Novi Sad J. Math.
Vol. 40, No. 3, 2010, 11–25
Proc. 3rd Novi Sad Algebraic Conf.
(eds. I. Dolinka, P. Marković)

ON THE COSET LAWS FOR SKEW LATTICES IN RINGS[1]

Karin Cvetko-Vah[2], João Pita Costa[3]

Abstract

Skew lattices are the most successful generalization of lattices to the noncommutative case to date. Roughly speaking, each skew lattice can be seen as a lattice of rectangular bands. A coset decomposition can be given to each pair of comparable maximal rectangular bands. The internal structure of skew lattices is revealed by their coset structure. In the present paper we study the coset structure of skew lattices in rings and present certain coset laws that describe the connections among the coset decompositions given by distinct pairs of maximal rectangular bands.

AMS Mathematics Subject Classification (2010): 06F99, 15B99

Key words and phrases: skew lattices in rings, coset structure, coset laws, index of a coset

Introduction

The study of noncommutative lattices began in 1949 with Pascual Jordan's paper [7] that was motivated by certain questions in quantum mechanics. Skew lattices turned out to be the most fruitful class of noncommutative lattices and were thus studied the most. The foundations of the modern theory of skew lattices can be found in Jonathan Leech's 1989 paper [9]. Special attention was devoted to the skew lattices in rings. The operations defined by $x \wedge y = xy$ and $x \vee y = x + y - xy$ succeeded to provide a rather large class of examples which have motivated many of the properties studied in the general case. A good survey on skew lattices can be found in Leech [11].

In the study of skew lattice there are two perspectives that complement each other. One perspective considers skew lattices as non-commutative lattices. The other perspective sees the skew lattices as double bands. Therefore we introduce the natural partial order and skew lattice varieties as well as Green's relations and other semigroup theoretical notions. Given a pair of comparable $\mathcal{D}$-classes

[1]The second author thanks the support of Fundação para a Ciência e Tecnologia with the reference SFRH / BD / 36694 / 2007.

[2]University of Ljubljana, Faculty of Mathematics and Physics, Jadranska 19, 1000 Ljubljana, Slovenia, e-mail: *karin.cvetko@fmf.uni-lj.si*

[3]University of Ljubljana, Faculty of Mathematics and Physics, Jadranska 19, 1000 Ljubljana, Slovenia, e-mail: *joaopitacosta@gmail.com*

each of the two classes induces a partition of the other class, and the blocks of these partitions are called *cosets*. (See Section 1 for precise definitions.) The coset structure reveals a new perspective that does not have a counterpart either in the theory of lattices or in the theory of bands. The study of the coset structure of skew lattices began with Leech [11] in 1996. It gives an introspective into *how* the $\mathcal{D}$-classes are glued into a lattice thus providing important additional information. In the present paper we present certain coset laws that reveal the relation among the coset decompositions given by distinct pairs of $\mathcal{D}$-classes.

In Section 2 we consider the coset structure of skew lattices in rings. In order to obtain a larger class of examples we shall work with a more general version of the join operation, namely the *cubic join*, defined in Section 1. We will be able to prove certain coset laws joining the partitions that different $\mathcal{D}$-classes induce on each other. In Section 3 we demonstrate the results of Section 2 in the case of skew lattices in rings of matrices, while in Section 4 we derive a combinatorial result for finite skew lattices in rings, namely given non comparable $\mathcal{D}$-classes A and B with the join class J and the meet class M we shall obtain $|A||B| = |M||J|$.

1 Preliminaries

A *skew lattice* $\mathbf{S}$ is a nonempty set S equipped with two associative binary operations $\wedge$ and $\vee$, called the *meet* and the *join*, that satisfy the absorption laws $(b \wedge a) \vee a = a = a \vee (a \wedge b)$ and their duals. Both operations are idempotent by the usual argument: $a \wedge a = a \wedge (a \vee (a \wedge b)) = a$.

Given nonempty sets L and R their direct product $L \times R$ is a skew lattice with the operations defined by $(x, y) \vee (x', y') = (x', y)$ and $(x, y) \wedge (x', y') = (x, y')$. A *rectangular skew lattice* is an isomorphic copy of such a skew lattice.

Skew lattices can be viewed as non-commutative generalizations of lattices or as *double bands*, where a *band* is a semigroup of idempotents. Recall that in a band $(B, \cdot)$, the Green's equivalence relations can be defined by:

$$\begin{array}{lcl} x\mathcal{R}y & \Leftrightarrow & (xy = y \ \& \ yx = x), \\ x\mathcal{L}y & \Leftrightarrow & (xy = x \ \& \ yx = y), \\ x\mathcal{H}y & \Leftrightarrow & (x\mathcal{R}y \ \& \ x\mathcal{L}y), \\ \mathcal{D} & = & \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}. \end{array}$$

Furthermore, a band $(B, \cdot)$ is called *regular* if it satisfies the identity $axaya = axya$.

Given a skew lattice $\mathbf{S}$ and $\mathcal{U} \in \{\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}\}$ denote by $\mathcal{U}_\vee$ the Green's relation corresponding to the band $(S, \vee)$ and by $\mathcal{U}_\wedge$ the relation corresponding to the band $(S, \wedge)$. Furthermore, the abbreviated notation $\mathcal{U}$ is used strictly for $\mathcal{U}_\wedge$ and the $\mathcal{U}$-class containing an element $x \in S$ is denoted by $\mathcal{U}_x$.

Leech's First Decomposition Theorem [9] states that in a skew lattice $\mathbf{S}$ relations $\mathcal{D}_\vee$ and $\mathcal{D}_\wedge$ coincide, $\mathcal{D}$ is a congruence, the $\mathcal{D}$-classes are exactly the maximal rectangular subalgebras of $\mathbf{S}$ and $\mathbf{S}/\mathcal{D}$ is a lattice.

The natural partial order $\geq$ is defined on a skew lattice $\mathbf{S}$ by $x \geq y$ if and only if $x \wedge y = y = y \wedge x$ or dually, $x \vee y = x = y \vee x$. The natural equivalence is compatible with the natural partial order. Relation $\mathcal{D}$ is also called the *natural equivalence* (Note that Leech's First Decomposition Theorem is a generalization of the Clifford-McLean Theorem for bands.)

A skew lattice is *right handed* if it satisfies the identity $x \wedge y \wedge x = y \wedge x$ and its dual $x \vee y \vee x = x \vee y$. These identities essentially assert that $x\mathcal{D}y$ implies $x \wedge y = y$ and $x \vee y = x$. *Left handed* skew lattices are defined by the opposite identities. Leech's Second Decomposition Theorem [9] yields that every skew lattice is *biregular*, that is both $(S, \vee)$ and $(S, \wedge)$ are regular bands, relations $\mathcal{R} = \mathcal{R}_\wedge = \mathcal{L}_\vee$ and $\mathcal{L} = \mathcal{L}_\wedge = \mathcal{R}_\vee$ are both congruencies and $\mathbf{S}$ factors as the fiber product (pull-back) of the right handed skew lattice $\mathbf{S}/\mathcal{L}$ by the left handed skew lattice $\mathbf{S}/\mathcal{R}$ over their common maximal lattice image $\mathbf{S}/\mathcal{D}$. (This is a skew lattice version of the Kimura Factorization Theorem [8] for regular bands.)

In skew lattices the following dualities hold: $x \vee y = x$ iff $x \wedge y = y$, and $x \vee y = y$ iff $x \wedge y = x$. A skew lattice $\mathbf{S}$ is *symmetric* if given x and y in S, $x \wedge y = y \wedge x$ if and only if $x \vee y = y \vee x$. On the other hand, a skew lattice $\mathbf{S}$ is *cancellative* if for all $x, y, z \in S$, $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ imply $y = z$, and $x \vee y = z \vee y$ and $x \wedge y = z \wedge y$ imply $x = z$. Cancellative skew lattices were introduced in [9] and extensively studied in [4].

Let $\mathbf{R} = (R, +, \cdot)$ be a ring and $E(R)$ the set of all idempotent elements in $\mathbf{R}$. Set $x \wedge y = xy$ and $x \vee y = x \circ y = x + y - xy$. If $S \subseteq E(R)$ is closed under both $\vee$ and $\wedge$ then $(S; \wedge, \vee)$ is a skew lattice. Maximal regular multiplicative bands need not to be closed under the circle operation. Indeed, examples are easily found within rectangular bands. Although, such examples are closed under the following cubic variant ∇ of $\vee$ defined by

$$x \nabla y = x + y + yx - xyx - yxy,$$

since in the rectangular case $x \nabla y$ reduces to yx and one obtains a *rectangular* skew lattice. By a *skew lattice in a ring* $\mathbf{R}$ we mean a set $S \subseteq E(\mathbf{R})$ that is closed under both multiplication and ∇, and forms a skew lattice for the two operations. (In particular, we have to make sure that ∇ is associative on $\mathbf{S}$.) Given a multiplicative band $\mathbf{B}$ in a ring $\mathbf{R}$ the relation between $\circ$ and ∇ is given by $e \nabla f = (e \circ f)^2$ for all $e, f \in B$. In the case of right [left] handed skew lattices the nabla operation reduces to the circle operation. Skew lattices in rings are always symmetric and cancellative. (In general, symmetry is implied by cancellation, see [4].) In the remainder of the paper we shall assume that $\mathbf{R}$ is a fixed ring and $\mathbf{S}$ is a skew lattice in $\mathbf{R}$.

Lemma 1 ([1])**.** *Let $\mathbf{S}$ be a skew lattice and $a_1, a_2, u, v \in S$ such that $\mathcal{D}_u \leq \mathcal{D}_{a_i} \leq \mathcal{D}_v$ in the lattice $\mathbf{S}/\mathcal{D}$. Then $a_1 v a_2 = a_1 a_2$ and $a_1 \nabla u \nabla a_2 = a_1 \nabla a_2$.*

Proof. We obtain

$$a_1 v a_2 = (a_1 v a_1) v (a_2 v a_2) = a_1 (v a_1 v a_2 v) a_2 = (a_1 v a_1)(a_2 v a_2) = a_1 a_2$$

by regularity; $a_1 \nabla u \nabla a_2 = a_1 \nabla a_2$ follows from the regularity of ∇. □

Lemma 1 implies that if $\mathbf{S}$ is a skew lattice consisting of exactly two $\mathcal{D}$-classes $A > B$ then $BaB = B$ and $A \nabla b \nabla A = A$, for all $a \in A$ and $b \in B$. On the other hand, $B \nabla a \nabla B$ and AbA are in general non-trivial and $B \nabla a \nabla B = \{\, b \nabla a \nabla b : b \in B \,\}$ and $AbA = \{\, aba : a \in A \,\}$, for all $a \in A$, $b \in B$. (Note that by Lemma 1 given $b', b'' \in B$, $u = b' \nabla a \nabla b''$ and $b \in B$ such that $b < u$ we obtain $u = b \nabla u \nabla b = b \nabla b' \nabla a \nabla b'' \nabla b = b \nabla a \nabla b$.)

To describe the coset structure, we need to focus our attention on *primitive* skew lattices, ie. skew lattices with exactly two $\mathcal{D}$-classes. Let $\mathbf{S}$ be a primitive skew lattice with $\mathcal{D}$-classes $A > B$. Given $b \in B$ the subset $AbA = \{\, aba : a \in A \,\}$ is called a *coset of* A *in* B. Similarly, a coset of B in A is any subset of A of the form $B \nabla a \nabla B = \{\, b \nabla a \nabla b : b \in B \,\}$, for $a \in A$.

For $a \in A$ the set

$$aBa = \{\, aba : b \in B \,\} = \{\, b \in B : b \leq a \,\}$$

is the *image of* a *in* B. Dually, for $b \in B$ the set $b \nabla A \nabla b = \{\, a \in A : b \leq a \,\}$ is the *image* of b in A. Leech's Theorem [11] yields that the $\mathcal{D}$-class B is partitioned by the cosets of A and the image set in B of any element $a \in A$ is a transversal of cosets of A in B; furthermore, given cosets A_i in A and B_j in B there is a natural bijection of cosets $\phi_{ji} : A_i \to B_j$, called the *coset bijection*, such that $\phi_{ji}(x) = y$ iff $x \geq y$. Moreover, both operations $\cdot$ and ∇ are determined by the coset bijections. If $\mathbf{S}$ is right handed then given $x \in A_i$ and $y \in B_j$ we obtain

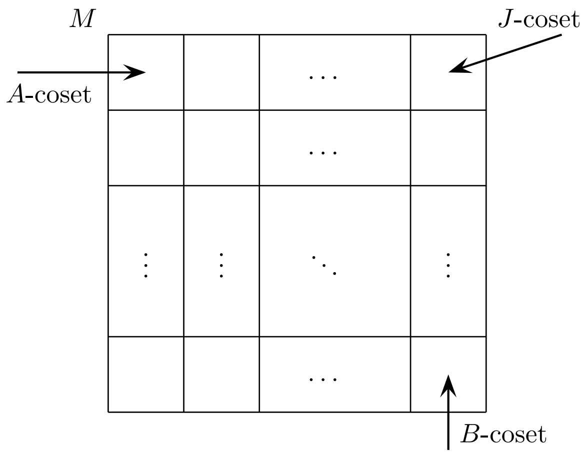
$$\phi_{ji}(x) = yx \text{ and } \phi_{ji}^{-1}(y) = y \nabla x.$$

Let A and B be incomparable $\mathcal{D}$-classes in $\mathbf{S}$, $J = A \nabla B$ and $M = AB$. By a *skew diamond* $\{\, J > A, B > M \,\}$ we refer to the sub-skew lattice in $\mathbf{S}$ with the universe $M \cup A \cup B \cup J$.

Theorem 2 (Leech, [10])**.** *Let* $\mathbf{S}$ *be a skew lattice and* $\{\, J > A, B > M \,\}$ *a skew diamond in* $\mathbf{S}$*. Given* $j \in J$ *there exist* $a \in A$ *and* $b \in B$ *such that* $a \vee b = j = b \vee a$*. Dually, given* $m \in M$ *there exist* $a \in A$ *and* $b \in B$ *such that* $ab = m = ba$*.*

The double partition of either J or M by A-cosets and B-cosets can be refined by the coset partition which J and M directly induce on each other. The double partition coincides with the partition by J-M cosets exactly when the skew lattice is symmetric.

Theorem 3 (Leech, [10])**.** *A skew lattice* $\mathbf{S}$ *is symmetric if and only if for any skew diamond* $\{\, J > A, B > M \,\}$ *in* $\mathbf{S}$ *the partition of* J *by intersections of the* A*-cosets with the* B*-cosets equals the partition of* J *by the* M*-cosets and a dual assertion holds for the meet class* M*.*



We finish this section with a couple of examples of skew lattices in rings.

Example. Let F be a field, $n \in \mathbb{N}$ and $S \subseteq M_n(F)$ the set of all matrices of the form

$$\begin{bmatrix} 0_k & S & ST \\ 0 & I_l & T \\ 0 & 0 & 0_m \end{bmatrix},$$

where I_l is an $l \times l$ identity matrix for some $l \in \{0, 1, \ldots, n\}$, 0_k and 0_m are the $k \times k$ and $m \times m$ zero matrices, respectively, while S and T are arbitrary matrices over F of the suiting dimensions. Then $\mathbf{S} = (S, \cdot, \circ)$ is a multiplicative band that is closed under the circle operation and hence forms a skew lattice. Furthermore, $\mathbf{S}$ is right-handed, hence $\circ$ coincides with ∇ on $\mathbf{S}$.

Example. Let F be a field and consider the matrix ring $M_4(F)$. Denote

$$A = \left\{ a = \begin{bmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;:\; a_{ij} \in F \right\},$$

$$B = \left\{ b = \begin{bmatrix} 1 & b_{12} & 0 & b_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \;:\; b_{ij} \in F \right\},$$

$$M = \left\{ m = \begin{bmatrix} 1 & m_{12} & m_{13} & m_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;:\; m_{ij} \in F \right\}$$

and

$$J = \left\{ j = \begin{bmatrix} 1 & 0 & 0 & j_{14} \\ 0 & 1 & 0 & j_{24} \\ 0 & 0 & 1 & j_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} : j_{ij} \in F \right\}.$$

It is an easy exercise to check that $S = A \cup B \cup M \cup J$ is closed under both multiplication and the circle operation. It forms a right-handed skew lattice, so again $\circ = \nabla$. Furthermore, $\{ J > A, B > M \}$ is a skew diamond.

2 The coset laws

It was shown in [2] that cancellative skew lattices satisfy two specific coset laws. In this section we are going to present these and other coset laws in the context of skew lattices in rings where the idea first came up.

We begin with a technical result that shall be useful in the continuation. Recall that given $\mathcal{D}$-classes $X > Y$, an X-coset Y_i in Y and a Y-coset X_j in X there exist $x \in X_j$ and $y \in Y_i$ such that $x > y$.

Lemma 4. *Let* $\mathbf{S}$ *be a skew lattice and* $\{ J > A, B > M \}$ *a skew diamond in* $\mathbf{S}$. *If* $m \in M$ *and* $j \in J$ *are such that* $m < j$ *then there exist* $a \in A$ *and* $b \in B$ *such that* $m < a, b < j$, $ab = ba = m$ *and* $a\nabla b = j = b\nabla a$.

Proof. Take any $a' < j$ in A, $b' < j$ in B and set $a = m\nabla a'\nabla m$, $b = m\nabla b'\nabla m$. Observe that $m < a, b < j$. Furthermore, $ab = abmab = m = bamba = ba$ and similarly $a\nabla b = j = b\nabla a$. □

Let S be a skew lattice with two comparable $\mathcal{D}$-classes $X > Y$. For all $x \in X$ and $y \in Y$, $XyX = \{ wyw : w \in X \}$ is the coset of X in Y and, regardless of associativity, $Y\nabla x\nabla Y = \{ y + x - xyx : y \in Y \}$ is the coset of Y in X for the nabla operation [1]. As skew lattices in rings are always symmetric the Theorem 3 characterization can be rephrased as:

(i) $JmJ = (AmA) \bigcap (BmB)$, for all $m \in M$;

(ii) $M\nabla j\nabla M = (A\nabla j\nabla A) \bigcap (B\nabla j\nabla B)$, for all $j \in J$.

Theorem 3 yields the following coset laws:

Theorem 5. *Let* $\mathbf{S}$ *be a skew lattice and* $\{ J > A, B > M \}$ *a skew diamond in* $\mathbf{S}$. *Given* $m, m' \in M$ *and* $j, j' \in J$ *the following identities hold on* $\mathbf{S}$*:*

(i) $JmJ = Jm'J$ *if and only if* $AmA = Am'A$ *and* $BmB = Bm'B$*;*

(ii) $M\nabla j\nabla M = M\nabla j'\nabla M$ *if and only if* $A\nabla j\nabla A = A\nabla j'\nabla A$ *and* $B\nabla j\nabla B = B\nabla j'\nabla B$.

The following result was first observed in [3]. We restate the proof for the sake of completeness.

Lemma 6. *Let $X > Y$ be $\mathcal{D}$-classes in* $\mathbf{S}$ *and let Y_j be an X-coset in Y and X_i a Y-coset in X. If elements $y_1, y_2 \in Y_j$, $x_1, x_2 \in X$ are such that $y_1 \leq x_1$ and $y_2 \leq x_2$ hold then $y_1 - x_1 = y_2 - x_2$.*

Proof. Consider the coset bijection $\phi_{ji} : X_i \to Y_j$. Since $\phi_{ji}(x_1) = y_1$ and $\phi_{ji}(x_2) = y_2$ one obtains $x_2 = y_2 \nabla x_1 \nabla y_2 = y_2 + x_1 - x_1 y_2 x_1 = y_2 + x_1 - y_1$. □

Lemma 6 shows us that given any pair of cosets $Y_j \subset Y$ and $X_i \subset X$ a constant $c(Y_j, X_i) \in R$ exists such that $\phi_{ji}(x) = x + c(Y_j, X_i)$ for all $x \in X_i$. Such constants $c(Y_j, X_i)$ are called the *coset constants*. In a sense the coset constants "code" the respective coset bijections.

Lemma 7. *Let* $\mathbf{S}$ *be a skew lattice and $\{ J > A, B > M \}$ a skew diamond in* $\mathbf{S}$. *Given any $m \in M$ and $j \in J$ there exists $b \in B$ such that*

$$AmA = AbA \text{ and } A \nabla j \nabla A = A \nabla b \nabla A.$$

Proof. Let $m' \in JmJ$ be such that $m' < j$. By Lemma 4 there exist $a \in A$ and $b \in B$ such that $m' = ab$ and $j = a \nabla b$. Hence, $AmA = Am'A = AabA = AbA$ and $A \nabla j \nabla A = A \nabla a \nabla b \nabla A = A \nabla b \nabla A$. □

Proposition 8. *Given a skew diamond $\{ J > A, B > M \}$ in* $\mathbf{S}$, *$x \in A$ and $y \in B$,*

$$yxy = y + c(BxB, M \nabla y \nabla M), \tag{1}$$

$$x \nabla y \nabla x = x - c(JxJ, A \nabla y \nabla A). \tag{2}$$

Proof. Let $x \in A$ and $y \in B$. One obtains $yxy \leq y$ and $x \leq x \nabla y \nabla x$. From here the assertions follow. □

Corollary 9. *Given a skew diamond $\{ J > A, B > M \}$ in* $\mathbf{S}$, *$x \in A$ and $y \in B$,*

$$c(BxB, M \nabla y \nabla M) = c(JxJ, A \nabla y \nabla A), \tag{3}$$

$$c(JyJ, B \nabla x \nabla B) = c(AyA, M \nabla x \nabla M). \tag{4}$$

Proof. Let $x \in A$ and $y \in B$. Proposition 8 yields $yxy = y + c(BxB, M \nabla y \nabla M)$ and $x \nabla y \nabla x = x - c(JxJ, A \nabla y \nabla A)$. On the other hand,

$$x \nabla y \nabla x = x + y - yxy = x - c(BxB, M \nabla y \nabla M)$$

and (3) follows. (4) is proved by an analogous argument. □

Now that we have developed the necessary tools we are ready to state another set of coset laws for skew lattices in rings.

Theorem 10. *Let* $\mathbf{S}$ *be a skew lattice in a ring. The following coset equivalences hold:*

(i) *For all skew diamonds* $\{ J > A, B > M \}$ *in* $\mathbf{S}$ *and* $x, x' \in A$, $M \nabla x \nabla M = M \nabla x' \nabla M$ *if and only if* $B \nabla x \nabla B = B \nabla x' \nabla B$.

(ii) *For all skew diamonds* $\{ J > A, B > M \}$ *in* $\mathbf{S}$ *and* $x, x' \in A$, $BxB = Bx'B$ *if and only if* $JxJ = Jx'J$.

Proof. Suppose that $M \nabla x \nabla M = M \nabla x' \nabla M$. Then given $b \in B$,

$$\begin{aligned} b \nabla x \nabla b &= b - c(JbJ, B \nabla x \nabla B) \\ &= b - c(AbA, M \nabla x \nabla M) \\ &= b - c(AbA, M \nabla x' \nabla M) \\ &= b - c(JbJ, B \nabla x' \nabla B) \\ &= b \nabla x' \nabla b \end{aligned}$$

and $B \nabla x \nabla B = B \nabla x' \nabla B$ follows.

On the other hand, assume that $B \nabla x \nabla B = B \nabla x' \nabla B$ and let $m \in M$. Hence, $m \nabla x \nabla m = m - c(AmA, M \nabla x \nabla M)$. As $AmA = AabA = AbA$ for some $a \in A$ and $b \in B$ it follows that

$$\begin{aligned} m - c(AmA, M \nabla x \nabla M) &= m - c(AbA, M \nabla x \nabla M) \\ &= m - c(JbJ, B \nabla x \nabla B) \\ &= m - c(JbJ, B \nabla x' \nabla B) \\ &= m - c(AbA, M \nabla x' \nabla M) \\ &= m - c(AmA, M \nabla x' \nabla M) \\ &= m \nabla x' \nabla m. \end{aligned}$$

Hence, $M \nabla x \nabla M = M \nabla x' \nabla M$. The case (ii) is similar. □

Remark 11. Let $\mathbf{S}$ be a skew lattice in a ring $\mathbf{R}$ such that $\mathbf{S}/\mathcal{D}$ is countable, and let $\mathcal{C}$ be a given maximal chain in the distributive lattice $\mathbf{S}/\mathcal{D}$. Denote $\mathcal{C}_0 = \mathcal{C}$. For $i \geq 1$ let $\mathcal{C}_i$ denote the lattice that is a union of $\mathcal{C}_{i-1}$ and a given $\mathcal{D}$-class D_i that is not contained in $\mathcal{C}_{i-1}$ and is such that there exists $\mathcal{D}$-classes M_i, A_i and J_i in $\mathcal{C}_{i-1}$, both M_i and J_i adjacent to both A_i and D_i in $\mathbf{S}/\mathcal{D}$, and $\{ J_i > A_i, D_i > M_i \}$ being a skew diamond. Since $\mathbf{S}/\mathcal{D}$ is a countable distributive lattice it follows that *any* $\mathcal{D}$-class D can be accessed from any maximal chain $\mathcal{C}$ in finitely many steps of the kind just described. Theorem 10 now implies that the coset decompositions of pairs of adjacent $\mathcal{D}$-classes along some [any] maximal chain in $\mathbf{S}/\mathcal{D}$ completely determine *all* coset decompositions of pairs of adjacent $\mathcal{D}$-classes in $\mathbf{S}/\mathcal{D}$.

Based on Lemma 4 we can now state the identities connecting the coset constants arising in a skew diamond.

Lemma 12. *Given* $m \in M$ *and* $j \in J$ *such that* $m < j$ *there exist* $a \in A$ *and* $b \in B$ *such that*

$$\begin{aligned} c(JmJ, M \nabla j \nabla M) &= c(AmA, M \nabla a \nabla M) + c(BmB, M \nabla b \nabla M) \\ &= c(JaJ, A \nabla j \nabla A) + c(JbJ, B \nabla j \nabla B). \end{aligned}$$

Proof. By Lemma 4 there exist $a \in A$ and $b \in B$ such that $m < a, b < j$. Thus $j = a \nabla b = a + b - m$ and

$$\begin{aligned} jmj - j &= m - j \\ &= m - (a + b - m) \\ &= (m - a) + (m - b) \\ &= (ama - a) + (bmb - b). \end{aligned}$$

The second equality can now be derived directly from Corollary 9 using Lemma 7. □

Lemma 12 allows us to give an alternative proof of Theorem 5 above.

Proof of Theorem 5. Let $m, m' \in M$ be such that $JmJ = Jm'J$. As $JmJ \subseteq AmA$ and $JmJ \subseteq BmB$, both $AmA = Am'A$ and $BmB = Bm'B$ follow.

On the other hand, let $m, m' \in M$ be such that $AmA = Am'A$ and $BmB = Bm'B$. Take $j \in J$, $j > m$ and let $a \in A$ and $b \in B$ be as in Lemma 12. Then

$$\begin{aligned} jmj &= j + c(JmJ, M\nabla j\nabla M) \\ &= j + c(AmA, M\nabla a\nabla M) + c(BmB, M\nabla b\nabla M) \\ &= j + c(Am'A, M\nabla a\nabla M) + c(Bm'B, M\nabla b\nabla M) \\ &= j + c(Jm'J, M\nabla j\nabla M) \\ &= jm'j. \end{aligned}$$

□

Within a chain of three components $A > B > C$ each coset constant from A to C is obtained as the sum of any corresponding coset constants from A to B and from B to C as follows:

$$cba - a = (cba - ba) + (ba - a)$$

This fact hints us for the third and final coset laws presented bellow.

Theorem 13. *For all skew diamonds $\{ J > A, B > M \}$ in a skew lattice* $\mathbf{S}$ *the following hold.*

(i) Let $m, m' \in M$. Then $JmJ = Jm'J$ if and only if given any $a \in A$ both $J(m\nabla a\nabla m)J = J(m'\nabla a\nabla m')J$ and $AmA = Am'A$.

(ii) Let $j, j' \in J$. Then $M\nabla j\nabla M = M\nabla j'\nabla M$ if and only if given any $a \in A$ both $M\nabla(jaj)\nabla M = M\nabla(j'aj')\nabla M$ and $A\nabla j\nabla A = A\nabla j'\nabla A$.

Proof. If $JmJ = Jm'J$ then $m' = jmj$ for some $j \in J$. Let $a \in A$. Then

$$m'\nabla a\nabla m' = m' + a - am'a = jmj + a - ajmja.$$

Let $y \in J$ be such that $a < y$. Hence $y(m'\nabla a\nabla m')y = yjmjy + yay - yajmjay = y(m + a - ama)y$ by Lemma 1. On the other hand $am'a = ajmja \in AmA$ and therefore $AmA = Am'A$.

To prove the converse, suppose that $J(m\nabla a\nabla m)J = J(m'\nabla a\nabla m')J$ and $AmA = Am'A$ hold. Then there exist $a \in A$ and $j \in J$ such that $m' = ama$ and $m\nabla a\nabla m = j(m'\nabla a\nabla m')j$. Hence $m + a - ama = jm'j + jaj - jamaj$. Multiplying the latter by j on both sides yields $jmj = jm'j$ and $JmJ = Jm'J$ follows.

Again, (ii) is proved by an analogous argument. □

3 The coset laws in rings of matrices

Let F be a field with characteristic different from 2, $n \in \mathbb{N}$ and $\mathbf{S}$ a right handed skew lattice in $M_n(F)$. If $\mathbf{S}$ has two comparable $\mathcal{D}$-classes $A > B$ then given $a \in A$ and $b \in B$, $bA = \{ba : a \in A\}$ is the coset of A in B and $B \circ a = \{b + a - ba : b \in B\}$ is the coset of B in A. (Recall that in the right handed case ∇ reduces to the circle operation.)

The standard form for right handed skew lattices in $M_n(F)$ was described in [3], based on the standard form for pure bands in matrix rings that was developed by Fillmore at al. in [5] and [6]. It is described as follows. Let $E_1 < ... < E_m$ be a maximal chain of $\mathcal{D}$-classes of the skew lattice $\mathbf{S}$. Then a basis for F^n exists such that in this basis:

(i) for any two matrices $a \in E_i$, $b \in E_j$, $i > j$, a block decomposition exists such that a and b have block forms

$$a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} I & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(ii) for non-comparable $\mathcal{D}$-classes A and B with the meet class M and the join class J a block decomposition exists such that we may assume that $m_0 \in M$, $a_0 \in A$, $b_0 \in B$ and $j_0 \in J$, where

$$m_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, a_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, j_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, given any matrices $m \in M$, $j \in J$, $a \in A$ and $b \in B$ they have block forms

$$m = \begin{bmatrix} I & m_{12} & m_{13} & m_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, a = \begin{bmatrix} I & 0 & a_{13} & a_{14} \\ 0 & I & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b = \begin{bmatrix} I & b_{12} & 0 & b_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } j = \begin{bmatrix} I & 0 & 0 & j_{14} \\ 0 & I & 0 & j_{24} \\ 0 & 0 & I & j_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding coset constants are then given by

$$c(mA, M \circ a) = \begin{bmatrix} 0 & m_{12} & 0 & m_{12}a_{24} \\ 0 & -I & 0 & -a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$c(aJ, A \circ j) = \begin{bmatrix} 0 & 0 & a_{13} & a_{13}j_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & -j_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$c(mB, M \circ b) = \begin{bmatrix} 0 & 0 & m_{13} & m_{13}b_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & -b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$c(bJ, B \circ j) = \begin{bmatrix} 0 & b_{12} & 0 & b_{12}j_{24} \\ 0 & -I & 0 & -j_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From this one can clearly observe the identity $c(aB, M \circ b) = c(aJ, A \circ b)$ since

$$c(aB, M \circ b) = ab - b = \begin{bmatrix} 0 & 0 & a_{13} & a_{13}b_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & -a_{13} \\ 0 & 0 & 0 & 0 \end{bmatrix} = a - a \circ y = c(aJ, A \circ b)$$

Dually, $c(B \circ a, bM) = c(M \circ a, yA)$ is clear, hinting the coset laws of Theorem 10. In the case of the other two coset laws, the coset constant equations are easily seen from the following calculations,

$$c(mJ, M \circ j) = mj - j = \begin{bmatrix} 0 & m_{12} & m_{13} & m_{12}j_{24} + m_{13}j_{34} \\ 0 & -I & 0 & -j_{24} \\ 0 & 0 & -I & -j_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} c(mA, M \circ a) + c(mB, M \circ b) &= (ma - a) + (mb - b) \\ &= \begin{bmatrix} 0 & m_{12} & m_{13} & m_{12}a_{24} + m_{13}b_{34} \\ 0 & -I & 0 & -a_{24} \\ 0 & 0 & -I & -b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

If $j = a \circ b$ then $j_{24} = a_{24}$ and $j_{34} = b_{34}$. Hence,

$$c(mJ, M \circ j) = c(mA, M \circ a) + c(mB, M \circ b),$$

the identity that follows from the property of symmetry.

In what follows we present the computations from which the coset laws of the previous section were first observed.

Letting $m = \begin{bmatrix} I & x & y & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ we get

$$mj_0 = \begin{bmatrix} I & x & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, ma_0 = \begin{bmatrix} I & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } mb_0 = \begin{bmatrix} I & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where one can observe that mJ depends on x and y whereas mA depends on x and mB depends on y. Hence it is clear that the following coset law holds for all $m \in M$,

$$mJ = m'J \text{ if and only if } mA = m'A \text{ and } mB = m'B.$$

On the other hand, letting

$$j = \begin{bmatrix} I & 0 & 0 & x \\ 0 & I & 0 & y \\ 0 & 0 & I & z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we get

$$m_0 \circ j = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & y \\ 0 & 0 & I & z \\ 0 & 0 & 0 & 0 \end{bmatrix}, a_0 \circ j = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & z \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } b_0 \circ j = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & y \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence for all $j \in J$

$$M \circ j = M \circ j' \text{ if and only if } A \circ j = A \circ j' \text{ and } B \circ j = B \circ j'.$$

In order to observe the Theorem 10 coset laws let

$$a = \begin{bmatrix} I & 0 & x & y \\ 0 & I & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and notice that

$$m_0 \circ a = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } b_0 \circ a = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & z \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

as well as

$$ab_0 = \begin{bmatrix} I & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } aj_0 = \begin{bmatrix} I & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

According to these computations, given $a, a' \in A$ one obtains

$$M \circ a = M \circ a' \text{ if and only if } B \circ a = B \circ a',$$

and

$$aB = a'B \text{ if and only if } aJ = a'J.$$

To see that the coset laws that derive from Theorem 13 hold let

$$m = \begin{bmatrix} I & x & y & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} I & 0 & w & u \\ 0 & I & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$(m\nabla a)j_0 = \begin{bmatrix} I & 0 & y & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is interestingly independent of a. On the other hand,

$$mj_0 = \begin{bmatrix} I & x & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad ma_0 = \begin{bmatrix} I & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$mJ = m'J \text{ iff for all } a \in A,\ (m\nabla a)J = (m'\nabla a)J \text{ and } mA = m'A.$$

Dually, we obtain

$$M\nabla j = M\nabla j' \text{ iff for all } a \in A,\ M\nabla(aj) = M\nabla(aj') \text{ and } A\nabla j = A\nabla j'.$$

4 Counting cosets

Let $\mathbf{S}$ be a finite skew lattice with two incomparable $\mathcal{D}$-classes, say $X > Y$. Fixing $x \in X$ and $y \in Y$ the respective image sets are given by

$$xYx = \{\, y \in Y : x \geq y \,\} \text{ and } y\nabla X\nabla y = \{\, x \in X : x \geq y \,\}.$$

Due to the fact that the image sets are transversals of the coset partition of a $\mathcal{D}$-class, $|y\nabla X\nabla y|$ is the number of X-cosets in Y and $|xYx|$ is the number of Y-cosets in X. Therefore, all image sets of elements from one class have equal size, ie.,

$$\text{for all } y, y' \in Y,\ |\{\, x \in X : x \geq y \,\}| = |\{\, x \in X : x \geq y' \,\}|$$

and dually, given $x, x' \in X$ their image sets in Y also have the same power.

Consider the family $\{X_i : i \leq |y\nabla X\nabla y|\}$ of Y-cosets in X and the family $\{Y_j : j \leq |xYx|\}$ of X-cosets in Y. As all cosets X_i have equal cardinality the number of Y-cosets in X can be expressed by $|X|/|X_i|$, that is, $|X| = |y\nabla X\nabla y||X_i|$ and, dually, $|Y| = |xYx||Y_j|$. Furthermore, we have $|X_i| = |Y_j|$ for all i, j and therefore

$$|xYx| = \frac{|y\nabla X\nabla y||Y|}{|X|}.$$

Theorem 14. *Let $\mathbf{S}$ be a finite skew lattice in a ring and $\{\, J > A, B > M \,\}$ a skew diamond in $\mathbf{S}$. Then*

$$|A||B| = |J||M|.$$

Proof. Fix $a \in A$, $b \in B$, $j \in J$ and $m \in M$. Theorem 10 yields that the number of M-cosets in A is equal to the number of B-cosets in J. Thus the respective image sets are equipotent, ie. $|m\nabla A\nabla m| = |b\nabla J\nabla b|$. Dually, we obtain $|aMa| = |jBj|$. Therefore $\frac{|A|}{|M|} = \frac{|m\nabla A\nabla m|}{|aMa|}$, $\frac{|B|}{|J|} = \frac{|jBj|}{|b\nabla J\nabla b|}$ and thus

$$\frac{|A||B|}{|J||M|} = \frac{|m\nabla A\nabla m||jBj|}{|aMa||b\nabla J\nabla b|} = 1.$$

□

References

[1] Cvetko-Vah, K., Pure Skew Lattices in Rings, Semigroup Forum 68 (2004), 268–279.

[2] Cvetko-Vah, K., Skew lattices in rings, Dissertation, University of Ljubljana, 2005.

[3] Cvetko-Vah, K., On the structure of semigroups of idempotent matrices, Linear Algebra Appl. 426 (2007), 204–213.

[4] Cvetko-Vah, K., Kinyon, M., Leech, J., Spinks, M., Cancellation in skew lattices, Order, to appear. DOI: 10.1007/s11083-010-9151-7

[5] Fillmore, P., MacDonald, G., Radjabalipour, M., Radjavi, H., Towards a classification of maximal unicellular bands, Semigroup Forum 49 (1994), 195–215.

[6] Fillmore, P., MacDonald, G., Radjabalipour, M., Radjavi, H., Principal ideal bands, Semigroup Forum 59 (1999), 362–373.

[7] Jordan, P., Über nichtkommutative Verbände, Arch. Math. 2 (1949), 56–59.

[8] Kimura, N., The Structure of Idempotent Semigroups, Pacific J. Math. 8 (1958), 257–275.

[9] Leech, J., Skew Lattices in rings, Algebra Univers. 26 (1989), 48–72.

[10] Leech, J., The geometric structure of skew lattices, Trans. Amer. Math. Soc. 335 (1993), 823–842.

[11] Leech, J., Recent developments in the theory of skew lattices, Semigroup Forum 52 (1996), 7–24.

Received by the editors April 1, 2010. Revised September 14, 2010.