Novi Sad J. Math.
Vol. 40, No. 3, 2010, 83–87
Proc. 3rd Novi Sad Algebraic Conf.
(eds. I. Dolinka, P. Marković)

FINITE IRREFLEXIVE HOMOMORPHISM-HOMOGENEOUS BINARY RELATIONAL SYSTEMS[1]

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Abstract. A structure is called homogeneous if every isomorphism between finite substructures of the structure extends to an automorphism of the structure. Recently, P. J. Cameron and J. Nešetřil introduced a relaxed version of homogeneity: we say that a structure is homomorphism-homogeneous if every homomorphism between finite substructures of the structure extends to an endomorphism of the structure. In this paper we characterize all finite homomorphism-homogeneous relational systems with one irreflexive binary relation.

AMS Mathematics Subject Classification (2010): 05C20

Key words and phrases: finite digraphs, homomorphism-homogeneous structures

1. Introduction

A structure is *homogeneous* if every isomorphism between finite substructures of the structure extends to an automorphism of the structure. For example, finite and countably infinite homogeneous directed graphs were described in [2]. In their recent paper [1] the authors discuss a generalization of homogeneity to various types of morphisms between structures, and in particular introduce the notion of homomorphism-homogeneous structures.

Definition 1.1 (Cameron, Nešetřil [1]). *A structure $\mathcal{A}$ is called* homomorphism-homogeneous *if every homomorphism between finite substructures of $\mathcal{A}$ extends to an endomorphism of $\mathcal{A}$.*

In the seminal paper [1] the authors have, among other things, shown that an undirected graph without loops is homomorphism-homogeneous if and only if it is isomorphic to k disjoint copies of K_n for some $k, n \geqslant 1$. In this paper we generalize this result to arbitrary finite relational systems with one irreflexive

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binary relation. It turns out that such a relational system is homomorphism-homogeneous if and only if it is isomorphic to one of the following:

- k disjoint copies of K_n for some $k, n \geqslant 1$; or
- k disjoint copies of C_3 for some $k \geqslant 1$, where C_3 denotes the oriented 3-cycle.

Finally, let us mention that in [4] the authors of this paper have characterized finite homomorphism-homogeneous relational systems with one reflexive binary relation. In comparison to the irreflexive case, the characterization of finite homomorphism-homogeneous reflexive binary relational systems is quite involved.

2. Preliminaries

A binary relational system is an ordered pair (V, E) where $E \subseteq V^2$ is a binary relation on V. A binary relational system (V, E) is *reflexive* if $(x, x) \in E$ for all $x \in V$, *irreflexive* if $(x, x) \notin E$ for all $x \in V$, *symmetric* if $(x, y) \in E$ implies $(y, x) \in E$ for all $x, y \in V$ and *antisymmetric* if $(x, y) \in E$ implies $(y, x) \notin E$ for all distinct $x, y \in V$.

Binary relational systems can be thought of in terms of digraphs (hence the notation (V, E)). Then V is the set of *vertices* and E is the set of *edges* of the binary relational system/digraph (V, E). Edges of the form (x, x) are called *loops*. If $(x, x) \in E$ we also say that *x has a loop*. Instead of $(x, y) \in E$ we often write $x \to y$ and say that *x dominates y*, or that *y is dominated by x*. By $x \sim y$ we denote that $x \to y$ or $y \to x$, while $x \rightleftarrows y$ denotes that $x \to y$ and $y \to x$. If $x \rightleftarrows y$, we say that x and y form a *double edge*. We shall also say that a vertex x is *incident with a double edge* if there is a vertex $y \neq x$ such that $x \rightleftarrows y$.

Digraphs (V, E) where E is a symmetric binary relation on V are usually referred to as *graphs*. *Proper digraphs* are digraphs (V, E) where E is an antisymmetric binary relation. In this paper, digraphs (V, E) where E is neither antisymmetric nor symmetric will be referred to as *improper digraphs*. In an improper digraph there exists a pair of distinct vertices x and y such that $x \rightleftarrows y$ and another pair of distinct vertices u and v such that $u \to v$ and $v \not\to u$.

A digraph $D' = (V', E')$ is a *subdigraph* of a digraph $D = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We write $D' \leqslant D$ to denote that D' is isomorphic to a subdigraph of D. For $\emptyset \neq W \subseteq V$ by $D[W]$ we denote the digraph $(W, E \cap W^2)$ which we refer to as the *subdigraph of D induced by W*.

Vertices x and y are *connected in D* if there exists a sequence of vertices $z_1, \ldots, z_k \in V$ such that $x = z_1 \sim \ldots \sim z_k = y$. A digraph D is *weakly connected* if each pair of distinct vertices of D is connected in D. A digraph D is *disconnected* if it is not weakly connected. A *connected component of D* is a maximal set $S \subseteq V$ such that $D[S]$ is weakly connected. The number of connected components of D will be denoted by $\omega(D)$.

Vertices x and y are *doubly connected in D* if there exists a sequence of vertices $z_1, \ldots, z_k \in V$ such that $x = z_1 \rightleftarrows \ldots \rightleftarrows z_k = y$. Define a binary relation $\theta(D)$ on $V(D)$ as follows: $(x, y) \in \theta(D)$ if and only if $x = y$ or x and y are doubly connected. Clearly, $\theta(D)$ is an equivalence relation on $V(D)$ and $\omega(D) \leqslant |V(D)/\theta(D)|$. We say that a digraph D is *θ-connected* if $\omega(D) = |V(D)/\theta(D)|$, and that it is *$\theta$-disconnected* if $\omega(D) < |V(D)/\theta(D)|$. Note that a θ-connected digraph need not be connected, and that a θ-disconnected digraph need not be disconnected; a digraph D is θ-connected if every connected component of D contains precisely one $\theta(D)$-class, while it is θ-disconnected if there exists a connected component of D which consists of at least two $\theta(D)$-classes. In particular, every proper digraph with at least two vertices is θ-disconnected, and every graph is θ-connected.

Let K_n denote the complete irreflexive graph on n vertices. Let $\mathbf{1}$ denote the trivial digraph with only one vertex and no edges, and let $\mathbf{1}^\circ$ denote the digraph with only one vertex with a loop. An *oriented cycle with n vertices* is a digraph C_n whose vertices are 1, 2, ..., n, $n \geqslant 3$, and whose edges are $1 \to 2 \to \ldots \to n \to 1$.

For digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$, by $D_1 + D_2$ we denote the *disjoint union* of D_1 and D_2. We assume that $D + O = O + D = D$, where $O = (\emptyset, \emptyset)$ denotes the *empty digraph.* The disjoint union $\underbrace{D + \ldots + D}_{k}$ consisting of $k \geqslant 1$ copies of D will be abbreviated to $k \cdot D$. Moreover, we let $0 \cdot D = O$.

Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. We say that $f : V_1 \to V_2$ is a *homomorphism* between D_1 and D_2 and write $f : D_1 \to D_2$ if

$$x \to y \text{ implies } f(x) \to f(y), \text{ for all } x, y \in V_1.$$

An *endomorphsim* is a homomorphism from D into itself. A mapping $f : V_1 \to V_2$ is an *isomorphism* between D_1 and D_2 if f is bijective and

$$x \to y \text{ if and only if } f(x) \to f(y), \text{ for all } x, y \in V_1.$$

Digraphs D_1 and D_2 are *isomorphic* if there is an isomorphism between them. We write $D_1 \cong D_2$. An *automorphsim* is an isomorphism from D onto itself.

A digraph D is *homomorphism-homogeneous* if every homomorphism $f : W_1 \to W_2$ between finite induced subdigraphs of D extends to an endomorphism of D (see Definition 1.1).

3. Finite irreflexive binary relational systems

Cameron and Nešetřil have shown in [1] that a finite irreflexive graph is homomorphism-homogeneous if and only if it is isomorphic to $k \cdot K_n$ for some $k, n \geqslant 1$. It was shown in [3, Theorem 3.10] that a finite irreflexive proper digraph is homomorphism-homogeneous if and only if it is isomorphic to $k \cdot \mathbf{1}$ for some $k \geqslant 1$ or $k \cdot C_3$ for some $k \geqslant 1$. In this section we show that these are the only finite homomorphism-homogeneous irreflexive binary relational systems by showing that no finite irreflexive improper digraph is homomorphism-homogeneous.

Lemma 3.1. *Let D be a finite homomorphism-homogeneous irreflexive improper digraph. Then every vertex of D is incident with a double edge.*

Proof. Let $x \rightleftarrows y$ be a double edge in D and let v be an arbitrary vertex of D. The mapping

$$f : \begin{pmatrix} x \\ v \end{pmatrix}$$

is a homomorphism between finite induced subdigraphs of D, so it extends to an endomorphism f^* of D by the homogeneity requirement. Then $x \rightleftarrows y$ implies $v = f^*(x) \rightleftarrows f^*(y)$, and $f^*(y) \neq v$ since D is irreflexive. □

Lemma 3.2. *Let D be a finite homomorphism-homogeneous irreflexive improper digraph and let $S \in V(D)/\theta(D)$ be an arbitrary equivalence class of $\theta(D)$. Then $D[S] \cong K_n$ for some $n \geqslant 2$.*

Proof. Lemma 3.1 implies that $|S| \geqslant 2$ for every $S \in V(D)/\theta(D)$.

Suppose that there is an $S \in V(D)/\theta(D)$ such that $D[S]$ is not a complete graph. Then there exist $u, v \in S$ such that $u \not\rightarrow v$ or $v \not\rightarrow u$. Let $z_1, z_2, \ldots, z_k \in V(D)$ be the shortest sequence of vertices of D such that

$$u = z_1 \rightleftarrows z_2 \rightleftarrows \ldots \rightleftarrows z_k = v.$$

Then $k \geqslant 3$ since $u \not\rightleftarrows v$, and the fact that $z_1, z_2, \ldots, z_k$ is the shortest such sequence implies that $z_1 \not\rightleftarrows z_3$. The mapping

$$f_1 : \begin{pmatrix} z_1 & z_3 \\ z_2 & z_3 \end{pmatrix}$$

is a homomorphism between finite induced subdigraphs of D, so it extends to an endomorphism f_1^* of D by the homogeneity requirement. Let $x_1 = f_1^*(z_2)$. It is easy to see that $x_1 \notin \{z_1, z_2, z_3\}$ and $x_1 \rightleftarrows y$ for all $y \in \{z_2, z_3\}$. Consider now the mapping

$$f_2 : \begin{pmatrix} z_1 & z_3 & x_1 \\ z_2 & z_3 & x_1 \end{pmatrix}.$$

which is clearly a homomorphism between finite induced subdigraphs of D. It extends to an endomorphism f_2^* of D. Let $x_2 = f_2^*(z_2)$. Again, it is easy to see that $x_2 \notin \{z_1, z_2, z_3, x_1\}$ and that $x_2 \rightleftarrows y$ for all $y \in \{z_2, z_3, x_1\}$. Analogously, the mapping

$$f_3 : \begin{pmatrix} z_1 & z_3 & x_1 & x_2 \\ z_2 & z_3 & x_1 & x_2 \end{pmatrix}$$

is a homomorphism between finite induced subdigraphs of D, so it extends to an endomorphism f_3^* of D. Let $x_3 = f_3^*(z_2)$. Again, $x_3 \notin \{z_1, z_2, z_3, x_1, x_2\}$ and $x_2 \rightleftarrows y$ for all $y \in \{z_2, z_3, x_1, x_2\}$. We can continue with this procedure as many times as we like, which contradicts the fact that D is a finite digraph. □

Proposition 3.1. *There does not exist a finite homomorphism-homogeneous irreflexive improper digraph.*

Proof. Suppose that D is a finite homomorphism-homogeneous irreflexive improper digraph. Then there exist vertices $x, y \in V(D)$ such that $x \to y$ and $y \not\to x$. Let $S = x/\theta(D)$ and $T = y/\theta(D)$. By Lemma 3.2, $S \cap T = \emptyset$. Let $T = \{y, t_1, \dots, t_k\}$. Since $D[T]$ is a complete graph (Lemma 3.2 again), the mapping

$$f : \begin{pmatrix} x & t_1 & \dots & t_k \\ y & t_1 & \dots & t_k \end{pmatrix}$$

is a homomorphism between finite induced subdigraphs of D, so it extends to an endomorphism f^* of D by the homogeneity requirement. Let us compute $f^*(y)$. From $f^*(t_1) \in T$ it follows that $f^*(T) \subseteq T$. Moreover, $f^*|_T$ is injective since there are no loops in D. Therefore, $f^*|_T : T \to T$ is a bijection. But $f^*(t_i) = t_i$ for all $i \in \{1, \dots, k\}$, so it follows that $f^*(y) = y$. Now, $x \to y$ implies $f^*(x) \to f^*(y)$, that is, $y \to y$, which is impossible since there are no loops in D. □

Corollary 3.1. *Let D be a finite irreflexive binary relational system. Then D is homomorphism-homogeneous if and only if it is isomorphic to one of the following:*

(1) $k \cdot K_n$ for some $k, n \geqslant 1$;

(2) $k \cdot C_3$ for some $k \geqslant 1$.

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Received by the editors January 4, 2010. Revised August 12, 2010.