\partial x\partial t}(f_\eta^{-1}(t),\zeta)\frac{\partial F_\varepsilon}{\partial z}(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta)) \mid\Big)_{\varepsilon,\eta,\rho} \in |A|\,.$$

But it is easy to see that

$$\left(\sup_{t\in[-b,b],\zeta\in L}\left|(f_\eta^{-1})'(t)\frac{\partial F_\varepsilon}{\partial \xi}(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}(f_\eta^{-1}(t),\zeta))\right|\right)_{(\varepsilon,\eta,\rho)} \in |A|\,.$$

For the other term the only part needing some new explanations is to prove that

$$\left(\sup_{t\in[-b,b],\zeta\in L}\frac{\partial^2 v_{\varepsilon,\eta,\rho}}{\partial x\partial t}(f_\eta^{-1}(t),\zeta)\right)_{(\varepsilon,\eta,\rho)} = \left(\sup_{t\in[-b,b],\zeta\in L} F_\varepsilon(f_\eta^{-1}(t),\zeta,v'_{\varepsilon,\eta,\rho}\left(f_\eta^{-1}(t),\zeta\right)\right)_{(\varepsilon,\eta,\rho)} \in |A|\,.$$

But here we use the fact that $(h_\eta^{-1}-f_\eta^{-1})_\eta \in \mathcal{N}_\tau(\mathbb{R})$ to find η_0 such that

$$\forall 0<\eta<\eta_0, \left\|f_\eta^{-1}-h_\eta^{-1}\right\|_{[-b,b]} < 1. \tag{16}$$

We proved in the proof of Theorem 8 that $\left(P_{K_\eta,\alpha}(v_{\varepsilon,\eta,\rho})\right)_{(\varepsilon,\eta,\rho)} \in |A|$ and because of (16) we have that $f_\eta^{-1}(L)\times[-b,b]\subset K_\eta$, which settles this case. For higher derivatives the reasoning involves the same estimate and presents no new obstacles. So this proves that

$$\forall\alpha,\ \left(P_{K_\eta,\alpha}\left(\int_{f_\eta^{-1}(t)}^{h_\eta^{-1}(t)}\int_t^{h_\eta(\xi)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\zeta d\xi\right)\right)_{(\varepsilon,\eta,\rho)} \in I_A.$$

Similar arguments apply to prove that

$$\forall\alpha,\ \left(P_{K_\eta,\alpha}\left(\int_{f_\eta^{-1}(t)}^{x}\int_{f_\eta(\xi)}^{h_\eta(\xi)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\zeta d\xi\right)\right)_{(\varepsilon,\eta,\rho)} \in I_A.$$

So we have proved that

$$P_{K_\eta,\alpha}\left(u_{1,\varepsilon,\eta,\rho}(x,t)-v_{1,\varepsilon,\eta,\rho}(x,t)\right)_{(\varepsilon,\eta,\rho)} \equiv \left(\int_{f_\eta^{-1}(t)}^{x}\int_t^{f_\eta(\xi)}\left[F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta))-F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\right]d\zeta d\xi\right)_{(\varepsilon,\eta,\rho)} \mod I_A.$$

We define

$$\begin{aligned}\sigma_{\varepsilon,\eta,\rho}(x,t) &= u_{\varepsilon,\eta,\rho}(x,t) - v_{\varepsilon,\eta,\rho}(x,t)\\ &\quad - \int_{f_\eta^{-1}(t)}^{x}\int_t^{f_\eta(\xi)} \left[F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta)) - F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\right] d\zeta d\xi.\end{aligned}$$

So, by the above arguments we just proved that $\left(P_{K_\eta,\alpha}\left(\sigma_{\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in I_A$. We now define $w_{\varepsilon,\eta,\rho}(x,t) = u_{\varepsilon,\eta,\rho}(x,t) - v_{\varepsilon,\eta,\rho}(x,t)$. Keeping the same notations as in the proof of Theorem 8, we want to prove that $\forall n$, $\left(P_{K_\eta,n}\left(w_{\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in I_A$. Let us first prove that $\left(P_{K_\eta,0}(w_{\varepsilon,\eta,\rho})\right)_{(\varepsilon,\eta,\rho)} \in I_A$. First we have

$$\begin{aligned}&F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta)) - F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\\ &= w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\int_0^1 \frac{\partial F_\varepsilon}{\partial z}(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta)+\theta(w'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\theta,\end{aligned}$$

then

$$\begin{aligned}&w_{\varepsilon,\eta,\rho}(x,t) = \sigma_{\varepsilon,\eta,\rho}(x,t) +\\ &\int_{f_\eta^{-1}(t)}^{x}\int_t^{f_\eta(\xi)} w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\left(\int_0^1 \frac{\partial F_\varepsilon}{\partial z}(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta)+\theta(w'_{\varepsilon,\eta,\rho}(\xi,\zeta))d\theta\right) d\zeta d\xi.\end{aligned}$$

Now we have

$$\forall\eta,\ \cup_{(x,t)\in K_\eta}\{(\xi,\zeta)\mid \xi\in[x,f_\eta^{-1}(t)], t\le\zeta\le f_\eta(\xi)\} \subset L_\eta = [\alpha_{K,\eta},\beta_{K,\eta}]\times[-b,b],$$

so that setting $l_\eta = \sup_{L_\eta\times\mathbb{R}}\left|\frac{\partial F}{\partial z}\right|$ we have

$$|w_{\varepsilon,\eta,\rho}(x,t)| \le l_\eta \int_{\alpha_{K,\eta}}^{\beta_{K,\eta}}\int_t^{f_\eta(x)} \left|w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\right| d\zeta d\xi + |\sigma_{\varepsilon,\eta,\rho}(x,t)|,$$

then
(17)

$$\forall (x,t)\in K_\eta, |w_{\varepsilon,\eta,\rho}(x,t)| \le l_\eta \int_{\alpha_{K,\eta}}^{\beta_{K,\eta}}\int_t^{f_\eta(x)} \left|w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\right| d\zeta d\xi + \|\sigma_{\varepsilon,\eta,\rho}\|_{K_\eta}.$$

We have

$$\begin{aligned}w'_{\varepsilon,\eta,\rho}(x,\zeta) &= \frac{\partial}{\partial t}\left(u_{0,\varepsilon,\eta,\rho} - v_{0,\varepsilon,\eta,\rho}\right)(x,\zeta)\\ &\quad + \int_x^{f_\eta^{-1}(\zeta)} F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta))\,\mathrm{d}\xi - \int_x^{h_\eta^{-1}(\zeta)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\,\mathrm{d}\xi,\end{aligned}$$

then

$$w'_{\varepsilon,\eta,\rho}(x,\zeta) = \frac{\partial w_{0,\varepsilon,\eta,\rho}}{\partial t}(x,\zeta) + \int_x^{f_\eta^{-1}(\zeta)} \left(F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta)) - F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\right) \mathrm{d}\xi - \int_{f_\eta^{-1}(\zeta)}^{h_\eta^{-1}(\zeta)} F_\varepsilon(\xi,\zeta,v'_{\varepsilon,\eta,\rho}(\xi,\zeta))\,\mathrm{d}\xi.$$

Thus we have

$$\left|w'_{\varepsilon,\eta,\rho}(x,\zeta)\right| \leq i_{\varepsilon,\eta,\rho} + \int_x^{f^{-1}(\zeta)} \left|w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\right| \left| \int_0^1 \frac{\partial F_\varepsilon}{\partial z}(\xi,\zeta,u'_{\varepsilon,\eta,\rho}(\xi,\zeta)) + \theta w''_{\varepsilon,\eta,\rho}(\xi,\zeta))\,\mathrm{d}\theta \right| \mathrm{d}\xi$$

where

$$(i_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} = \left(P_{K_\eta,1}(w_{0,\varepsilon,\eta,\rho}) + \left\|f_\eta^{-1} - h_\eta^{-1}\right\|_{[-a,a]} \left\|F_\varepsilon\right\|_{[\lambda_\eta,\mu_\eta]\times[-a,a]\times\mathbb{R}}\right)_{\varepsilon,\eta,\rho} \in I_A.$$

We deduce

$$\left|w'_{\varepsilon,\eta,\rho}(x,\zeta)\right| \leq i_{\varepsilon,\eta,\rho} + c_1 r_\varepsilon^p \int_x^{f_\eta^{-1}(t)} \left|w'_{\varepsilon,\eta,\rho}(\xi,\zeta)\right| \mathrm{d}\xi.$$

Put $e_{\varepsilon,\eta,\rho}(x) = \sup_{\zeta\in[-a,a]} \left|w'_{\varepsilon,\eta,\rho}(x,\zeta)\right|$. Then

$$\left|w'_{\varepsilon,\eta,\rho}(x,t)\right| \leq i_{\varepsilon,\eta,\rho} + c_1 r_\varepsilon^p \int_x^{f_\eta^{-1}(t)} e_{\varepsilon,\eta,\rho}(\xi)\,\mathrm{d}\xi.$$

We deduce that

$$\forall x \in [\lambda_\eta,\mu_\eta], e_{\varepsilon,\eta,\rho}(x) \leq i_{\varepsilon,\eta,\rho} + c_1 r_\varepsilon^p \int_x^{f_\eta^{-1}(t)} e_{\varepsilon,\eta,\rho}(\xi)\,\mathrm{d}\xi.$$

Thus, according to Gronwall's lemma,

$$0 \leq e_{\varepsilon,\eta,\rho}(x) \leq i_{\varepsilon,\eta,\rho} \exp\left(\int_x^{f_\eta^{-1}(t)} c_1 r_\varepsilon^p \,\mathrm{d}\xi. \right) \leq i_{\varepsilon,\eta,\rho} c_1 r_\varepsilon^p \left(\mu_\eta - \lambda_\eta\right),$$

then

$$0 \leq \left\| w'_{\varepsilon,\eta,\rho} \right\|_{\infty,K_\eta} \leq i_{\varepsilon,\eta,\rho} c_1 r_\varepsilon^p \left(\mu_\eta - \lambda_\eta\right)$$

and consequently $\left(P_{K_\eta,1}(w_{\varepsilon,\eta,\rho})\right)_{(\varepsilon,\eta,\rho)} \in I_A$. According (17), we deduce

$$\left(P_{K_\eta,0}(w_{\varepsilon,\eta,\rho})\right)_{(\varepsilon,\eta,\rho)} \in I_A.$$

Which implies the 0th order estimate. According to Proposition 1, we deduce $(w_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)} \in \mathcal{N}(\mathbb{R}^2)$; consequently u does not depend on the choice of the representative $(f_\eta)_\eta$ of the class $f = [f_\eta] \in \mathcal{G}_\tau(\mathbb{R})$. □

5. Non-characteristic non-Lipschitz problem with irregular data

5.1. Notations

We take $\Lambda = \Lambda_1 \times \Lambda_3 = (0,1] \times (0,1]$, and $\lambda = (\varepsilon, \rho)$; ($\eta$ is fixed and $f_\eta = f$). Let $(r_\varepsilon)_\varepsilon$ be in $(\mathbb{R}_*^+)^{(0,1]}$ such that $\lim\limits_{\varepsilon \to 0} r_\varepsilon = +\infty$. Set

$$(A'2) \qquad \left\{ \begin{array}{c} f \in \mathrm{C}^\infty(\mathbb{R}),\ f \text{ strictly increasing},\ f(\mathbb{R}) = \mathbb{R}, \\ \forall x \in \mathbb{R}, f'(x) \neq 0, \end{array} \right.$$

$$(18) \qquad \begin{cases} a_K = 2\max(f^{-1}(a), \left|f^{-1}(-a)\right|), \\ K_a = K_1 \times K_2 \text{ with } K_1 = [-a_K/2, a_K/2] \text{ and } K_2 = [-a, a]\,. \end{cases}$$

$$\begin{cases} \exists\, (l_\rho)_{(\rho)} \in \mathbb{R}_*^{(0,1]} \text{ such that } \forall K_2 \Subset \mathbb{R}, \forall \alpha_2 \in \mathbb{N}, \exists D_2 = D_{K_2,\alpha_2,\rho} \in \mathbb{R}_+^*, \exists q \in \mathbb{N}, \\ \max \left[\sup\limits_{K_2} \left| D^{\alpha_2} \psi_\rho(f^{-1}(t)) \right|, \sup\limits_{K_2} \left| D^{\alpha_2} \Upsilon_\rho(t) \right| \right] \leq D_2 \left(l_\rho\right)^{-q}. \end{cases}$$

We take $\mathcal{C} = A/I_A$ the ring overgenerated by $(\varepsilon)_{(\varepsilon,\rho)}$, $(\rho)_{(\varepsilon,\rho)}$, $(r_\varepsilon)_{(\varepsilon,\rho)}$, $(l_\rho)_{(\varepsilon,\rho)}$, $(e^{r_\varepsilon})_{(\varepsilon,\rho)}$ elements of $(\mathbb{R}_*^+)^{(0,1]\times(0,1]}$. Then $\mathcal{A}\left(\mathbb{R}^2\right) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}^2), (P_{K,l})_{K \Subset \mathbb{R}^2, l \in \mathbb{N}}\right)$ and, in the same way, $\mathcal{A}(\mathbb{R}) = \mathcal{X}(\mathbb{R})/\mathcal{N}(\mathbb{R})$ is built on $\mathcal{C}$ with $(\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}), (P_{K,l})_{K \Subset \mathbb{R}, l \in \mathbb{N}}\right)$. Then we can rewrite the previous section and get similar results. We have the same definitions as previously and we obtain the same theorems, the same proofs replacing f_η by f. As previously, we can prove that Problem (P_{gen}) has a generalized solution $u = [u_{\varepsilon,\rho}]$ in the algebra $\mathcal{A}\left(\mathbb{R}^2\right)$.

5.2. Comparison with classical solutions

Even if the data are as irregular as distributions, it may happen that the initial formal ill-posed problem (P_{form}) has nonetheless a local smooth solution as it will be seen in the following example 3. We are going to prove that this solution is exactly the restriction (according to the sheaf theory sense) of the generalized one.

The generalized solution to Problem (P_{gen}) is defined from the integral representation (1). Thus, we are going to study the relationship between this generalized function and the classical solutions to (P_{form}) (when they exist) on a domain Ω such that $\forall\,(x,y) \in \Omega,\ D(x,y,g) \subset \Omega$. This justified to choose $\Omega = \left]f^{-1}(\mu), f^{-1}f^{-1}(\nu)\right[\times]\mu,\nu[$ when $(\mu,\nu) \in \mathbb{R}^2$ with $\mu < 0 < \nu$.

Remark 9. *If the non-regularized problem (P_{form}) has a smooth solution v on Ω then, necessarily, we have $\Omega \subset \mathbb{R}^2 \backslash \mathrm{singsupp}\,(u)$.*

Recall that there exists a canonical sheaf embedding of $\mathrm{C}^\infty(\cdot)$ into $\mathcal{A}\,(\cdot)$, through the morphism of algebra

$$\sigma_O : \mathrm{C}^\infty\,(O) \to \mathcal{A}\,(O)\,, f \mapsto [f_{\varepsilon,\rho}]\,(\text{where } O \text{ is any open subset of } \mathbb{R}^2 \text{ and } f_{\varepsilon,\rho} = f).$$

The presheaf $\mathcal{A}$ allows restriction and, as usually, we denote by $u|_O$ the restriction on O of $u \in \mathcal{A}\left(\mathbb{R}^2\right)$.

Theorem 15. *Let $u = [u_{\varepsilon,\rho}]$ be the solution to Problem (P_{gen}). Let Ω be an open subset of $\mathbb{R}^2$ such that $\Omega \subset \mathbb{R}^2 \backslash \mathrm{singsupp}\,(u)$. Assume that $\Omega = \bigcup\limits_{\varepsilon \in \Lambda_1} \Omega_\varepsilon$ with $(\Omega_\varepsilon)_\varepsilon$ is an increasing family of open subsets of $\mathbb{R}^2$ such that $\Omega_\varepsilon =]g(a_\varepsilon), g(b_\varepsilon)[\times]a_\varepsilon, b_\varepsilon[$ when $(a_\varepsilon, b_\varepsilon) \in \mathbb{R}^2$ with $a_\varepsilon < 0 < b_\varepsilon$. Assume that problem ($P_{form}$) has a smooth solution v on Ω such that $\sup\limits_{(x,y)\in\Omega_\varepsilon} |v'(x,y)| < r_\varepsilon - 1$ for any ε. Then v (element of $\mathrm{C}^\infty\,(\Omega)$ canonically embedded in $\mathcal{A}(\Omega)$) is the restriction (according to the sheaf theory sense) of u to Ω, $v = u|_\Omega$.*

Proof. We clearly have $\forall\,(x,y) \in \Omega, \exists\varepsilon_0, \forall\varepsilon \le \varepsilon_0, (x,y) \in \Omega_\varepsilon$. Then $D(x,y,g) \subset \Omega_\varepsilon \subset \Omega$; we have

$$v(x,y) = v_0(x,y) - \iint_{D(x,y,g)} F(\xi,\zeta,v'(\xi,\zeta))d\xi d\zeta$$

then

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial v_0}{\partial t}(x,t) - \int_x^{f^{-1}(t)} F(\xi,t,v'(\xi,t))\,\mathrm{d}\xi,$$

We take a representative of u in the family $(u_{\varepsilon,\rho})_{(\varepsilon,\rho)}$; we have

$$\forall\,(x,y) \in \Omega,\ u_{\varepsilon,\rho}(x,y) = u_{0,\varepsilon,\rho}(x,y) - \iint_{D(x,y,g)} F_\varepsilon(\xi,\zeta,u'_{\varepsilon,\rho}(\xi,\zeta))d\xi d\zeta$$

and $v_0(x,y) = u_{0,\varepsilon,\rho}(x,y)$. Moreover

$$\frac{\partial u_{\varepsilon,\rho}}{\partial t}(x,t) = \frac{\partial u_{0,\varepsilon,\rho}}{\partial t}(x,t) - \int_x^{f^{-1}(t)} F_\varepsilon(\xi,t,u'_{\varepsilon,\rho}(\xi,t))\,\mathrm{d}\xi,$$

Set $(w_{\varepsilon,\rho})_{(\varepsilon,\rho)} = \left(u_{\varepsilon,\rho}|_{\Omega} - v\right)_{(\varepsilon,\rho)}$ and take $K \Subset \Omega$. There exists ε_1 such that, for all $\varepsilon < \varepsilon_1$, $K \Subset \Omega_\varepsilon$. According to the definition of Ω_ε, there exists a, $0 < a < (b_\varepsilon - a_\varepsilon)/2$, such that $K \subset Q_a \subset \Omega$ with $Q_a = \left[f^{-1}(a_\varepsilon + a), f^{-1}(b_\varepsilon - a)\right] \times [a_\varepsilon + a, b_\varepsilon - a]$. Take $(x, y) \in K$, then $D(x, y, g) \subset Q_a$. Note that, for $(\xi, \varsigma, z) \in \Omega_\varepsilon \times]-r_\varepsilon + 1, r_\varepsilon - 1[$, we have $F(\xi, \varsigma, z) = F_\varepsilon(\xi, \varsigma, z)$ by construction of F_ε and values of v' are in $]-r_\varepsilon + 1, r_\varepsilon - 1[$. Thus v' and $u'_{\varepsilon,\rho}$ are solutions of the same integral equation, which admits a unique solution since F_ε is a smooth function of its arguments. Thus, for all $\varepsilon \leq \varepsilon_1$, v' and $u'_{\varepsilon,\rho}$ are equal on Ω_ε. Moreover, we have (like 6)

$$|w_{\varepsilon,\rho}(x,t)| \leq c_1 r_\varepsilon^p \int_{f^{-1}(b_\varepsilon - a)}^{f^{-1}(a_\varepsilon + a)} \int_{f(x)}^{t} \left|w'_{\varepsilon,\rho}(\xi,\zeta)\right| \,\mathrm{d}\xi\,\mathrm{d}\zeta,$$

then $w_{\varepsilon,\rho} = 0$. We deduce that v and $u_{\varepsilon,\rho}$ are solutions of the same integral equation, which admits a unique solution. Thus $(P_{K,n}(v))_{(\varepsilon,\rho)} \in |A|$ for any $K \Subset \Omega$ and $n \in \mathbb{N}$. Then v (identified with $\left[(v)_{(\varepsilon,\rho)}\right]$) belongs to $\mathcal{A}(\Omega)$. Moreover, for all $\varepsilon \leq \varepsilon_1$, $\sup_{(x,y)\in Q_a} |w_{\varepsilon,\rho}(x,y)| = 0$, hence $(P_{K,l}(w_{\varepsilon,\rho}))_{(\varepsilon,\rho)} \in |I_A|$ for any $l \in \mathbb{N}$ as $w_{\varepsilon,\rho}$ vanishes on K. Thus $(w_{\varepsilon,\rho})_{(\varepsilon,\rho)} \in \mathcal{N}(\Omega)$ and $v = u|_{\Omega}$ as claimed. □

6. Example

Example 3. *Assume that $(\varepsilon, \eta, \rho) \in \Lambda = (0,1]^3$. Consider the problem*

$$(P_{form}) \begin{cases} \dfrac{\partial^2 u}{\partial x \partial t} = \left(\dfrac{\partial u}{\partial t}\right)^2, \\ u|_{(Ox)} = 0, \\ \left.\dfrac{\partial u}{\partial t}\right|_{(Ox)} = vp(\frac{1}{1-x}). \end{cases}$$

This problem is classically highly ill-posed. We build $\mathcal{A}\left(\mathbb{R}^2\right)$ like in (3.3). Let be (P_{gen}) the generalized associated problem as it is done in Subsection 3.5.

$$P_{gen} \begin{cases} \dfrac{\partial^2 u}{\partial x \partial t} = \mathcal{F}(u'), \\ \mathcal{R}_f(u) = 0, \\ \mathcal{R}_f(u') = \psi \end{cases}$$

where $\mathcal{F}$ is associated to $F(\cdot,\cdot,u') = (u')^2$ via the family $(g_\varepsilon)_\varepsilon$ and $f_\eta(x) = \eta x$. The generalized functions $\varphi = [\varphi_\rho] \in \mathcal{A}(\mathbb{R})$, $\psi = [\psi_\rho] \in \mathcal{A}(\mathbb{R})$ are constructed *from*

$$\begin{aligned}\psi_\rho(x) &= \left(\theta_\rho * vp(\frac{1}{1-\cdot})\right)(x)\\ &= \langle vp(\frac{1}{1-z}), z \mapsto \theta_\rho(x-z)\rangle\\ &= \lim_{\varepsilon\to 0}\int_{|1-z|>\varepsilon}\frac{\theta_\rho(x-z)}{1-z}dz,\\ \varphi_\rho(x) &= 0\end{aligned}$$

where $(\theta_\rho)_\rho$ *is a chosen family of mollifiers. Then* ψ_ρ *regularize* $vp(\frac{1}{1-x})$*. To solve Problem* (P_{gen}) *associated to* (P_{form}) *we can consider the family of problems*

$$(P_{(\varepsilon,\eta,\rho)})\begin{cases}\dfrac{\partial^2}{\partial x\partial t}u_{(\varepsilon,\eta,\rho)}(x,t) = \left(u'_{(\varepsilon,\eta,\rho)}(x,t))g_\varepsilon(u'_{(\varepsilon,\eta,\rho)}(x,t))\right)^2,\\ u_{(\varepsilon,\eta,\rho)}(x,\eta x) = 0,\\ u'_{(\varepsilon,\eta,\rho)}(x,\eta x) = \psi_\rho(x).\end{cases}$$

If $u_{\varepsilon,\eta,\rho}$ *is a solution to* $\left(P_{(\varepsilon,\eta,\rho)}\right)$ *then* $u = [u_{\varepsilon,\eta,\rho}]$ is the solution to (P_{gen}).
We have the restriction

$$vp(\frac{1}{1-x})\Big|_{(Ox)} = \left(x \mapsto \frac{1}{1-x}\right).$$

Let $(P_{gen,\eta})$ *be the generalized associated problem to the problem*

$$(P_{form,\eta})\begin{cases}\dfrac{\partial^2 u}{\partial x\partial t} = \left(\dfrac{\partial u}{\partial t}\right)^2,\\ u|_{(t=\eta x)} = 0,\\ \dfrac{\partial u}{\partial t}\Big|_{(t=\eta x)} = vp(\frac{1}{1-x}).\end{cases}$$

To solve Problem $(P_{gen,\eta})$ *associated to* $(P_{form,\eta})$ *we can consider the same family of problems* $\left(P_{(\varepsilon,\eta,\rho)}\right)$*. Then* $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\rho)}$ *is a representative of the solution* u_η *to* $(P_{gen,\eta})$*. On* $\Omega =]-\infty,1[\times\mathbb{R}$ *and for* η *fixed, problem* $(P_{form,\eta})$ *has the classical solution* v *in* $\mathrm{C}^\infty(\Omega)$ *where*

$$v(x,t) = \frac{t}{(1-x)},$$

and Theorem 15 shows that the restriction of $u_\eta \in \mathcal{A}\left(\mathbb{R}^2\right)$ *to* Ω *is precisely* v*. The local classical solution* v *which blows-up for* $x = 1$ *extends to a global generalized solution* u_η *which absorbs this blow-up.*

7. Appendix

7.1. Global smooth solutions to the Cauchy problem

We build the solution of the Cauchy problem by means of successive approximation techniques (See [16]).

Here $F_\varepsilon = F$, $f_\eta = f$, $\varphi_\rho = \varphi$ and $\psi_\rho = \psi$. Then we denote by (P_∞) the problem $\left(P_{(\varepsilon,\eta,\rho)}\right)$, by (P_i) the problem $\left(I_{(\varepsilon,\eta,\rho)}\right)$ and by $D(x,t,f)$ the domain $D(x,t,f_\eta)$.

Theorem 16. *Let $u \in \mathrm{C}^0(\mathbb{R}^2)$. The function u is a solution to (P_∞) if and only if u is a solution to (P_i).*

Proof. The existence of f^{-1} is ensured by $(H1)$. Hypothesis $(H1)$ also ensures that the domain $D(x,t,f)$ is bounded. If u is a solution to (P_∞), suppose that $t \geq f(x)$. We have

$$\iint\limits_{D(x,t,f)} \left(\frac{\partial^2 u}{\partial x \partial t}(\xi,\eta)\,\mathrm{d}\xi \right) \mathrm{d}\eta = \int_{f(x)}^{t} \frac{\partial u}{\partial t}(f^{-1}(\eta),\eta)\,\mathrm{d}\eta - \int_{f(x)}^{t} \frac{\partial u}{\partial t}(x,\eta)\,\mathrm{d}\eta$$
$$= \Upsilon(t) - \Upsilon(f(x)) - u(x,t) + \varphi(x),$$

where Υ denotes a primitive of $\psi \circ f^{-1}$. Then

$$u(x,t) = u_0(x,t) - \iint\limits_{D(x,t,f)} F(\xi,\eta,u'(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where $u_0(x,t) = \Upsilon(t) - \Upsilon(f(x)) + \varphi(x)$. We obtain the same result if we suppose $t \leq f(x)$. Thus u satisfies (P_i). If u satisfies (P_i), suppose that $t \geq f(x)$ we can write

$$u(x,t) = u_0(x,t) - \int_{x}^{f^{-1}(t)} \left(\int_{f(\xi)}^{t} F(\xi,\eta,u'(\xi,\eta))\,\mathrm{d}\eta \right) \mathrm{d}\xi.$$

As $u \in \mathrm{C}^0(\mathbb{R}^2)$ we have

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)(x,t) = F(x,t,u'(x,t)).$$

Let us calculate again $u(x,t)$ in the following way:

$$u(x,t) = u_0(x,t) - \int_{f(x)}^{t} \left(\int_{x}^{f^{-1}(\eta)} F(\xi,\eta,u'(\xi,\eta))\,\mathrm{d}\xi \right) \mathrm{d}\eta.$$

As $u \in \mathrm{C}^0(\mathbb{R}^2)$ we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)(x,t) = F(x,t,u'(x,t)).$$

Finally, the partial derivatives can be exchanged and we have

$$\frac{\partial^2 u}{\partial x \partial t}(x,t) = F(x,t,u'(x,t)).$$

Furthermore,

$$u\left(x, f(x)\right) = u_0\left(x, f(x)\right) = \varphi(x),$$
$$u'\left(x, f(x)\right) = u_0'\left(x, f(x)\right) = \psi \circ f^{-1}\left(f(x)\right) = \psi\left(x\right).$$

These results remain unchanged if we suppose $t \leq f(x)$, so u satisfies (P_∞). We can show by induction, that u is therefore of class C^∞. For more details, we refer the reader to [11], [12], in which similar calculation is made. We have, of course, the following corollary. □

Corollary 17. *If u is a solution to (P_i) (or to (P_∞)), then u belongs to* $\mathrm{C}^\infty(\mathbb{R}^2)$.

7.2. Uniqueness of the solution

Theorem 18. *From hypothesis $(H1)$ it follows that problem (P_∞) has a unique solution in* $\mathrm{C}^\infty(\mathbb{R}^2)$.

Proof. According to Theorem 16, solving problem (P_∞) amounts to solving problem (P_i), that is searching for $u \in \mathrm{C}^0(\mathbb{R}^2)$ satisfying (1). For every compact subset of $\mathbb{R}^2$, we can find $a > 0$, large enough, so that this compact subset is contained in $K_a = \left[f^{-1}(-a), f^{-1}(a)\right] \times [-a, a]$. Let us assume always that $t \geq f(x)$ and let us make the change of variables $X = x - f^{-1}(-a)$, $T = t + a$. The relation (1) can be written as

$$u(X + f^{-1}(-a), T - a) = u_0(X + f^{-1}(-a), T - a)$$
$$- \iint\limits_{D(X+f^{-1}(-a),T-a,f)} F(\xi + f^{-1}(-a), \eta - a, u'(\xi + f^{-1}(-a), \eta - a))\, \mathrm{d}\xi\, \mathrm{d}\eta,$$

whose form is

$$U(X,T) = U_0(X,T) - \iint\limits_{\mathfrak{D}(X,T,g)} \mathfrak{F}(\xi, \eta, U'(\xi,\eta))\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{1.5}$$

with $g(X) = f(X + f^{-1}(-a)) + a$; K_a turns into the compact subset $Q_a = \left[0, \left(f^{-1}(a) - f^{-1}(-a)\right)\right] \times [0, 2a]$. The equation of (γ) can then be written as $T = g(X)$ and $g(0) = 0$. So we now have $X \geq 0$ and $T \geq g(X)$. According to hypothesis $(H1)$, we can put

$$m_a = \sup_{(\xi,\eta)\in Q_a;\ z\in\mathbb{R}} \left| \frac{\partial \mathfrak{F}}{\partial z}(\xi, \eta, z) \right|.$$

We consider the sequence of approximations $(U_n)_{n\in\mathbb{N}}$ defined by

$$(I) \qquad \forall n \in \mathbb{N}^*,\ U_n(X,T) = U_0(X,T) - \iint\limits_{\mathfrak{D}(X,T,g)} \mathfrak{F}(\xi, \eta, U'_{n-1}(\xi,\eta))\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

Using the auxiliary series $\sum_{n\geq 1}(U_n(X,T)-U_{n-1}(X,T))$ and $\sum_{n\geq 1}\left(U'_n(X,T)-U'_{n-1}(X,T)\right)$ we show the uniform convergence of $(U_n)_n$ on every compact subset $K \Subset \mathbb{R}^2$, toward a continuous function U satisfying (I), that the uniform limit U is derivable on every compact subset $K \Subset \mathbb{R}^2$ and that the sequence $(U'_n)_{n\in\mathbb{N}}$ converges uniformly on every compact subset $K \Subset \mathbb{R}^2$, to the function U'on every compact subset $K \Subset \mathbb{R}^2$. Consider W to be another solution to (I), $\Delta = W-U$, $\Delta' = W'-U'$. Let $(X,T) \in Q_\lambda$. As $\mathfrak{D}(X,T,g) \subset Q_\lambda$ and $T \geq g(X)$, we have

$$|\Delta(X,T)| \leq m_\lambda \int_0^{2\lambda}\int_0^{T} |\Delta'(\xi,\eta)|\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{An1}$$

and

$$|\Delta'(X,T)| \leq m_\lambda \int_0^{2\lambda} |\Delta'(\xi,T)|\,\mathrm{d}\xi.$$

Thus $\Delta' = 0$ and $W = U$. Note that we use, in an essential way, the hypothesis $(H1)$ in the proof. For more details, we refer the reader to [11], [12], in which similar Picard's procedure is used. □

Remark 10. *We deduce easily the following estimates useful in the sequel. For every compact subset $K \Subset \mathbb{R}^2$, there exists a compact subset*

$$K_a = \left[f^{-1}(-a), f^{-1}(a)\right] \times [-a,a] \Subset \mathbb{R}^2$$

containing K, such that

(1.8)

$$m_a = \sup_{(x,t)\in K_a;\ r\in\mathbb{R}} \left|\frac{\partial F}{\partial z}(x,t,r)\right| ; \Phi_a = \|F((\ .\ ,\ .\ ,0)\|_{\infty,K_a} + m_a \|u'_0\|_{\infty,K_a} ;$$

$$\|u\|_{\infty,K} \leq \|u\|_{\infty,K_a} \leq \|u_0\|_{\infty,K_a} + \frac{\Phi_a}{m_a}\exp\left(2a\left(f^{-1}(a)-f^{-1}(-a)\right)m_a\right). \tag{1.9}$$

References

[1] Allaud, E., Dévoué, V., Generalized solutions to a charasteristic Cauchy problem. Preprint. `http://fr.arxiv.org/abs/0811.2256`

[2] Antonevich, A.B., Radyno, Ya.V., On a general method for constructing algebras of generalized functions. Soviet Math. Dokl. 43(3) (1992), 680–684.

[3] Biagioni, H.A., A nonlinear theory of generalized functions. Lecture notes in Math. 1421, Berlin, Heidelberg, New York: Springer, 1990.

[4] Colombeau, J.-F., Elementary introduction to new generalized functions. North Holland Math. Studies 113. Amsterdam: North-Holland, 1984.

[5] Colombeau, J.-F., New Generalized Functions and Multiplication of Distributions. Amsterdam, Oxford, New-York: North-Holland, 1984.

[6] Delcroix, A., Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions. Novi Sad J. Math. Vol. 35 No. 2 (2005), 27–40.

[7] Delcroix, A., Some properties of (C, E, P)-algebras : Overgeneration and 0^{th}-order estimates. Preprint. http://fr.arxiv.org/abs/0810.831

[8] Delcroix, A., Dévoué, V., Marti, J.-A., Generalized solutions of singular differential problems. Relationship with classical solutions. J. Math. Anal. Appl. 353 (1) (2009), 386–402. http://dx.doi.org/10.1016/j.jmaa.2008.11.077

[9] Delcroix, A., Dévoué, V., Marti, J.-A., Well posed differential problems in Algebras of Generalized Functions. Accepted by Applicable Analysis (2010).

[10] Delcroix, A., Scarpalézos, D., Topology on Asymptotic Algebras of Generalized Functions and Applications. Monatsh. Math. 129 (2000), 1–14.

[11] Dévoué, V., Sur les singularités de certains problèmes différentiels. Thèse de doctorat, Université des Antilles et de la Guyane (2005). http://tel.ccsd.cnrs.fr/tel-00012098

[12] Dévoué, V., On generalized solutions to the wave equation in canonical form. Dissertationes math. 443 (2007), 1–69.

[13] Dévoué, V., Generalized solutions to a non Lipschitz Cauchy problem. J. Appl. Anal. Vol. 15, No. 1 (2009), 1–32.

[14] Dévoué, V., Generalized solutions to a non Lipschitz Goursat problem. Diff. Equa. & Appl. Vol. 1, No. 2 (2009), 153–178.

[15] Egorov, Y.V., A contribution to the theory of generalized functions. Russian Math. Surveys 43, n°5, 1-49. (1990).

[16] Garabedian, P.R., Partial differential equations. John Wiley & Sons Inc. (1964).

[17] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R.. Geometric Theory of Generalized Functions with Applications to General Relativity. Kluwer Academic Press, 2001.

[18] Lions, J.L., Quelques méthodes de résolution des problèmes aux limites non linéaires. Paris: Dunod, 2002.

[19] Marti, J.-A. Fundamental structures and asymptotic microlocalization in sheaves of generalized functions. Integral Transforms Spec. Funct. 6(1-4) (1998), 223–228.

[20] Marti, J.-A., $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-Sheaf structure and applications. In: Nonlinear theory of generalized functions. pp.175–186. (eds. M. Grosser et al.), Research Notes in Mathematics. Chapman & Hall/CRC, 1999.

[21] Marti, J.-A., Multiparametric Algebras and Characteristic Cauchy Problem. Nonlinear algebraic analysis and applications, Proceeding of the International Conference on Generalized Functions (ICGF 2000). CSP (2004), 181–192.

[22] Marti, J.-A., Nonlinear algebraic analysis of delta shock wave to Burgers' equation. Pacific J. Math. 210(1) (2003), 165–187.

[23] Marti, J.-A., Nuiro, P.S., Analyse algébrique d'un problème de Dirichlet non linéaire et singulier. Topological Methods in Non Linear Analysis, vol 13 (1999), 301-311.

[24] Marti, J.-A., Nuiro, P.S., Valmorin, V.S. A nonlinear Goursat problem with irregular data. Integral Transforms Spec. Funct. 6(1-4) (1998), 229–246.

[25] Nedeljkov, M., Pilipović, S., Scarpalezos, D., The linear theory of Colombeau generalized functions. Pitman Research Notes in Mathematics Series 385. Longman, 1998.

[26] Nedeljkov, M., Oberguggenberger, M., Pilipović, S., Generalized solution to a semilinear wave equation. Nonlinear Anal. 61 (2005), 461–475.

[27] Oberguggenberger, M., Multiplication of Distributions and Applications to Partial Differential Equations. Longman Scientific & Technical, 1992.

[28] Scarpalézos, D., Colombeau's generalized functions: Topological structures; Microlocal properties. A simplified point of view. Part I. Bull. Cl. Sci. Math. Nat. Sci. Math. 25 (2000), 89–114.

[29] Scarpalézos, D., Colombeau's generalized functions: Topological structures; Microlocal properties. A simplified point of view. Part II. Publ. Inst. Math. (Beograd) (N.S.) 76(90) (2004), 111–125.

[30] Taylor, M.E., Partial Differential Equations III. Nonlinear Equations. Applied Mathematical Sciences 117. New York: Springer-Verlag, 1996.

Received by the editors October 9, 2010