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ON A SEMIGROUP VARIETY OF GYÖRGY POLLÁK

Edmond W. H. Lee[1]

Abstract. Let $\mathcal{P}$ be the variety of semigroups defined by the identity $xyzx \approx x^2$. By a result of György Pollák, every subvariety of $\mathcal{P}$ is finitely based. The present article is concerned with subvarieties of $\mathcal{P}$ and the lattice they constitute, where the main result is a characterization of finitely generated subvarieties of $\mathcal{P}$. It is shown that a subvariety of $\mathcal{P}$ is finitely generated if and only if it contains finitely many subvarieties, and the identities defining these varieties are described. Specifically, it is decidable when a finite set of identities defines a finitely generated subvariety of $\mathcal{P}$. It follows that the finitely generated subvarieties of $\mathcal{P}$ constitute an incomplete lattice while the non-finitely generated subvarieties of $\mathcal{P}$ constitute an interval. It is also shown that given any pair of finitely generated subvarieties of $\mathcal{P}$, a finite semigroup that generates their meet is computable.

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1. Introduction

All varieties in this article are varieties of semigroups. The reader is referred to [3, 15] for any undefined notation and terminology. A variety $\mathcal{V}$ is said to be *quasilinear* if for each word w, the variety $\mathcal{V}$ satisfies an identity of the form $w \approx w'$ for some linear word w'. Recently, Dolinka and Đapić [4] described all quasilinear varieties. There are precisely nine quasilinear varieties that are band varieties [4, Proposition 1]. But the situation for non-band varieties is more complicated since it follows from [4, Theorem 5] that there are infinitely many non-band quasilinear varieties, all of which satisfy the identity

$$xyx \approx x^2. \tag{1}$$

Dolinka and Đapić also deduced that each quasilinear variety is both finitely based and finitely generated.

It is well and long known that every band variety is finitely based [2, 5, 6]. However, the finite basis property of every non-band quasilinear variety is

[1]Department of Mathematics, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada, e-mail: *ewl@sfu.ca*

actually a consequence of a stronger result of Pollák [12], which states that every semigroup that satisfies the identity

$$xyzx \approx x^2 \tag{2}$$

is finitely based. Since the identity (1) implies the identity (2), every non-band quasilinear variety is also finitely based.

Let $\mathcal{P}$ be the variety defined by the identity (2) and $\mathfrak{L}(\mathcal{P})$ be the lattice of subvarieties of $\mathcal{P}$. The lattice $\mathfrak{L}(\mathcal{P})$ has a very complex structure since a result of Vernikov and Volkov [14] implies that every finite lattice is embeddable in it. But the lattice $\mathfrak{L}(\mathcal{P})$ is only countably infinite due to the finite basis property of all subvarieties of $\mathcal{P}$. On the other hand, although all quasilinear varieties are finitely generated, not all subvarieties of $\mathcal{P}$ are finitely generated. In fact, infinitely many subvarieties of $\mathcal{P}$ are non-finitely generated [10, Proposition 4.2]. Apart from this result, not much is known about the subvarieties of $\mathcal{P}$ with respect to the finite generation property. The present article aims to shed some light in this direction.

A *diverse identity* is an identity of the form $x_1x_2\cdots x_n \approx w$ where w is a word that is not formed by any permutation of the variables $x_1, x_2, \ldots, x_n$.

Theorem 1. *The following statements on any subvariety $\mathcal{V}$ of $\mathcal{P}$ are equivalent*:

(a) *$\mathcal{V}$ is finitely generated*;

(b) *any basis for $\mathcal{V}$ contains some diverse identity*;

(c) *any basis for $\mathcal{V}$ contains an identity not implied by the identities* (2) *and*

$$xy \approx yx; \tag{3}$$

(d) *$\mathcal{V}$ contains finitely many subvarieties.*

Remark 2. It follows from Theorem 1 that a subvariety of $\mathcal{P}$ is finitely generated if and only if it contains finitely many subvarieties. This result, however, does not hold for varieties in general since there exist finitely generated varieties with infinitely many subvarieties [10] and non-finitely generated varieties with finitely many subvarieties [8, 9]. In fact, there exist finitely generated varieties with continuum many subvarieties [7].

Corollary 3. *Given any finite set Σ of identities, it is decidable if the subvariety of $\mathcal{P}$ defined by Σ is finitely generated. In particular, the variety $\mathcal{P}$ is non-finitely generated.*

Remark 4. Since every subvariety of $\mathcal{P}$ is finitely based, the finiteness of the set Σ of identities in Corollary 3 does not compromise the possibility of Σ defining any subvariety of $\mathcal{P}$. But as noted in [15, Problem 2.3], the problem of deciding when a finite set of identities defines a finitely generated variety remains open in general.

Let $\mathcal{C}$ be the subvariety of $\mathcal{P}$ defined by the commutativity identity (3). For each $n \geq 2$, let $\mathcal{D}_n$ be the subvariety of $\mathcal{P}$ defined by the diverse identity

$$x_1 x_2 \cdots x_n \approx x_1^2 x_n^2. \tag{4_n}$$

It is easy to show that the inclusion $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ holds for any $n \geq 2$.

Proposition 5. (i) *The finitely generated subvarieties of* $\mathcal{P}$ *constitute the incomplete sublattice* $\bigcup\{\mathfrak{L}(\mathcal{D}_n) \mid n \geq 2\}$ *of* $\mathfrak{L}(\mathcal{P})$.

(ii) *The non-finitely generated subvarieties of* $\mathcal{P}$ *constitute the subinterval* $[\mathcal{C}, \mathcal{P}]$ *of* $\mathfrak{L}(\mathcal{P})$.

Consequently, $\mathfrak{L}(\mathcal{P}) = [\mathcal{C}, \mathcal{P}] \cup \bigcup\{\mathfrak{L}(\mathcal{D}_n) \mid n \geq 2\}$.

For any semigroup $\mathbf{S}$, let $\langle \mathbf{S} \rangle$ denote the variety generated by $\mathbf{S}$. In general, if $\mathbf{S}_1$ and $\mathbf{S}_2$ are any finite semigroups, then the meet $\langle \mathbf{S}_1 \rangle \cap \langle \mathbf{S}_2 \rangle$ is a variety that may or may not be finitely generated. Even if the variety $\langle \mathbf{S}_1 \rangle \cap \langle \mathbf{S}_2 \rangle$ is known to be finitely generated, the task of computing a finite generating semigroup is nontrivial. However, this task can be accomplished if $\mathbf{S}_1$ and $\mathbf{S}_2$ are finite semigroups in the variety $\mathcal{P}$.

Theorem 6. *Given any finite semigroups* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *in the variety* $\mathcal{P}$, *a finite semigroup that generates the meet* $\langle \mathbf{S}_1 \rangle \cap \langle \mathbf{S}_2 \rangle$ *is computable.*

Theorem 1, Proposition 5, and Theorem 6 are established in the next three sections. Corollary 3 follows from the equivalence of statements (a) and (b) in Theorem 1.

2. Regarding Theorem 1

Let $\mathcal{X}^*$ be the free monoid over a countably infinite set $\mathcal{X}$ of variables. Elements of $\mathcal{X}^*$ are referred to as *words*. The *content* of a word w, denoted by $\mathsf{c}(w)$, is the set of variables occurring in w. A word is *linear* if each of its variables has multiplicity one. A *permutation identity* is an identity $u \approx v$ where u and v are distinct linear words such that $\mathsf{c}(u) = \mathsf{c}(v)$. Note that a nontrivial identity $u \approx v$ is diverse if and only if it is a non-permutation identity where either u or v is a linear word.

Lemma 7. (i) *If a set* Σ *of identities implies some diverse identity, then the set* Σ *must contain some diverse identity.*

(ii) *An identity* $u \approx v$ *is non-diverse if and only if the identities* (2) *and* (3) *imply the identity* $u \approx v$.

Proof. (i) Let Σ be any set of non-diverse identities. Then the set Σ can only contain the following identities: trivial identities, permutation identities, and identities formed by nonlinear words. It is then easy to show that the set Σ cannot imply any diverse identity.

(ii) Since the identities (2) and (3) are non-diverse, it follows from part (i) that they can only imply non-diverse identities. Conversely, let $u \approx v$ be any non-diverse identity. It is clear that the identities (2) and (3) imply any permutation identity. Therefore it suffices to assume that both u and v are nonlinear words, whence $u = u_1xu_2xu_3$ and $v = v_1yv_2yv_3$ for some $x, y \in \mathcal{X}$ and $u_i, v_i \in \mathcal{X}^*$. Since

$$\begin{aligned} u &\overset{(3)}{\approx} x^2u_1u_2u_3 \overset{(2)}{\approx} xy^2xu_1u_2u_3 \overset{(3)}{\approx} yx^2u_1u_2u_3y \overset{(2)}{\approx} y^2 \\ &\overset{(2)}{\approx} y^3v_1v_2v_3y \overset{(3)}{\approx} y^4v_1v_2v_3 \overset{(2)}{\approx} y^2v_1v_2v_3 \overset{(3)}{\approx} v, \end{aligned}$$

the identities (2) and (3) imply the non-diverse identity $u \approx v$. □

Lemma 8. *Let* $\mathbf{S}$ *be any finite semigroup in the variety* $\mathcal{P}$ *with* $|\mathbf{S}| = n \geq 2$. *Then* $\mathbf{S}$ *satisfies the diverse identity* (4_n) *and hence belongs to the variety* $\mathcal{D}_n$.

Proof. Suppose that $a_1, a_2, \ldots, a_n \in \mathbf{S}$. Then it follows from [1, Proposition 3.7.1] that $a_1a_2\cdots a_n = b_1b_2^2b_3$ for some $b_1, b_2, b_3 \in \mathbf{S}$. Since

$$\begin{aligned} a_1^2a_n^2 &\overset{(2)}{=} a_1a_2\cdots a_na_1^2a_n^2a_1a_2\cdots a_n = b_1(b_2^2b_3a_1^2a_n^2b_1b_2^2)b_3 \\ &\overset{(2)}{=} b_1b_2^2b_3 = a_1a_2\cdots a_n, \end{aligned}$$

the semigroup $\mathbf{S}$ satisfies the identity (4_n). □

Lemma 9. *Let* $u \approx v$ *be any diverse identity. Suppose that* w *is any shortest linear word in* $\{u, v\}$. *Then the identities* (2) *and* $u \approx v$ *imply the following*:

(a) *the identity* (4_n) *with* $n = |w| + 4$;

(b) *some permutation identity.*

Proof. Without loss of generality, assume that $u = x_1x_2\cdots x_m$ is a shortest linear word in $\{u, v\}$ and that $y_1, y_2, y_3, z_1, z_2 \notin \mathsf{c}(uv)$.

Case 1. v is nonlinear. Then $v = v_1xv_2xv_3$ for some $v_i \in \mathcal{X}^*$ and $x \in \mathcal{X}$. Let φ be the substitution $x \mapsto z_1xz_2$ and let $v_i' = v_i\varphi$. Then

$$\begin{aligned} y_1(u\varphi)y_2 &\approx y_1(v\varphi)y_2 = y_1v_1'z_1(xz_2v_2'z_1x)z_2v_3'y_2 \\ &\overset{(2)}{\approx} (y_1v_1'z_1xy_1)(y_2xz_2v_3'y_2) \overset{(2)}{\approx} y_1^2y_2^2 \end{aligned}$$

so that the identities (2) and $u \approx v$ imply the identity $y_1(u\varphi)y_2 \approx y_1^2y_2^2$. Since the word $y_1(u\varphi)y_2$ is linear and

$$|y_1(u\varphi)y_2| = |u\varphi| + 2 = \begin{cases} m+2 & \text{if } x \notin \mathsf{c}(u), \\ m+4 & \text{if } x \in \mathsf{c}(u), \end{cases}$$

it follows that the identities (2) and $u \approx v$ imply (a). Now the words $y_1y_2y_3(u\varphi)$ and $y_1y_3y_2(u\varphi)$ are also linear. Since

$$\begin{aligned} y_1y_2y_3(u\varphi) &\approx y_1y_2y_3(v\varphi) = y_1y_2y_3v_1'z_1(xz_2v_2'z_1x)z_2v_3' \\ &\overset{(2)}{\approx} (y_1y_2y_3v_1'z_1xy_1^2)xz_2v_3' \overset{(2)}{\approx} y_1y_3y_2v_1'z_1(xy_1^2x)z_2v_3' \\ &\overset{(2)}{\approx} y_1y_3y_2v_1'z_1xz_2v_2'z_1xz_2v_3' = y_1y_3y_2(v\varphi) \\ &\approx y_1y_3y_2(u\varphi), \end{aligned}$$

the identities (2) and $u \approx v$ imply (b).

CASE 2. v is linear. Since $|u| \leq |v|$, some variable z in v does not occur in u. Let v_z be the word obtained from v by replacing z with z^2. Then the identity $u \approx v$ implies the identity $u \approx v_z$ where v_z is a nonlinear word. The same argument in Case 1 can be repeated to show that the identities (2) and $u \approx v_z$ imply (a) and (b). □

Lemma 10 (Malyshev [11]). *Any variety that satisfies some diverse identity and some permutation identity contains finitely many subvarieties.*

Recall that a variety $\mathcal{V}$ is *locally finite* if each finitely generated semigroup in $\mathcal{V}$ is finite.

Lemma 11. *Let $\mathcal{V}$ be any locally finite variety whose subvarieties satisfy the ascending chain condition. Then $\mathcal{V}$ is a finitely generated variety.*

Proof. For each $n \geq 1$, let $\mathcal{V}_n$ be the variety generated by the $\mathcal{V}$-free semigroup over n variables. The varieties in the chain $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots$ are finitely generated because the variety $\mathcal{V}$ is locally finite. By the ascending chain condition, there exists some sufficiently large r such that $\mathcal{V}_r = \mathcal{V}_{r+1} = \cdots$. Consequently, $\mathcal{V} = \mathcal{V}_r$. □

Proof of Theorem 1. The theorem holds vacuously if the variety $\mathcal{V}$ is trivial. Therefore assume that the variety $\mathcal{V}$ is nontrivial, and let Σ be any basis for $\mathcal{V}$.

(a) IMPLIES (b). Suppose that the variety $\mathcal{V}$ is generated by some nontrivial semigroup of order n. Then by Lemma 8, the variety $\mathcal{V}$ satisfies the diverse identity (4_n) so that Σ implies (4_n). By Lemma 7(i), some identity in Σ is diverse.

(b) IMPLIES (c). Suppose that the set Σ contains some diverse identity $u \approx v$. Since the identities (2) and (3) are non-diverse identities, it follows from Lemma 7(i) that the identities (2) and (3) do not imply the identity $u \approx v$.

(c) IMPLIES (d). Suppose that the identities (2) and (3) do not imply some identity in Σ. Then it follows from Lemma 7(ii) that the variety $\mathcal{V}$ satisfies some diverse identity. By Lemma 9, the variety $\mathcal{V}$ also satisfies some permutation identity. Hence by Lemma 10, the variety $\mathcal{V}$ contains finitely many subvarieties.

(d) IMPLIES (a). By [13, Theorem P], the variety $\mathcal{V}$ is locally finite. If the variety $\mathcal{V}$ contains finitely many subvarieties, then it is finitely generated by Lemma 11. □

3. Regarding Proposition 5

Proof of Proposition 5. (i) Let $\mathcal{V}$ be any subvariety of $\mathcal{D}_n$ for any $n \geq 2$ and let Σ be any basis for $\mathcal{V}$. Since Σ implies the diverse identity (4_n), it follows from Lemma 7(i) that Σ contains some diverse identity. Hence the variety $\mathcal{V}$ is finitely generated by Theorem 1. Conversely, it follows from Lemma 8 that any finitely generated subvariety of $\mathcal{P}$ is a subvariety of $\mathcal{D}_n$ for some $n \geq 2$. Consequently, the finitely generated subvarieties of $\mathcal{P}$ constitute the set

$$\mathfrak{D} = \bigcup \{\mathfrak{L}(\mathcal{D}_n) \mid n \geq 2\}.$$

Since $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ for any $n \geq 2$, it is easy to show that $\mathfrak{D}$ is a lattice. For each $n \geq 1$, let $\mathcal{P}_n$ be the variety generated by the $\mathcal{P}$-free semigroup over n variables. Then $\mathcal{P} = \bigvee\{\mathcal{P}_n \mid n \geq 1\}$ where each $\mathcal{P}_n$ is finitely generated. However, the variety $\mathcal{P}$ is non-finitely generated by Corollary 3 so that the lattice $\mathfrak{D}$ is incomplete.

(ii) This follows from the equivalence of statements (a) and (c) in Theorem 1. □

4. Regarding Theorem 6

Suppose that $\mathbf{S}$ is any semigroup in the variety $\mathcal{D}_n$ for some $n \geq 5$. Then the semigroup $\mathbf{S}$ is defined within $\mathcal{D}_n$ by identities that are formed by words of length less than n, that is, $\mathbf{S}$ is defined within $\mathcal{D}_n$ by some set of identities from

$$\Psi_n = \{u \approx v \mid \mathsf{c}(u), \mathsf{c}(v) \subseteq \mathcal{X}_{2n-2}, |u|, |v| < n\},$$

where $\mathcal{X}_{2n-2} = \{x_1, x_2, \ldots, x_{2n-2}\}$.

Lemma 12. *Let* $\mathbf{S}$ *be any finite semigroup in the variety* $\mathcal{P}$*. Then a finite basis for* $\mathbf{S}$ *is computable.*

Proof. By Lemma 8, the semigroup $\mathbf{S}$ belongs to the variety $\mathcal{D}_n$ for some $n \geq 2$. Since $\mathcal{D}_2 \subseteq \mathcal{D}_3 \subseteq \cdots$, it can further be assumed that $n \geq 5$. Hence $\mathbf{S}$ is defined within $\mathcal{D}_n$ by some set of identities from Ψ_n. Let $\Psi_n^{\mathbf{S}}$ be the set of all identities from Ψ_n that are satisfied by $\mathbf{S}$. Then $\{(2), (4_n)\} \cup \Psi_n^{\mathbf{S}}$ is a basis for $\mathbf{S}$. Since $\mathbf{S}$ and Ψ_n are both finite, the set $\Psi_n^{\mathbf{S}}$ is also finite and computable. □

Lemma 13. *Let* Σ *be any finite set of identities and* $\mathcal{V}$ *be the subvariety of* $\mathcal{P}$ *defined by* Σ*. Suppose that* $\mathcal{V}$ *is finitely generated. Then a finite semigroup that generates the variety* $\mathcal{V}$ *is computable.*

Proof. By Theorem 1, the set Σ contains some diverse identity $u \approx v$. By Lemma 9, the variety $\mathcal{V}$ satisfies the identity (4_n) for some $n \in \{|u|+4, |v|+4\}$ and so is contained in the variety $\mathcal{D}_n$. Hence the variety $\mathcal{V}$ is defined within $\mathcal{D}_n$ by some set of identities from Ψ_n. Let Ψ_n' be set of all identities from Ψ_n that are not satisfied by the variety $\mathcal{V}$. Then any semigroup in $\mathcal{V}$ that does not satisfy any identity in Ψ_n' will generate the variety $\mathcal{V}$.

Let $\mathbf{F}$ be the $\mathcal{V}$-free semigroup over $\mathcal{X}_{2n-2}$. Let $u \approx v \in \Psi_n'$. Since the variety $\mathcal{V}$ does not satisfy the identity $u \approx v$, some semigroup $\mathbf{S}$ in $\mathcal{V}$ also does not satisfy the identity $u \approx v$. Since u and v are words over $\mathcal{X}_{2n-2}$, the semigroup $\mathbf{S}$ can be chosen to be finitely generated by at most $2n-2$ elements. Therefore the semigroup $\mathbf{S}$ is a homomorphic image of the semigroup $\mathbf{F}$, whence $\mathbf{F}$ does not satisfy the identity $u \approx v$. The semigroup $\mathbf{F}$ is clearly computable and so is the required semigroup that generates the variety $\mathcal{V}$. □

Proof of Theorem 6. For each $i \in \{1, 2\}$, let $\mathbf{S}_i$ be any finite semigroup in the variety $\mathcal{P}$. By Lemma 12, a finite basis Σ_i for the semigroup $\mathbf{S}_i$ is computable. Then the variety $\mathcal{V} = \langle \mathbf{S}_1 \rangle \cap \langle \mathbf{S}_2 \rangle$ is defined by $\Sigma = \Sigma_1 \cup \Sigma_2$ and is finitely generated by Proposition 5(i). By Lemma 13, a finite semigroup that generates the variety $\mathcal{V}$ is computable. □

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