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THE RELATIONSHIP BETWEEN PROPER AND INNER HYPERSUBSTITUTIONS FOR VARIETIES OF RINGS

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Abstract. A hypersubstitution maps an algebra to an algebra of the same type, by replacing the operations by term operations. A hypersubstitution is called proper with respect to a variety if it is a mapping on this variety and it is called inner if it is an identity mapping on this variety. Proper as well as inner hypersubstitutions characterize a variety. Each inner hypersubstitution is a proper one but not conversely. In the present paper, we characterize the relationship between proper and inner hypersubstitutions for varieties of rings satisfying $x^{n+1} \approx x$, in particular for $n = 6$.

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1. Introduction

The problem of transforming one algebra into another in one step occurs in both theoretical and applied computer science. One way to realize such a transformation algebraically is by derived algebras, whereby the fundamental operations of an algebra are replaced by term operations of that algebra. This procedure can be expressed using the concept of a hypersubstitution, introduced by Denecke, Lau, Pöschel, and Schweigert ([5]). If $\mathcal{A}= (A; (f_i^A)_{i\in I})$ is an algebra of type $\tau = (n_i)_{i\in I}$ and σ is a hypersubstitution (of type τ) then the algebra $\sigma(\mathcal{A}):= (A; (\sigma(f_i)^A)_{i\in I})$ is called a derived algebra. The derived algebra has the same carrier set as the original one but instead of the operations f_i^A for $i \in I$, the algebra $\sigma(\mathcal{A})$ has the term operations $\sigma(f_i)^A$ for $i \in I$. Let us consider a variety V of algebras of type τ. In general, V has need not be closed under σ, i.e. $\sigma(V) := \{\sigma(\mathcal{A}) \mid \mathcal{A} \in V\}$ is in most cases not a subset of V. If $\sigma(V)$ is contained in V then σ is called a proper hypersubstitution with respect to V ([10]). Each hypersubstitution is a proper one with respect to a solid variety ([8]). But if V is not solid then there are hypersubstitutions which are not proper. If $\sigma(\mathcal{A})$ agrees

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with $\mathcal{A}$ for all $\mathcal{A} \in V$ then σ is called an inner hypersubstitution of V ([10]). Of course, each inner hypersubstitution of V is a proper one with respect to V. But the converse is not true. In several papers, proper hypersubstitutions of varieties are studied (see also [7]). In particular in [4], proper hypersubstitutions of varieties of bands are determined. In the present paper, we want to study proper hypersubstitutions with respect to varieties of rings. Note that there is no non-trivial solid variety of rings since the addition satisfies the commutative law (see also [3]). Let us now recall the basic concepts of the theory of hypersubstitutions. For more background see [7]. We fix an infinite alphabet $X := \{x_1, x_2, \ldots\}$ and a type $\tau = (n_i)_{i \in I}$. For $i \in I$, we denote by f_i the corresponding n_i-ary operation symbol. Then $W_\tau(X)$ denotes the set of all terms of type τ over the alphabet X. For $1 \leq n \in \mathbb{N}$, a term of type τ over the n-element alphabet $X_n := \{x_1, \ldots, x_n\}$ is called n-ary. In particular, a term of arity one is called a unary term, and a term of arity two is called binary. For each algebra $\mathcal{A}$, an n-ary term t induces an n-ary term operation t^A on the algebra $\mathcal{A}$. For a natural number $i \geq 1$, we denote by $c_{x_i}(t)$ the number of occurrence of x_i in the term t. A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which assigns each operation symbol f_i an n_i-ary term is called a hypersubstitution of type τ. It is not difficult to see that the algebra $\mathcal{A}= (A; (f_i^A)_{i \in I})$ and the derived algebra $\sigma(\mathcal{A}):= (A; (\sigma(f_i)^A)_{i \in I})$ have the same type. A hypersubstitution σ can be extended in a natural way to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ by the following inductive definition

(i) $\widehat{\sigma}[x_j] := x_j$ for $1 \leq j \in \mathbb{N}$

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ for composite terms $f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X)$.

(Here, $\sigma(f_i)(\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ means that we replace $x_1, \ldots, x_{n_i}$ in $\sigma(f_i)$ by $\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]$.) One can define a product $\circ_h$ on the set $Hyp(\tau)$ of all hypersubstitutions of type τ by letting $\sigma_1 \circ_h \sigma_2$ be the hypersubstitution which maps each operation symbol f_i to the term $\widehat{\sigma}_1[\sigma_2(f_i)]$. This operation $\circ_h$ is associative. There is an identity element in the semigroup $(Hyp(\tau); \circ_h)$, namely the hypersubstitution which maps the operation symbol f_i to its fundamental term $f_i(x_1, \ldots, x_{n_i})$ for $i \in I$. So $Hyp(\tau)$ forms a monoid under $\circ_h$. Additionally, for a fixed variety V of type τ we denoted by IdV the set of all identities in V. Then the elements of the set

$$P(V) := \{\sigma \in Hyp(\tau) \mid u \approx v \in IdV \Rightarrow \widehat{\sigma}[u] \approx \widehat{\sigma}[v] \in IdV\}$$

are exactly the proper hypersubstitutions with respect to V. Note that $P(V)$ forms a submonoid of the monoid of all hypersubstitutions. The elements of the set

$$P_0(V) := \{\sigma \in Hyp(\tau) \mid \sigma(f_i) \approx f_i(x_1, \ldots, x_{n_i}) \in IdV, i \in I\}$$

are exactly the inner hypersubstitutions with respect to V. If $P(V) = P_0(V)$ then V is called unsolid (see [1]). The relation $\sim_V$ is defined in [10] ($\sigma_1 \sim_V \sigma_2$ if $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for all $i \in I$). The number of equivalence classes of the relation $\sim_V$ on $P(V)$ is called the degree of V. The degree of particular varieties is studied in [6]. If the degree of a variety V is one then V is unsolid.

In the next section, we will characterize the relationship between proper and inner hypersubstitutions with respect to any variety of rings satisfying $x \approx x^n$ $(2 \leq n \in \mathbb{N})$. Moreover, we will verify that it is enough to consider the varieties of rings generated by any subdirectly irreducible ring. Section 3 is devoted the varieties of rings satisfying $x \approx x^7$. These twelve varieties of rings are of particular interest since each ring with special involution satisfies $x \approx x^7$ ([11]). We give the identity basis for each of the twelve subvarieties of the variety of all rings satisfying $x \approx x^7$. For a particular atom in this lattice we determine all proper hypersubstitutions, giving an algorithm to decide if a hypersubstitution is proper with respect to this variety. The described methods can also be used to determine $P(V)$ for other varieties V of rings.

2. Characterization of the relationship

The mapping $\widehat{\sigma}$ has as its domain the set $W_\tau(X)$ of terms. It replaces each vertices, labelled with an operation symbol f_i in the tree of a term by an n_i-ary term t. It can happen that t contains operation symbols different from f_i. From the point of view of theoretical and applied computer science it is often interesting to replace any operation symbol f_i in a term by a term containing only f_i as operation symbol. This can be realized by hypersubstitutions σ with the following additional property: for each $i \in I$, $\sigma(f_i)$ contains only f_i as operation symbol (or $\sigma(f_i) \in X_{n_i}$). The set of all such hypersubstitutions will be denoted by $Hyp^{oc}(\tau)$.

Lemma 2.1. *$Hyp^{oc}(\tau)$ forms a monoid.*

Proof. Clearly, the identity element $\sigma_{id} : f_i \mapsto f_i(x_1, \ldots, x_{n_i})$, $i \in I$, belongs to $Hyp^{oc}(\tau)$. Let $\sigma_1, \sigma_2 \in Hyp^{oc}(\tau)$ and $i \in I$. Then $\sigma_2(f_i) \in X_{n_i}$ or f_i is the only operation symbol in $\sigma_2(f_i)$. If $\sigma_2(f_i) \in X_{n_i}$ then $\sigma_1 \circ_h \sigma_2(f_i) = \widehat{\sigma}_1[\sigma_2(f_i)] = \sigma_2(f_i) \in X_{n_i}$. If $\sigma_2(f_i) = f_i(t_1, \ldots, t_{n_i})$ we assume that $\widehat{\sigma}_1[t_j] \in X_{n_i}$ or f_i is the only operation symbol in $\widehat{\sigma}_1[t_j]$ for $1 \leq j \leq n_i$ then $\sigma_1 \circ_h \sigma_2(f_i) = \widehat{\sigma}_1[\sigma_2(f_i)] = \sigma_1(f_i)(\widehat{\sigma}_1[t_1], \ldots, \widehat{\sigma}_1[t_{n_i}])$, i.e. $\sigma_1 \circ_h \sigma_2(f_i) \in X_{n_i}$ or f_i is the only operation symbol in $\sigma_1 \circ_h \sigma_2(f_i)$. This shows that $\sigma_1 \circ_h \sigma_2 \in Hyp^{oc}(\tau)$. □

In the following we fix the type $\tau = (2, 2, 1, 0)$ and consider varieties of rings. We will use $+$ and $\cdot$ as the binary operations, $-$ as the unary operation as well as 0 as the 0-ary operation. Let us denote by $\mathcal{V}^R(p^k)$ the variety of rings generated by the p^k -element field $GF(p^k)$ for any prime number p and any natural number $k \geq 1$.

The next theorem characterizes the relationship between inner and proper hypersubstitution with respect to $\mathcal{V}^R(p^k)$.

Theorem 2.2. *Let p be a prime number, $k \geq 1$ be a natural number. Then*

$$Hyp^{oc}(\tau) \cap P(\mathcal{V}^R(p^k)) = P_0(\mathcal{V}^R(p^k)).$$

Proof. It is easy to see that $P_0(\mathcal{V}^R(p^k)) \subsetneqq Hyp^{oc}(\tau) \cap P(\mathcal{V}^R(p^k))$. We will discuss the other inclusion. Let $\sigma \in Hyp^{oc}(\tau) \cap P(\mathcal{V}^R(p^k))$. Assume that $c_{x_1}(\sigma(+)) \not\equiv 1 \ mod \ p$. Then there is an $a \in \{0, 2, 3, ..., p-1\}$ such that $c_{x_1}(\sigma(+)) \equiv a \ mod \ p$. We apply σ to the ring identity $x + 0 \approx x$ and obtain $c_{x_1}(\sigma(+))x \approx x \in Id(\mathcal{V}^R(p^k))$. Since $c_{x_1}(\sigma(+)) \equiv a \ mod \ p$ and $px \approx 0 \in Id(\mathcal{V}^R(p^k))$, the identity $c_{x_1}(\sigma(+))x \approx x$ provides $ax \approx x$ and finally $x \approx 0 \in Id(\mathcal{V}^R(p^k))$, a contradiction. In the same way we show that $c_{x_2}(\sigma(+)) \equiv 1 \ mod \ p$. This gives $\sigma(+) \approx c_{x_1}(\sigma(+))x_1 + c_{x_2}(\sigma(+))x_2 \approx x_1 + x_2 \in Id(\mathcal{V}^R(p^k))$. Now, we want to show $c_{x_1}(\sigma(\cdot)) \equiv c_{x_2}(\sigma(\cdot)) \equiv 1 \ mod \ p^k - 1$. We apply σ to the commutative law $xy \approx yx \in Id(\mathcal{V}^R(p^k))$. Then we get $x^{c_{x_1}(\sigma(\cdot))}y^{c_{x_2}(\sigma(\cdot))} \approx y^{c_{x_1}(\sigma(\cdot))}x^{c_{x_2}(\sigma(\cdot))} \in Id(\mathcal{V}^R(p^k))$. We have $c_{x_1}(\sigma(\cdot)), c_{x_2}(\sigma(\cdot)) > 0$. Otherwise, there is an $i \in \{1, 2\}$ with $x^{c_{x_i}(\sigma(\cdot))} \approx y^{c_{x_i}(\sigma(\cdot))} \in Id(\mathcal{V}^R(p^k))$, a contradiction. Then there are $a, b \in \mathbb{N}^+ := \mathbb{N}\backslash\{0\}$ with $a, b < p^k$ such that $c_{x_1}(\sigma(\cdot)) \equiv a \ mod \ p^k - 1$ and $c_{x_2}(\sigma(\cdot)) \equiv b \ mod \ p^k - 1$. Assume that $c_{x_1}(\sigma(\cdot)) \not\equiv c_{x_2}(\sigma(\cdot)) \ mod \ p^k - 1$. We replace y by $x^{p^k - 1}$ in the previous identity $x^{c_{x_1}(\sigma(\cdot))}y^{c_{x_2}(\sigma(\cdot))} \approx y^{c_{x_1}(\sigma(\cdot))}x^{c_{x_2}(\sigma(\cdot))}$. Then we get $x^{c_{x_1}(\sigma(\cdot))} \approx x^{c_{x_2}(\sigma(\cdot))} \in Id(\mathcal{V}^R(p^k))$. Since $x^{p^k} \approx x \in Id(\mathcal{V}^R(p^k))$, we have $x^a \approx x^{c_{x_1}(\sigma(\cdot))} \approx x^{c_{x_2}(\sigma(\cdot))} \approx x^b \in Id(\mathcal{V}^R(p^k))$. This contradicts $a \not\equiv b \ mod \ p^k - 1$. Hence $a \equiv c_{x_1}(\sigma(\cdot)) \equiv c_{x_2}(\sigma(\cdot)) \ mod \ p^k - 1$. Assume that $a \not\equiv 1 \ mod \ p^k - 1$. We apply σ to the associative law $x(yz) \approx (xy)z \in Id(\mathcal{V}^R(p^k))$. Then we get $x^a(y^a z^a)^a \approx (x^a y^a)^a z^a \in Id(\mathcal{V}^R(p^k))$. We replace y and z by $x^{p^k - 1}$ in the previous identity. Then we get $x^a \approx x^{a^2} \in Id(\mathcal{V}^R(p^k))$. We want to show that $\hat{\sigma}[x^k] \approx x^{ak}$ for $k \in \mathbb{N}^+$ by induction. We have $\hat{\sigma}[x^2] \approx \hat{\sigma}[xx] \approx x^a x^a \approx x^{2a} \in Id(\mathcal{V}^R(p^k))$. Suppose that $\hat{\sigma}[x^n] \approx x^{na} \in Id(\mathcal{V}^R(p^k))$ for some $n \in \mathbb{N}^+$. Then $\hat{\sigma}[x^{n+1}] \approx \hat{\sigma}[x^n x] \approx \sigma(\cdot)(\hat{\sigma}[x^n], x) \approx \sigma(\cdot)(x^{na}, x) \approx (x^{na})^a x^a \approx x^{na^2} x^a \approx x^{na} x^a \approx x^{(n+1)a} \in Id(\mathcal{V}^R(p^k))$. This show that $\hat{\sigma}[x^k] \approx x^{ka} \in Id(\mathcal{V}^R(p^k))$ for all $k \in \mathbb{N}^+$. We apply σ to the identity $x^{p^k} \approx x \in Id(\mathcal{V}^R(p^k))$. Then we get $x^{ap^k} \approx x \in Id(\mathcal{V}^R(p^k))$. Since $x^{p^k} \approx x \in Id(\mathcal{V}^R(p^k))$, the identity $x^{ap^k} \approx x$ provides $x^a \approx x \in Id(\mathcal{V}^R(p^k))$, a contradiction. This shows that $c_{x_1}(\sigma(\cdot)) \equiv c_{x_2}(\sigma(\cdot)) \equiv 1 \ mod \ p^k - 1$ and $\sigma(\cdot) \approx x_1^{c_{x_1}(\sigma(\cdot))} \cdot x_2^{c_{x_2}(\sigma(\cdot))} \approx x_1 \cdot x_2 \in Id(\mathcal{V}^R(p^k))$. Finally, we consider $\sigma(-)$. We have two possibilities. First, suppose that the number of occurrences of " $-$ " in $\sigma(-)$ is odd. Then $\sigma(-) \approx -x \in Id(\mathcal{V}^R(p^k))$. Next, suppose that the number of occurrence of " $-$ " in $\sigma(-)$ is even. Here we have $\sigma(-) \approx x \in Id(\mathcal{V}^R(p^k))$. We want to show that $\sigma(-) \approx -x \in Id(\mathcal{V}^R(p^k))$. In the case $p = 2$, we have $x \approx -x \in Id(\mathcal{V}^R(p^k))$. Then we get immediately $\sigma(-) \approx -x \in Id(\mathcal{V}^R(p^k))$. We consider the case $p \neq 2$. Assume that $\sigma(-) \approx x \in Id(\mathcal{V}^R(p^k))$. We apply σ to the ring identity $x + (-x) \approx 0$. Then we get $2x \approx x + x \approx 0 \in Id(\mathcal{V}^R(p^k))$. Since $px \approx 0 \in Id(\mathcal{V}^R(p^k))$ and $p \neq 2$, the identity $2x \approx 0$ provides $x \approx 0 \in Id(\mathcal{V}^R(p^k))$, a contradiction. So $\sigma(-) \approx -x \in Id(\mathcal{V}^R(p^k))$. Altogether, this shows that $\sigma \in P_0(\mathcal{V}^R(p^k))$. □

Some $\vee$-semilattices of varieties of rings generated by varieties of type $\mathcal{V}^R(p^k)$ are of particular interest (see also [2]). We are going to characterize the relationship between inner and proper hypersubstitutions for all varieties in such $\vee$-semilattices. For this we show that the proper hypersubstitutions with respect to the join of two varieties are the proper hypersubstitutions with respect to both varieties.

Lemma 2.3. *Let V and W be varieties of type τ. Then*

(i) $P(V \vee W) = P(V) \cap P(W)$;

(ii) $P_0(V \vee W) = P_0(V) \cap P_0(W)$.

Proof. (i) It is easy to see that $P(V \vee W) \subseteq P(V)$ and $P(V \vee W) \subseteq P(W)$, i.e. $P(V \vee W) \subseteq P(V) \cap P(W)$. Now we show the converse inclusion. Let $\sigma \in P(V) \cap P(W)$. Further let $u \approx v \in Id(V \vee W) = IdV \cap IdW$. From $u \approx v \in IdV$ and $\sigma \in P(V)$ it follows $\widehat{\sigma}[u] \approx \widehat{\sigma}[v] \in IdV$ and similarly $\widehat{\sigma}[u] \approx \widehat{\sigma}[v] \in IdW$. Thus $\widehat{\sigma}[u] \approx \widehat{\sigma}[v] \in IdV \cap IdW = Id(V \vee W)$. This shows that $\sigma \in P(V \vee W)$.

(ii) It is easy to check that $P_0(V \vee W) \subseteq P_0(V) \cap P_0(W)$. Conversely, let $\sigma \in P_0(V) \cap P_0(W)$ and let f an n-ary operation symbol. Then $\sigma(f) \approx f(x_1, ..., x_n) \in IdV \cap IdW = Id(V \vee W)$. This shows that $\sigma \in P_0(V \vee W)$. □

Lemma 2.3 can be extended for finite sets varieties. Using Theorem 2.2, we obtain a characterization of the relationship between inner and proper hypersubstitutions within a $\vee$-semilattice of varieties of ring generated by varieties of type $\mathcal{V}^R(p^k)$.

Theorem 2.4. *Let $1 \leq n$ be a natural number, $p_1, \ldots, p_n$ be prime numbers and $k_1, \ldots, k_n \geq 1$ be natural numbers. Then*

$$P(\bigvee_{i=1}^{n} \mathcal{V}^R(p_i^{k_i})) \cap Hyp^{oc}(\tau) = P_0(\bigvee_{i=1}^{n} \mathcal{V}^R(p_i^{k_i})).$$

Proof. For $1 \leqq i \leqq n$, we have $P(\mathcal{V}^R(p_i^{k_i})) \cap Hyp^{oc}(\tau) = P_0(\mathcal{V}^R(p_i^{k_i}))$. Hence

$$\begin{aligned} P_0(\bigvee_{i=1}^{n} \mathcal{V}^R(p_i^{k_i})) &= \bigcap_{i=1}^{n} P_0(\mathcal{V}^R(p_i^{k_i})) \\ &= \bigcap_{i=1}^{n} (P(\mathcal{V}^R(p_i^{k_i}) \cap Hyp^{oc}(\tau)) \\ &= \bigcap_{i=1}^{n} P(\mathcal{V}^R(p_i^{k_i})) \cap Hyp^{oc}(\tau) \\ &= P(\bigvee_{i=1}^{n} \mathcal{V}^R(p_i^{k_i})) \cap Hyp^{oc}(\tau) \end{aligned}$$

by Lemma 2.3. □

So, the relationship between $P_0(V)$ and $P(V)$ is characterized for each subvariety V of the variety of rings satisfying $x \approx x^{n+1}$ for any natural number $n \geq 1$ since it is generated by the varieties $\mathcal{V}^R(p^k)$ where p runs over all prime numbers such that $p-1$ divides n and k runs over all natural numbers ≥ 1 with $p^k \leq n$. The next section is devoted to the case $n = 6$.

3. Rings satisfying $x \approx x^7$

The subvariety lattice of the variety of rings satisfying $x \approx x^7$ has twelve elements denoted by $V_1 := TR$, $V_2 := \mathcal{V}^R(7)$, $V_3 := \mathcal{V}^R(3)$, $V_4 := V_2 \vee V_3$, $V_5 := \mathcal{V}^R(2)$, $V_6 := V_2 \vee V_5$, $V_7 := V_5 \vee V_3$, $V_8 := V_4 \vee V_7$, $V_9 := \mathcal{V}^R(4)$, $V_{10} := V_6 \vee V_9$, $V_{11} := V_7 \vee V_9$, and V_{12} be the variety of all rings satisfying $x^7 \approx x$. It is the direct product of the following three lattices (see [2]):

1. $\{TR, \mathcal{V}^R(2), \mathcal{V}^R(4)\}$
2. $\{TR, \mathcal{V}^R(3)\}$
3. $\{TR, \mathcal{V}^R(7)\}$.

The following figure illustrates this lattice.

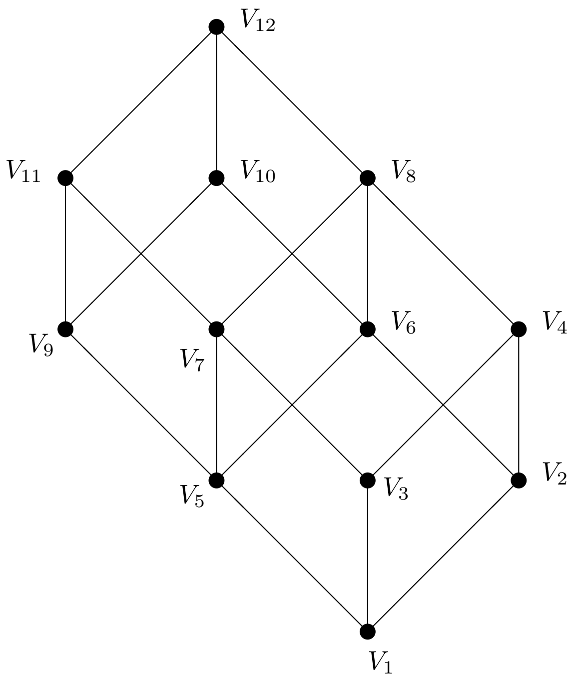

Figure 1: The lattice of all varieties of rings satisfying $x^7 \approx x$

It is well known that each of these twelve subvarieties of rings satisfying $x \approx x^7$, is finitely based. We give a minimal identity basis for each one. It will happen that we need only the seven ring identities, $x \approx x^7$ and one additional identity in each case. It will cause no confusion in this section if we write $V(u \approx v)$ for the variety of all rings satisfying $x \approx x^7$ and the additional identity $u \approx v$. Since the identity $x \approx x^7$ is redundant in three cases we write $V^*(u \approx v)$ for the variety of all rings satisfying the seven ring identities and the additional identity $u \approx v$. Clearly, V_1 is the trivial variety TR and V_{12} is the variety $V^*(x \approx x^7)$ of all rings satisfying $x \approx x^7$.

Theorem 3.1. *We have* $V_2 = V(7x \approx 0)$, $V_3 = V(3x \approx 0)$, $V_4 = V(21x \approx 0)$, $V_5 = V^*(x^2 \approx x)$, $V_6 = V(7y(x^2 - x) \approx 0)$, $V_7 = V(3y(x^2 - x) \approx 0)$, $V_8 = V(21y(x^2 - x) \approx 0)$, $V_9 = V^*(x^4 \approx x)$, $V_{10} = V(7y(x^4 - x) \approx 0)$, $V_{11} = V(3y(x^4 - x) \approx 0)$ *and* $V_{12} = V^*(x \approx x^7)$.

Proof. Obviously, the varieties listed in the assertion are subvarieties of $V^*(x^7 \approx x)$ and the trivial variety TR is different to $V(u \approx v)$ for all identities $u \approx v$ listed in the assertion. Moreover, by straightforward calculations one can show that for any different identities $u \approx v$ and $s \approx t$, listed in the assertion, there is a $k \in \{2, 3, 4, 7\}$ such that $GF(k) \in V(u \approx v)$ and $GF(k) \notin V(s \approx t)$ or vice versa, where $GF(k)$ denotes the k-element field. Furthermore, it is easy to check that $V^*(x^2 \approx x) = V(x^2 \approx x)$, $V^*(x^4 \approx x) = V(x^4 \approx x)$ and $V^*(x^7 \approx x) = V(x^7 \approx x)$. Hence the eleven varieties defined by the identities listed in the assertion are pairwise distinct. Consequently, they are the non-trivial subvarieties of $V^*(x^7 \approx x)$. In particular, the following 20 inclusions are easy to verify:

$TR \subseteq V(7x \approx 0)$

$V^*(x^2 \approx x) \subseteq V(7y(x^2 - x) \approx 0)$

$V^*(x^4 \approx x) \subseteq V(7y(x^4 - x) \approx 0)$

$V(3x \approx 0) \subseteq V(21x \approx 0)$

$V(3y(x^2 - x) \approx 0) \subseteq V(21y(x^2 - x) \approx 0)$

$V(3y(x^4 - x) \approx 0) \subseteq V^*(x^7 \approx x)$

$TR \subseteq V^*(x^2 \approx x) \subseteq V^*(x^4 \approx x)$

$V(3x \approx 0) \subseteq V(3y(x^2 - x) \approx 0) \subseteq V(3y(x^4 - x) \approx 0)$

$V(7x \approx 0) \subseteq V(7y(x^2 - x) \approx 0) \subseteq V(7y(x^4 - x) \approx 0)$

$V(21x \approx 0) \subseteq V(21y(x^2 - x) \approx 0) \subseteq V^*(x^7 \approx x)$

$TR \subseteq V(3x \approx 0)$

$V^*(x^2 \approx x) \subseteq V(3y(x^2 - x) \approx 0)$

$V^*(x^4 \approx x) \subseteq V(3y(x^4 - x) \approx 0)$

$V(7x \approx 0) \subseteq V(21x \approx 0)$

$V(7y(x^2 - x) \approx 0) \subseteq V(21y(x^2 - x) \approx 0)$

$V(7y(x^4 - x) \approx 0) \subseteq V^*(x^7 \approx x)$.

These inclusions establish the given correspondence between the varieties $V_2, \ldots, V_{12}$ and the varieties defined by the identities listed in the assertion. □

Clearly, $P(TR) = Hyp(\tau)$. But the determination of all proper hypersubstitutions with respect to any non-trivial variety of rings satisfying $x^7 \approx x$ requires a lengthy calculation. We will present such a calculation as we determine the set $P(V^*(x^2 \approx x))$ of all proper hypersubstitutions with respect to the variety $V^*(x^2 \approx x)$ of rings generated by the two-element field. It will happen that $V^*(x^2 \approx x)$ is unsolid.

Proposition 3.2. *$V^*(x^2 \approx x)$ is unsolid.*

Proof. Since $P_0(V^*(x^2 \approx x)) \subseteq P(V^*(x^2 \approx x))$ (see [10]), we have to show the converse inclusion. Let $\sigma \in P(V^*(x^2 \approx x))$. We have to show that $\sigma(+) \approx x + y$, $\sigma(\cdot) \approx x \cdot y$, and $\sigma(-) \approx -x$ are identities in $V^*(x^2 \approx x)$. Since σ is a proper hypersubstitution with respect to $V^*(x^2 \approx x)$, the application of σ to the identities $x + x \approx 0$, $x + 0 \approx x$, $0 + x \approx x$, $x \cdot x \approx x$, $x \cdot 0 \approx 0$, $0 \cdot x \approx 0$ and $x + (-x) \approx 0$, satisfied in $V^*(x^2 \approx x)$, all result in identities in $V^*(x^2 \approx x)$ (note that $x^2 \approx x$ implies $4x^2 \approx 2x$, $4x \approx 2x$, and thus $2x = x + x = 0$). In this way we obtain the following identities in $V^*(x^2 \approx x)$:

$\widehat{\sigma}[x + x] \approx \widehat{\sigma}[0]$ gives

$$\sigma(+)(x, x) \approx 0 \quad (1);$$

$\widehat{\sigma}[x + 0] \approx \widehat{\sigma}[x]$ gives

$$\sigma(+)(x, 0) \approx x \quad (2);$$

$\widehat{\sigma}[0 + x] \approx \widehat{\sigma}[x]$ gives

$$\sigma(+)(0, x) \approx x \quad (3);$$

$\widehat{\sigma}[x \cdot x] \approx \widehat{\sigma}[x]$ gives

$$\sigma(\cdot)(x, x) \approx x \quad (4);$$

$\widehat{\sigma}[x \cdot 0] \approx \widehat{\sigma}[0]$ gives

$$\sigma(\cdot)(x, 0) \approx 0 \quad (5);$$

$\widehat{\sigma}[0 \cdot x] \approx \widehat{\sigma}[0]$ gives

$$\sigma(\cdot)(0, x) \approx 0 \quad (6);$$

$\widehat{\sigma}[x + (-x)] \approx \widehat{\sigma}[0]$ gives

$$\sigma(+)(x, \sigma(-)) \approx 0 \quad (7).$$

Let us now consider the terms $\sigma(+)$, $\sigma(\cdot)$ and $\sigma(-)$. First, we deal with the term $\sigma(+)$. Clearly, there are $a, b, c \in \{0, 1\}$ such that $\sigma(+) \approx ax + by + c(x \cdot y) \in V^*(x^2 \approx x)$. Assume that $a + b + c \equiv 1$ *mod* 2. Then we replace the variable y by x in $(\sigma(+) = \widehat{\sigma}[x+y] =)\ \sigma(+)(x, y) \approx ax + by + c(x \cdot y)$ and obtain $\sigma(+)(x, x) \approx x$ since $2x \approx x \approx x^2 \in V^*(x^2 \approx x)$. This contradicts (1). Hence $a + b + c \equiv 0$ *mod* 2. Assume that $c = 1$. Then $a = 0$ and $b = 1$ or vice versa. If $a = 0$ and $b = 1$ then we replace y by 0 in $\sigma(+)(x, y) \approx 0x + y + 0(x \cdot y)$ obtaining $\sigma(+)(x, 0) \approx 0$. This contradicts (2). If $a = 1$ and $b = 0$ then we replace x by 0 in $\sigma(+)(x, y) \approx x + 0y + 0(x \cdot y)$ obtaining $\sigma(+)(0, y) \approx 0$. This contradicts (3). Hence $c = 0$ and thus $a = b = 1$. This shows that $\sigma(+) \approx x + y \in V^*(x^2 \approx x)$. Now we consider the term $\sigma(\cdot)$. Clearly, there are $a, b, c \in \{0, 1\}$ such that $\sigma(\cdot) \approx ax + by + c(x \cdot y) \in V^*(x^2 \approx x)$. Assume that $a + b + c \equiv 0$ *mod* 2. Then we replace the variable y by x in $\sigma(\cdot)(x, y) \approx ax + by + c(x \cdot y)$ and obtain $\sigma(\cdot)(x, x) \approx 0$ since $2x \approx x \approx x^2 \in V^*(x^2 \approx x)$. This contradicts (4). Hence $a + b + c \equiv 1$ *mod* 2. Assume that $a = 1$. Then we replace y by 0 in $\sigma(\cdot)(x, y) \approx x + by + c(x \cdot y)$ and obtain $\sigma(\cdot)(x, 0) \approx x$, a contradiction to (5). Hence $a = 0$. Similarly, we obtain $b = 0$. Hence $c = 1$, i.e. $\sigma(\cdot)(x, y) \approx x \cdot y \in$

$V^*(x^2 \approx x)$. Finally, there is an $a \in \{0,1\}$ such that $\sigma(-) \approx ax \in V^*(x^2 \approx x)$. Assume that $a = 0$. Then we have $\sigma(-)(x) \approx \widehat{\sigma}[-x] \approx \sigma(-) \approx 0$. This implies $\sigma(+)(x, \sigma(-)) \approx \sigma(+)(x, 0)$, throughout $\sigma(+)(x, 0) \approx 0$ by (2), contradicting (7). Hence $a = 1$, i.e. $\sigma(-) \approx x$. Altogether, this shows that $\sigma \in P_0(V^*(x^2 \approx x))$. Consequently, $P(V^*(x^2 \approx x)) = P_0(V^*(x^2 \approx x))$. □

On the other hand, one obtains the following result by some non-trivial calculations.

Remark 3.3. Each non-trivial subvariety of $V^*(x^7 \approx x)$ different from $V^*(x^2 \approx x)$ is not unsolid.

In the variety $V^*(x^2 \approx x)$, a binary term has a normal form $ax + by + c(x \cdot y)$ for some $a, b, c \in \{0,1\}$. That means that for each term $t \in W_\tau(X_2)$ there are $a, b, c \in \{0,1\}$ such that $t \approx ax + by + c(x \cdot y) \in IdV^*(x^2 \approx x)$. In the following, we give an algorithm to determine $a, b, c \in \{0,1\}$ with $t \approx ax + by + c(x \cdot y) \in IdV^*(x^2 \approx x)$ for a given term $t \in W_\tau(X_2)$. For this we define a mapping

$$s : W_\tau(X_2) \to \{0,1\}^3$$

by

(i) $s(0) := (0,0,0)$

(ii) $s(x) := (1,0,0)$

(iii) $s(y) := (0,1,0)$

(iv) If $u, v \in W_\tau(X_2)$ with $s(u) = (a_1, a_2, a_3)$ and $s(v) = (b_1, b_2, b_3)$ then

(iv$_1$) $s(u + v) = (c_1, c_2, c_3)$ with $c_i \equiv a_i + b_i \; mod \; 2$ for $i = 1, 2, 3$

(iv$_2$) $s(u \cdot v) = (c_1, c_2, c_3)$ with $c_i \equiv a_i b_i \; mod \; 2$ for $i = 1, 2$ and
$c_3 \equiv a_1(b_2 + b_3) + a_2(b_1 + b_3) + a_3(b_1 + b_2 + b_3) \; mod \; 2$

(iv$_3$) $s(-u) = s(u)$.

This inductive definition of the function s provides an algorithm to calculate the triple $s(t)$ for each binary term $t \in W_\tau(X_2)$. In particular, we have $s(ax + by + c(x \cdot y)) = (a, b, c)$ for $a, b, c \in \{0,1\}$. Using the following lemma, we can decide to which of the eight normal forms $ax + by + c(x \cdot y)$ $(a, b, c \in \{0,1\})$ a given term is equivalent.

Lemma 3.4. *Let $t \in W_\tau(X_2)$ with $s(t) = (a, b, c)$. Then $t \approx ax + by + c(x \cdot y) \in IdV^*(x^2 \approx x)$.*

Proof. We give a proof by induction on the definition of s. If $t \in \{0, x, y\}$ then the claim holds. Let $u, v \in W_\tau(X_2)$ with $s(u) = (a_1, a_2, a_3)$ and $s(v) = (b_1, b_2, b_3)$ and suppose that $u \approx a_1 x + a_2 y + a_3(x \cdot y) \in IdV^*(x^2 \approx x)$ and $v \approx b_1 x + b_2 y + b_3(x \cdot y) \in IdV^*(x^2 \approx x)$. Note that $x \approx -x \in IdV^*(x^2 \approx x)$. So we have $s(-u) = s(u) = (a_1 x + a_2 y + a_3(x \cdot y))$ and $-u \approx u \approx a_1 x + a_2 y + a_3(x \cdot y) \in$

$V^*(x^2 \approx x)$. Further we have $u+v \approx a_1x+a_2y+a_3(x\cdot y)+b_1x+b_2y+b_3(x\cdot y) \approx (a_1+b_1)x+(a_2+b_2)y+(a_3+b_3)(x\cdot y) \approx c_1x+c_2y+c_3(x\cdot y) \in IdV^*(x^2 \approx x)$ where c_1, c_2, $c_3 \in \{0,1\}$ and $c_i \equiv a_i+b_i \ mod\ 2$ for $i=1,2,3$. On the other hand we have $s(u+v)=(c_1,c_2,c_3)$ since $c_i \equiv a_i+b_i \ mod\ 2$ for $i=1,2,3$. Finally, we have $u\cdot v \approx (a_1x+a_2y+a_3(x\cdot y))\cdot(b_1x+b_2y+b_3(x\cdot y)) \approx a_1b_1x+a_2b_2y+[a_1(b_2+b_3)+a_2(b_1+b_3)+a_3(b_1+b_2+b_3)(x\cdot y)] \approx d_1x+d_2y+d_3(x\cdot y) \in IdV^*(x^2 \approx x)$ with d_1, d_2, $d_3 \in \{0,1\}$ and $d_i \equiv a_ib_i \ mod\ 2$ for $i=1,2$ and $d_3 \equiv a_1(b_2+b_3)+a_2(b_1+b_3)+a_3(b_1+b_2+b_3) \ mod\ 2$. On the other hand, it is obvious that $s(u\cdot v)=(d_1,d_2,d_3)$ holds. □

This lemma can be used to decide if a hypersubstitution σ is a proper one. Since $V^*(x^2 \approx x)$ is unsolid by Proposition 3.2, σ is proper with respect to $V^*(x^2 \approx x)$ if and only if $\sigma(+) \approx x+y$, $\sigma(\cdot) \approx x\cdot y$, and $\sigma(-) \approx x$ are identities in $V^*(x^2 \approx x)$. (Clearly, $\sigma(0) \approx 0 \in V^*(x^2 \approx x)$.) By Lemma 3.4, we have to check that $s(\sigma(+))=(1,1,0)$, $s(\sigma(\cdot))=(0,0,1)$, and $s(\sigma(-))=(1,0,0)$.

Example 3.5.

- The hypersubstitution $\sigma_1 \in Hyp(\tau)$ with $\sigma_1(+)=x\cdot(y\cdot(x\cdot(x+y))+x)$, $\sigma_1(\cdot)=x\cdot y$, $\sigma_1(-)=x$, and $\sigma_1(0)=0+0$ is not proper with respect to $V^*(x^2 \approx x)$ since $s(\sigma_1(+))=(1,0,1)$.
- The hypersubstitution $\sigma_2 \in Hyp(\tau)$ with $\sigma_2(+)=y\cdot(x\cdot(x+y)+x)$, $\sigma_2(\cdot)=x\cdot y$, $\sigma_2(-)=x$ and $\sigma_2(0)=0+0$ is proper with respect to $V^*(x^2 \approx x)$ since $s(\sigma_2(+))=(1,1,0)$, $s(\sigma_2(\cdot))=(0,0,1)$, and $s(\sigma_2(-))=(1,0,0)$.

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