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SEQUENTIAL APPROACH TO INTEGRABLE DISTRIBUTIONS

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Abstract. The equivalence of various conditions for integrability of distributions is proved. The list of equivalent conditions given by P. Dierolf and J. Voigt in [3] is extended by adding several conditions in terms of extensions of linear continuous functionals defined on $\mathcal{D}$, with the topology of $\mathcal{B}_0$, to the space $\mathcal{B}$.

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1. Introduction

P. Dierolf and J. Voigt in [3] essentially extended the list of known equivalent conditions (see [11], [12] and [4]) for distributions to be integrable, i.e to belong to the space $\mathcal{D}'_{L^1}$ defined as the dual of the space $\mathcal{B}_0$ of smooth functions vanishing at infinity together with all their derivatives. The enlarged list of conditions appeared to be very useful in the proof of the fact that the classical convolutions of distributions in the sense of C. Chevalley [2], L. Schwartz [12] and R. Shiraishi [13] are equivalent to the convolution in the sense of V. S. Vladimirov (see [14] and [15]) and to several other sequentially defined convolutions in $\mathcal{D}'$ (in $\mathcal{S}'$) (see [3] and [5]).

Later R. Wawak introduced in [16] the notions of improper integrals, improper integrable distributions and improper convolutions in $\mathcal{D}'$ and in $\mathcal{S}'$, generalizing the classical results.

In a more general situation, various equivalent conditions for integrability of ultradistributions were studied by S. Pilipović in [10] and then used in the proof of the equivalence of various definitions of the convolutions of ultradistributions of Beurling type in $\mathcal{D}'^{(M_p)}$ and tempered ultradistributions of Beurling type in $\mathcal{S}'^{(M_p)}$ (see [10], [6], [7] and [1]).

The result of P. Dierolf and J. Voigt from [3] (see Theorem 1 in section 3) gives a good insight into the notions of integrable distributions and the integral of a distribution and gives a possibility of a more elementary treatment of these notions. Since the space $\mathcal{B}$ is the bidual of the space $\mathcal{B}_0$, it is clearly possible to define the integral of an integrable distribution $f \in \mathcal{D}'_{L^1}$ as $\langle f, 1\rangle$. However the sequential approach allows us to define the integral as a direct extension of

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a linear continuous functional from the space $\mathcal{D}$, endowed with the topology of $\mathcal{B}_0$, to the space $\mathcal{B}$ via sequential limits of suitable approximating sequences.

It is worth noting that such an extension, denoted by $\widetilde{f}$, exists in spite of the fact that functions from $\mathcal{B} \setminus \mathcal{B}_0$ cannot be approximated by sequences of functions from $\mathcal{D}$ in the topology of $\mathcal{B}$, but in the topology of $\mathcal{E}$. The construction of $\widetilde{f}$ requires the restriction of the class of admissible sequences, approximating functions from $\mathcal{B}\setminus\mathcal{B}_0$, to sequences of functions of the special form (belonging to the class E or $\overline{E}$ of unit-sequences). Though the approximating sequences are not convergent in $\mathcal{B}$, the constructed extension $\widetilde{f}$ is continuous in the topology of $\mathcal{B}$.

We will show, using the two mentioned classes of unit-sequences, that one may extend $f \in \mathcal{D}'_{L^1}$ to a linear continuous functional $\tilde{f}$ on $\mathcal{B}$ which satisfies the three types of estimates considered in [3]. As a matter of fact the received conditions appear to be equivalent to the known conditions for integrability of distributions.

The obtained results are applicable in the study of the convolution of distributions (see [8] and [9]).

2. Preliminaries

We apply mainly the standard notation with a few exceptions. For instance, to mark that a given subset K of $\mathbb{R}^d$ is compact we will use the symbol $K \sqsubset \mathbb{R}^d$ instead of the symbol $K \subset\subset \mathbb{R}^d$ used usually in the literature.

For the convenience we denote by $\mathcal{A}^K$ the subset of a given subspace $\mathcal{A}$ of the space of continuous functions on $\mathbb{R}^d$ consisting of all functions $\psi \in \mathcal{A}$ such that supp $\psi \cap K = \emptyset$ for a given $K \sqsubset \mathbb{R}^d$ (see section 3).

It will also be convenient to consider, except the usual support, supp ψ, of a function ψ on $\mathbb{R}^d$ also its *unitary support*, meant as the set $\psi := \{x \in \mathbb{R}^d : \psi(x) = 1\}$ and denoted by s^1.

We use the standard notation for various spaces of functions and distributions on $\mathbb{R}^d$, usually without marking the space $\mathbb{R}^d$: L^∞, $\mathcal{C}^\infty$, $\mathcal{E}$, $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{D}_K$ (for $K \sqsubset \mathbb{R}^d$), $\mathcal{D}'$, $\mathcal{D}'_{L^1}$. The supremum norm in L^∞ is denoted by $\|\cdot\|_\infty$. For $k \in \mathbb{N}_0$, $K \sqsubset \mathbb{R}^d$ and a $\mathcal{C}^\infty-$function φ on $\mathbb{R}^d$, we define the seminorms:

$$p_{k,K}(\varphi) := \max_{0\le i\le k} \max_{x\in K} |\varphi^{(i)}(x)|$$

and the norms:

$$p_k(\varphi) := \max_{0\le i\le k} \|\varphi^{(i)}\|_\infty.$$

Recall that the sets $\mathcal{B}_0$; $\mathcal{B}$; and $\mathcal{D}_K$ consist of all $\mathcal{C}^\infty-$functions φ such that $|\varphi^{(i)}(x)| \to 0$ as $|x| \to \infty$ for $i \in \mathbb{N}_0^d$; $p_k(\varphi) < \infty$ for $k \in \mathbb{N}_0$; and supp $\varphi \subseteq K$, respectively. Moreover, we have $\mathcal{E} = \mathcal{C}^\infty$ and $\mathcal{D} = \cup_{K\sqsubset\mathbb{R}^d}\mathcal{D}_K$ in the sense of equalities of sets. The sets under consideration are endowed with the topologies defined by the respective families of seminorms: $\mathcal{B}_0$ and $\mathcal{B}$ by the family $\{p_k : k \in \mathbb{N}_0\}$; $\mathcal{E}$ by the family $\{p_{k,K} : \ k \in \mathbb{N}_0, K \sqsubset \mathbb{R}^d\}$; and $\mathcal{D}_K$ by the family

$\{p_{k,K}\colon\ k \in \mathbb{N}_0\}$ (for $K \sqsubset \mathbb{R}^d$). The space $\mathcal{D}$ is endowed with the inductive limit topology of the spaces $\mathcal{D}_K$.

From the Leibniz formula it follows that

$$p_k(\varphi\psi) \le 2^k p_k(\varphi) p_k(\psi), \qquad \varphi, \psi \in \mathcal{B},\ k \in \mathbb{N}_0. \tag{1}$$

Let θ be a fixed function such that $\theta \in \mathcal{D}$ and $\theta(x) = 1$ for $|x| \le 1$ and let θ_j $(j \in \mathbb{N})$ be the functions given by

$$\theta_j(x) := \theta(x/j) \qquad \text{for} \quad x \in \mathbb{R}^d,\ j \in \mathbb{N}. \tag{2}$$

It follows from inequality (1) that

$$p_k((1-\theta_j)\varphi) \le A_k(\theta) p_k(\varphi) \quad \text{and} \quad p_k(\theta_j\varphi) \le A_k(\theta) p_k(\varphi) \tag{3}$$

for $\varphi \in \mathcal{B}$, $k \in \mathbb{N}_0$ and $j \in \mathbb{N}$, where

$$A_k(\theta) := 2^k(1 + p_k(\theta)), \qquad k \in \mathbb{N}_0. \tag{4}$$

Definition 1. By a *unit-sequence* we mean a sequence $\{\eta_n\}$ of functions of the class $\mathcal{D}$, convergent to 1 in $\mathcal{E}$, such that

$$\sup_{n\in\mathbb{N}} \| \eta_n^{(k)} \|_\infty \le \sup_{n\in\mathbb{N}} p_k(\eta_n) =: B_k < \infty, \qquad k \in \mathbb{N}_0^d. \tag{5}$$

By a *special unit-sequence* we mean such a unit-sequence $\{\eta_n\}$ that for every bounded set $K \subset \mathbb{R}^d$ there is an $n_0 \in \mathbb{N}$ such that

$$\eta_n(x) = 1, \qquad x \in K,\ n \ge n_0.$$

Denote by E the class of all unit-sequences and by $\overline{E}$ the class of all special unit-sequences.

By (1) and (5), we have the following estimate:

$$\sup_{n\in\mathbb{N}} p_k(\eta_n\varphi) \le 2^k B_k p_k(\varphi) \tag{6}$$

for arbitrary $\{\eta_n\} \in E$, $\varphi \in \mathcal{B}$ and $k \in \mathbb{N}_0^d$.

Let us consider the following condition for an arbitrary class of sequences (e.g. of functions on $\mathbb{R}^d$) which plays an essential role in justifying consistency of various sequential definitions:

Condition $\mathcal{I}$. *A class $\mathcal{R}$ of sequences satisfies the following implication:*

$$\{\tau_n^1\}, \{\tau_n^2\} \in \mathcal{R} \Rightarrow \{\tau_n\} \in \mathcal{R},$$

where $\{\tau_n\}$ is the interlacement of $\{\tau_n^1\}$ and $\{\tau_n^2\}$, i.e. the sequence defined by $\tau_{2n-1} := \tau_n^1$ and $\tau_{2n} := \tau_n^2$ for $n \in \mathbb{N}$.

Clearly, the classes E of all unit-sequences and $\overline{E}$ of all special unit-sequences satisfy Condition $\mathcal{I}$.

Definition 2. For a fixed function $\varphi \in \mathcal{B}$, a distribution f will be called 1° *extendible for* φ and 2° *specially extendible for* φ, respectively, if the sequence $\{\langle f, \eta_n\varphi\rangle\}$ is Cauchy 1° for each unit-sequence $\{\eta_n\} \in E$ and 2° for each special unit-sequence $\{\eta_n\} \in \overline{E}$, respectively.

Definition 3. An $f \in \mathcal{D}'$ will be called 1° *extendible to* $\mathcal{B}$ and 2° *specially extendible to* $\mathcal{B}$ if the distribution f is 1° extendible for $\varphi \in \mathcal{B}$ and 2° specially extendible for $\varphi \in \mathcal{B}$, respectively. By 1° the *extension* $\tilde{f}$ on $\mathcal{B}$ of an extendible distribution $f \in \mathcal{D}'$ and by 2° the *special extension* $\tilde{f}$ on $\mathcal{B}$ of a specially extendible distribution $f \in \mathcal{D}'$, respectively, we mean the mapping $\tilde{f}\colon\ \mathcal{B} \to \mathbb{C}$, uniquely defined for every $\varphi \in \mathcal{B}$ by means of the formula:

$$\langle \tilde{f}, \varphi\rangle := \lim_{j\to\infty} \langle f, \eta_j\varphi\rangle, \qquad \varphi \in \mathcal{B} \tag{7}$$

1° for every $\{\eta_n\} \in E$ and 2° for every $\{\eta_n\} \in \overline{E}$, respectively.

Remark 3. Assume that a distribution f is 1° extendible or 2° specially extendible for a given function $\varphi \in \mathcal{B}$. Then f can be uniquely extended to the mapping $f_\varphi\colon\ \mathcal{D} \cup \{\varphi\} \to \mathbb{C}$ given by

$$\langle f_\varphi, \omega\rangle := \lim_{j\to\infty} \langle f, \eta_j\omega\rangle, \qquad \omega \in \mathcal{D} \cup (\varphi) \tag{8}$$

for 1° $\{\eta_n\} \in E$ and 2° $\{\eta_n\} \in \overline{E}$, respectively. In fact, the sequence $\{\langle f, \eta_n\varphi\rangle\}$ is Cauchy for every $\{\eta_n\} \in E$, so the more for every $\{\eta_n\} \in \overline{E}$. Moreover, the limit in (8) in case 1° does not depend on the choice of $\{\eta_n\} \in E$ and, in case 2°, on the choice of $\{\eta_n\} \in \overline{E}$, because the classes E and $\overline{E}$ satisfy Condition $\mathcal{I}$. Consequently, the left side of (8) is well defined for $\omega = \varphi$. Moreover, due to continuity of f on $\mathcal{D}$, we have $\langle f_\varphi, \phi\rangle = \lim_{j\to\infty}\langle f, \eta_j\phi\rangle = \langle f, \phi\rangle$ for all $\phi \in \mathcal{D}$, for all $\{\eta_n\} \in E$ in case 1° and for all $\{\eta_n\} \in \overline{E}$ in case 2°.

3. Main Theorems

Integrable distributions, meant as elements of the topological dual $\mathcal{B}_0'$ of $\mathcal{B}_0$, were described by P. Dierolf and J. Voigt in [3] as distributions satisfying several equivalent conditions. To formulate their result below it will be convenient to use the following notation:

$$\mathcal{D}^K := \{\phi \in \mathcal{D}:\ \text{supp}\ \ \phi \cap K = \emptyset\}, \qquad K \sqsubset \mathbb{R}^d. \tag{9}$$

Theorem 1. *Let $f \in \mathcal{D}'$. The following conditions are equivalent:*

(a) *there are an $m \in \mathbb{N}_0$ and a $C > 0$ such that*

$$|\langle f, \phi\rangle| \le C p_m(\phi), \qquad \phi \in \mathcal{D}; \tag{10}$$

(b) *there exists an* $m \in \mathbb{N}_0$ *such that for every* $\varepsilon > 0$ *there is a* $K \sqsubset \mathbb{R}^d$ *for which the following inequality holds:*

$$|\langle f, \phi \rangle| \le \varepsilon p_m(\phi), \qquad \phi \in \mathcal{D}^K; \tag{11}$$

(c) *there are an* $m \in \mathbb{N}_0$*, a* $C > 0$ *and a* $K \sqsubset \mathbb{R}^d$ *for which the following inequality holds:*

$$|\langle f, \phi \rangle| \le C p_m(\phi), \qquad \phi \in \mathcal{D}^K; \tag{12}$$

(d) $\{\langle f, \eta_n \rangle\}$ *is a Cauchy sequence for every* $\{\eta_n\} \in E$*;*

$(\overline{d})$ $\{\langle f, \eta_n \rangle\}$ *is a Cauchy sequence for every* $\{\eta_n\} \in \overline{E}$*;*

We complete the list of equivalent conditions listed in Theorem 1 by characterizing in the next theorem integrable distributions as linear continuous functionals on $\mathcal{D}$ which can be extended to the whole space $\mathcal{B}$ in such a way that the extensions satisfy on $\mathcal{B}$ the estimates (10), (11), (12) given in conditions (a), (b), (c) for $\mathcal{D}$. Denote, similarly to (9),

$$\mathcal{B}^K := \{\varphi \in \mathcal{B} : \text{ supp } \varphi \cap K = \emptyset\}, \qquad K \sqsubset \mathbb{R}^d.$$

Theorem 2. *Let* $f \in \mathcal{D}'$ *and let* $\tilde{f}$ *denote the extension of* f *to* $\mathcal{B}$ *defined by (7) in case* f *is an extendible distribution. Each of the following conditions is equivalent to each of the conditions listed in Theorem 1:*

(A) f *is extendible to* $\mathcal{B}$ *and* $\tilde{f} \in \mathcal{B}'$*, i.e. there are an* $m \in \mathbb{N}_0$ *and a* $C > 0$ *such that*

$$|\langle \tilde{f}, \varphi \rangle| \le C p_m(\varphi), \qquad \varphi \in \mathcal{B}; \tag{13}$$

$(\overline{A})$ f *is specially extendible to* $\mathcal{B}$ *and* $\tilde{f} \in \mathcal{B}'$*, i.e. there are an* $m \in \mathbb{N}_0$ *and a* $C > 0$ *such that inequality (13) holds;*

(B) f *is extendible to* $\mathcal{B}$ *and* $\tilde{f}$ *has the property: there exists an* $m \in \mathbb{N}_0$ *such that for every* $\varepsilon > 0$ *there is a* $K \sqsubset \mathbb{R}^d$ *for which the inequality holds:*

$$|\langle \tilde{f}, \varphi \rangle| \le \varepsilon p_m(\varphi), \qquad \varphi \in \mathcal{B}^K; \tag{14}$$

$(\overline{B})$ f *is specially extendible to* $\mathcal{B}$ *and* $\tilde{f}$ *has the property: there exists an* $m \in \mathbb{N}_0$ *such that for every* $\varepsilon > 0$ *there is a* $K \sqsubset \mathbb{R}^d$ *for which inequality (14) holds;*

(C) f *is extendible to* $\mathcal{B}$ *and* $\tilde{f}$ *has the property: there exist an* $m \in \mathbb{N}_0$*, a* $C > 0$ *and a* $K \sqsubset \mathbb{R}^d$ *for which the inequality holds:*

$$|\langle \tilde{f}, \varphi \rangle| \le C p_m(\varphi), \qquad \varphi \in \mathcal{B}^K; \tag{15}$$

($\overline{C}$) *f is specially extendible to $\mathcal{B}$ and $\tilde{f}$ has the property: there exist an $m \in \mathbb{N}_0$, a $C > 0$ and a $K \sqsubset \mathbb{R}^d$ such that inequality (15) holds.*

If any of the above conditions holds, then

$$\langle \tilde{f}, 1 \rangle = \lim_{n \to \infty} \langle f, \eta_n \rangle \tag{16}$$

for all $\{\eta_n\} \in E$ (and the more for all $\{\eta_n\} \in \overline{E}$).

Definition 4. A distribution $f \in \mathcal{D}'$ is called *integrable* (belongs to $\mathcal{D}'_{L^1}$) if it satisfies one of the equivalent conditions listed in Theorems 1 and 2. By the *integral* of a given $f \in \mathcal{D}'_{L^1}$ we mean the common number described by equality (16), i.e.

$$\int_{\mathbb{R}^d} f := \lim_{n \to \infty} \langle f, \eta_n \rangle = \langle \tilde{f}, 1 \rangle$$

for arbitrary $\{\eta_n\} \in E$. The correctness of the above definition is guaranteed by equality (16) in the above theorem.

4. Proofs

In the proof of Theorem 2 below we will use the equivalence of the conditions mentioned in Theorem 1 proved in [3].

Since the implications $(A) \Rightarrow (C)$, $(\overline{A}) \Rightarrow (\overline{C})$, $(A) \Rightarrow (\overline{A})$, $(C) \Rightarrow (\overline{C})$ and $(\overline{C}) \Rightarrow (c)$ are obvious, to show that conditions (A), $(\overline{A})$, (C) and $(\overline{C})$ are equivalent to each of the conditions in Theorem 1 it suffices to prove the implication $(b) \Rightarrow (A)$. In order to deduce that also conditions (B) and $(\overline{B})$ are equivalent to those listed in Theorem 1, it will be enough to prove the implications $(b) \Rightarrow (B)$ and $(\overline{B}) \Rightarrow (d)$, because the implication $(B) \Rightarrow (\overline{B})$ is evident.

The following consequence of the compactness of supports of functions θ_j of the form (2) will be used in the proof: for a given distribution f and integers $j \in \mathbb{N}$ and $m \in \mathbb{N}_0$ there exist a natural number $m' \geq m$ and a constant $C' > 0$, depending on θ_j, such that

$$|\langle f, \theta_j \phi \rangle| = |\langle \theta_j f, \phi \rangle| \leq C' \, p_{m',K'}(\phi) \leq C' p_{m'}(\phi) \tag{17}$$

for all $\phi \in \mathcal{D}$, where $K' := \operatorname{supp} \theta_j \sqsubset \mathbb{R}^d$.

Proof. $(b) \Rightarrow (B)$ Assume that condition (b) holds. Fix a sequence $\{\eta_n\} \in E$ and a function $\varphi \in \mathcal{B}$. If $\varphi = 0$, the assertion is evidently true, so assume that $\varphi \neq 0$. Then $0 < p_0(\varphi) \leq p_k(\varphi)$ for all $k \in \mathbb{N}_0$, so there is a $\lambda \in (0, 1)$ such that

$$p_k(\varphi) > \lambda, \qquad k \in \mathbb{N}_0. \tag{18}$$

Putting $\phi_n := \eta_n \varphi$ we have $\phi_n \in \mathcal{D}$ for $n \in \mathbb{N}$ and $p_{k,K}(\phi_n - \varphi) \to 0$ for arbitrary $k \in \mathbb{N}_0$ and $K \sqsubset \mathbb{R}^d$, i.e. $\phi_n \to \varphi$ in $\mathcal{E}$. First we are going to show that $\{< f, \phi_n >\}$ is a Cauchy sequence.

Fix $\varepsilon > 0$. Due to (b), there is an index $m \in \mathbb{N}_0$ such that for every $\varrho > 0$, in particular for $\varrho = \varrho_\varepsilon$ of the form

$$\varrho_\varepsilon := \frac{\lambda\,\varepsilon}{2^{m+2} p_m(\varphi) A_m(\theta) B_m}$$

with $A_m(\theta)$ and B_m defined in (4) and (5), there exists a $K_\varepsilon \sqsubset \mathbb{R}^d$ such that

$$|\langle f, \phi\rangle| \leq \varrho_\varepsilon p_m(\phi), \qquad \phi \in \mathcal{D}^K. \tag{19}$$

Choose an open and bounded $U \supset K_\varepsilon$ and fix an index $j \in \mathbb{N}$ such that $s^1(\theta_j) \supset U$. Since $(1-\theta_j)(\phi_r - \phi_s) \in \mathcal{D}^{K_\varepsilon}$ and, by (3), (1) and (5),

$$p_m((1-\theta_j)(\phi_r - \phi_s)) \leq 2^{m+1} p_m(\varphi) A_m(\theta) B_m, \qquad r, s \in \mathbb{N},$$

we conclude from (19) that

$$|\langle f, (1-\theta_j)(\phi_r - \phi_s)\rangle| < \varepsilon/2, \qquad r, s \in \mathbb{N}. \tag{20}$$

On the other hand, there are a natural $m' \geq m$ and a $C' > 0$ which fulfil (17) for all $\phi \in \mathcal{D}$. Hence the inequalities

$$|\langle f, \theta_j(\varphi_r - \varphi_s)\rangle| < p_{m',K'}(\varphi_r - \varphi_s) < \varepsilon/2 \tag{21}$$

hold for sufficiently large $r, s \in \mathbb{N}$, due to (17).

By (20) and (21), $\{< f, \varphi\eta_n >\}$ is a Cauchy sequence for arbitrary $\{\eta_n\}$ in E and, by Condition $\mathcal{I}$, the limit of the sequence does not depend on the choice of $\{\eta_n\} \in E$. Consequently, the formula

$$\langle \widetilde{f}, \varphi\rangle := \lim_{n\to\infty} < f, \eta_n\varphi >, \qquad \{\eta_n\} \in E \tag{22}$$

well defines $\widetilde{f}$ for the function $\varphi \in \mathcal{B}$ arbitrarily fixed, i.e. $\widetilde{f}$ is an extension of f to the space $\mathcal{B}$ uniquely defined by (7). Clearly, $\widetilde{f}$ is linear on $\mathcal{B}$ and $\widetilde{f}|_{\mathcal{D}} = f$.

To prove that $\widetilde{f}$ satisfies inequality (14) we fix again arbitrarily $\{\eta_n\} \in E$ and $\varepsilon > 0$ and let K_ε be the corresponding compact set chosen according to condition (b). Fix now $\varphi \in \mathcal{B}^{K_\varepsilon}, \varphi \neq 0$, i.e. assume as before that $\varphi \in \mathcal{B}$ and, in addition, that supp φ is disjoint with K_ε. Of course, we may use all we proved before without this additional condition. The present assumption implies that $\eta_n\varphi \in \mathcal{D}^{K_\varepsilon}$. Hence, in view of (19),

$$|\langle f, \eta_n\varphi\rangle| \leq \varrho_\varepsilon p_m(\eta_n\varphi), \qquad n \in \mathbb{N}.$$

The above inequality implies

$$|\langle f, \eta_n\varphi\rangle| \leq \lambda\varepsilon < \varepsilon p_m(\varphi), \qquad n \in \mathbb{N}, \tag{23}$$

in view of (6) and (18). By (22) and (23), it follows that

$$|\langle \widetilde{f}, \varphi\rangle| = \lim_{n\to\infty} |\langle f, \eta_n\varphi\rangle| < \varepsilon p_m(\varphi).$$

for arbitrary $\{\eta_n\} \in E$ and $\varphi \in \mathcal{B}^{K_\varepsilon}$. Inequality (14) and the considered implication is thus proved.

$(b) \Rightarrow (A)$ On the base of the preceding implication, we may use condition (B) already proved. Put $\varepsilon = 1$ and $K_\varepsilon = K_1$, fix an open bounded $U \supset K_1$ and an index $j \in \mathbb{N}$ such that $s^1(\theta_j) \supset U$. For an arbitrary $\varphi \in \mathcal{B}$ the functions $(1-\theta_j)\varphi$ are in $\mathcal{B}^{K_1}$ and $\theta_j \widetilde{f} = \theta_j f$ is a distribution of compact support. Therefore, by (14), (17) and (3), there are a natural index $m' \geq m$ and a positive constant C' such that

$$\begin{aligned} |\langle \widetilde{f}, \varphi\rangle| &\leq |\langle \widetilde{f}, (1-\theta_j)\varphi\rangle| + |\langle f, \theta_j\varphi\rangle| \\ &\leq p_m((1-\theta_j)\varphi) + C' p_{m'}(\varphi) \leq (B_m(\theta) + C')p_{m'}(\varphi) \end{aligned}$$

for all $\varphi \in \mathcal{B}$, so inequality (14) and continuity of the extension $\widetilde{f}$ are proved.

Of course, equality (16) is a particular case of the general definition of $\widetilde{f}$ in formula (7).

$(\overline{B}) \Rightarrow (d)$ Fix $\{\eta_n\} \in E$. Since $\eta_n \in \mathcal{D} \subset \mathcal{B}$ for $n \in \mathbb{N}$ and $\widetilde{f}|_{\mathcal{D}} = f$, we have

$$\langle \widetilde{f}, \eta_n\rangle = \langle f, \eta_n\rangle, \qquad n \in \mathbb{N}. \tag{24}$$

In turn fix $\varepsilon > 0$. According to $(\overline{B})$, there exists an $m \in \mathbb{N}_0$ such that for every $\epsilon > 0$, in particular for $\epsilon := \varepsilon$, there is a compact set K_ε so that the inequality holds:

$$|\langle \widetilde{f}, \varphi\rangle| \leq \frac{\varepsilon p_m(\varphi)}{4A_m(\theta)B_m}, \qquad \varphi \in \mathcal{B}^{K_\varepsilon},$$

where $A_m(\theta)$ and B_m are the constant from (4) and (5). In particular, by (7),

$$|\langle f, \phi\rangle| = |\langle \widetilde{f}, \phi\rangle| \leq \frac{\varepsilon p_m(\phi)}{4A_m(\theta)B_m}, \qquad \phi \in \mathcal{D}^{K_\varepsilon}. \tag{25}$$

As before choose an open bounded set $U \supset K_\varepsilon$ and fix an index $j \in \mathbb{N}$ such that $K' := \operatorname{supp} \theta_j \supset s^1(\theta_j) \supset U$.

As noticed at the beginning of the section, there are a positive integer $m' \geq m$ and a constant $C' > 0$ satisfying (17) for all $\phi \in \mathcal{D}$. Hence, by (17), we have

$$|\langle f, \theta_j\phi\rangle| = C'\, p_{m',K'}(\phi) \leq C'\, p_{m'}(\phi) \tag{26}$$

for $\phi \in \mathcal{B}$. Since $\eta_n \to 1$ in $\mathcal{E}$ as $n \to \infty$, we have

$$p_{m',K'}(\eta_r - \eta_s) < \frac{\varepsilon}{2C'}, \tag{27}$$

for r and s sufficiently large. Hence, as a consequence of (25), (26), (3), (5) and (27), we conclude

$$|\langle f, \eta_r\rangle - \langle f, \eta_s\rangle| \leq |\langle f, (1-\theta_j)(\eta_r - \eta_s)\rangle| + |\langle f, \theta_j(\eta_r - \eta_s)\rangle| < \varepsilon$$

for sufficiently large r and s.

This means that $\{\langle f, \eta_n\rangle\}$ is a Cauchy sequence and its limit does not depend on $\{\eta_n\} \in E$, because the class E satisfies Condition $\mathcal{I}$. □

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