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LATTICE IDENTITIES AND COLORED GRAPHS CONNECTED BY TEST LATTICES

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Abstract. Czédli [4] has recently given a pictorial approach to several properties of free lattices. Our goal is to generalize his construction and use it to prove some additional classical lattice theoretical results in a new, more visual way.

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1. Introduction

The present paper offers a new approach to the following two theorems.

Theorem 1 (Sachs [11]). *The class of finite equivalence lattices generates the variety of all lattices.*

Theorem 2 (Whitman [12]). *Free lattices satisfy the Whitman condition.*

These theorems are classical fundamental results of lattice theory, of course. Both theorems are strengthened in many different ways. Indeed, there are many more results on free lattices in Freese, Ježek and Nation [7]. Nowadays, when each lattice resp. finite lattice is known to be representable as a sublattice of an appropriate equivalence lattice resp. finite equivalence lattice, see Whitman [13] and Pudlák and Tůma [10], we tend to forget Sachs' result, Theorem 1.

We note that Theorem 2 and specifically Theorem 1 require fairly complex proofs. The alternative proof for Theorem 2 in [7] is somewhat shorter than Whitman's original argument due to Day's doubling construction, see [6]. Yet another proof for Theorem 2 is given in Czédli [4], but it uses Jónsson's deep representation theorem of type 3, see [8]. Sachs' original argument of Theorem 1 is based on Whitman's representation theorem [13], which was later strengthened by Jónsson [8] and Pudlák and Tůma [10].

Our approach is entirely different from and much more elementary than the previous ones. Even if the underlying idea of applying Mal'cev conditions to the variety of sets may sound non-elementary, this notion will not occur in the rest of the paper. This is possible since in a series of papers Czédli has developed

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a graph theoretical technique as a reasonable substitute for Mal'cev conditions, see [1], [2], [3], [5] and finally [4]. The technique in [4] will be intensively used here; however, we have to develope it further to reach our goal.

The construction of test lattices in Czédli [4] is based on two steps, see Section 3. Firstly we define a directed graph colored by lattice terms. Secondly we define a test lattice that will be the sublattice of the equivalence lattice over the vertex set of the graph. We will show that the construction works for every natural number $n \geq 4$, and even the statement of Lemma 8 in [4] remains true for every natural number $n \geq 4$, see Lemma 3. The crucial point is that the lattices corresponding to different natural numbers are the same, see Lemma 4. We can use this fact to eliminate Jónsson's deep representation theorem of type 3 from the proof of Lemma 9 in [4]. Then we use the (modified) construction to prove Theorem 1 and Theorem 2, see Section 4.

2. Preliminaries

First, let us recall the basic concepts used in [4]. For a fixed positive integer k by a *k-pointed* lattice we mean a lattice L with k distinguished elements. For $\mathbf{g} = (g_1, \ldots, g_k) \in \mathsf{L}^k$, the k-pointed lattice will be denoted by $(\mathsf{L}; \mathbf{g})$. If p and q are lattice terms, then both $p = q$ and $p \leq q$ are *lattice identities.* A lattice identity is said to be *trivial*, if it holds in all lattices. Let it be denoted by $p =_{\text{triv}} q$ or $p \leq_{\text{triv}} q$. Given a k-ary lattice term p, we will call a k-pointed lattice $(\mathsf{L}; \mathbf{g})$ a *p-lattice*, if

- $\{g_1, \ldots, g_k\}$ generates L and
- for any k-ary lattice term q the identity $p(\mathbf{g}) \leq q(\mathbf{g})$ holds in L iff $p \leq_{\text{triv}} q$.

We use the terminology *test lattice* if we do not want to specify p. That is, if $(\mathsf{L}; \mathbf{g})$ is a p-lattice for some p then it is also called a test lattice. For example, if L is freely generated by $\{g_1, \ldots, g_k\}$ then it is obviously a p-lattice for every k-ary lattice term p.

For the whole paper, we fix a positive integer k and a set of variables $\mathbf{V} = \{\alpha_1, \ldots, \alpha_k\}$. We do not differentiate between lattice terms modulo associativity and idempotency. However, for technical reasons, we do make a distinction modulo commutativity. We define $\mathbf{J} = \mathbf{J}(\alpha_1, \ldots, \alpha_k)$, the set of *join-irreducible* lattice terms and $\mathbf{M} = \mathbf{M}(\alpha_1, \ldots, \alpha_k)$, the set of *meet-irreducible* lattice terms. Let $\mathbf{J}$ and $\mathbf{M}$ be the smallest sets that satisfy the following conditions:

1. $\mathbf{V} \subseteq \mathbf{J}, \mathbf{M}$;
2. if $t_1, \ldots, t_\ell \in \mathbf{J}$ ($\ell > 1$) are pairwise disjoint elements then $t_1 \vee \cdots \vee t_\ell \in \mathbf{M}$;
3. if $t_1, \ldots, t_\ell \in \mathbf{M}$ ($\ell > 1$) are pairwise disjoint elements then $t_1 \wedge \cdots \wedge t_\ell \in \mathbf{J}$;

In case 2 and 3, the subterms $t_1, \ldots, t_\ell$ will be called *joinands* and *meetands*, respectively. The set $\mathbf{J}$ and the set $\mathbf{M}$ will also be called the set of *meet-reducible* and *join-reducible* lattice terms, respectively. Let $\mathbf{T}$ denote the set of join- or meet-irreducible lattice terms, that is $\mathbf{T} = \mathbf{J} \cup \mathbf{M}$.

We note that this definition of lattice terms differ from that in Czédli [4] and in Freese, Ježek and Nation [7], too. We also note that $\mathbf{T}$ does not contain every lattice term, e.g. $\alpha_1 \vee \alpha_1 \notin \mathbf{T}$ or $(\alpha_1 \vee \alpha_2) \vee \alpha_3 \notin \mathbf{T}$. One can easily check, that for an arbitrary lattice term t there exists a lattice term $t' \in \mathbf{T}$ such that $t =_{\text{triv}} t'$.

For a lattice term $t \in \mathbf{T}$ we define the *length* of t (denoted by length(t)) in the natural way. If t is a variable then length$(t) = 1$. If $t = t_1 \vee \cdots \vee t_\ell$ or $t = t_1 \wedge \cdots \wedge t_\ell$ then length$(t) = 1 + \text{length}(t_1) + \cdots + \text{length}(t_\ell)$.

For a lattice term $t \in \mathbf{T}$ we define the *color set* of t (denoted by $\mathcal{C}(t)$) by induction on length(t). The terminology will be clear soon.

1. If $t \in \mathbf{V}$ is a variable then $\mathcal{C}(t) = \{t\}$;
2. if $t = t_1 \vee \cdots \vee t_\ell \in \mathbf{M} \setminus \mathbf{V}$ is join-reducible then $\mathcal{C}(t) = \mathcal{C}(t_1) \cup \cdots \cup \mathcal{C}(t_\ell)$;
3. if $t = t_1 \wedge \cdots \wedge t_\ell \in \mathbf{J} \setminus \mathbf{V}$ is meet-reducible then $\mathcal{C}(t) = \{t\} \cup \bigcup_{t_j \notin \mathbf{V}} \mathcal{C}(t_j)$.

We note that $t \in \mathbf{M} \setminus \mathbf{V}$ in case 2 and $t \in \mathbf{J} \setminus \mathbf{V}$ in case 3 implies that $\ell > 1$ in both cases. We also note that $\mathcal{C}(t)$ is the subset of $\mathbf{J}$.

3. Graphs and lattices defined by lattice terms

For a lattice term $t \in \mathbf{T}$ and for every positive integer n we will define a (finite) colored directed graph $\mathsf{G}^n(t)$. It will contain neither loops nor multiple edges. It will have two distinguished vertices, the *left and right endpoints*, usually denoted by x_0 and x_1. In figures the endpoints will be placed on the left-hand side, resp. right-hand side, and the orientation of edges will not be indicated based on the convention that all edges are directed from left to right. We will color the edges of the graph with the elements of $\mathcal{C}(t)$. Unless otherwise stated, *all graphs will be understood in the above sense.* Note, that this construction is a generalization of the construction Czédli has given in [4] for $n = 4$.

Let $\mathrm{V} = \mathrm{V}(\mathsf{G}^n(t))$, $\mathrm{E} = \mathrm{E}(\mathsf{G}^n(t))$ and $\mathrm{col} : \mathrm{E} \to \mathcal{C}(t)$ denote the vertex set, the edge set and the coloring map of $\mathsf{G}^n(t)$, respectively. We note that here V is a (finite) set and E is an irreflexive and antisymmetric relation on V. The map col defines the colors of the edges. We call an edge $(a, b) \in \mathrm{E}$ a *covering edge* if there is no directed path from a to b except (a, b). For an edge $(a, b) \in \mathrm{E}$ let $\mathsf{S}(a, b)$ denote the smallest subgraph that contains all directed paths from a to b. If $\mathrm{col}(a, b) = s$ then we will use both notations (a, s, b) and $\mathsf{S}(a, s, b)$.

If G and H are graphs and (a, b) is an edge of G then we can *replace* (a, b) by H or we can *glue* H to (a, b). In both cases we suppose that $\mathrm{V}(\mathsf{G})$ and $\mathrm{V}(\mathsf{H})$ are disjoint sets. In the first case we erase the edge (a, b) and we identify a and b with the left and right endpoints of H, respectively. In the second case we keep (a, b) and identify a and b with the left and right endpoints of H, respectively. See both cases in Figure 1. Two graphs G_1 and G_2 are isomorphic iff there is a bijection φ between $\mathrm{V}(\mathsf{G}_1)$ and $\mathrm{V}(\mathsf{G}_2)$ that satisfies:

- $(a, b) \in \mathrm{E}(\mathsf{G}_1)$ iff $(\varphi(a), \varphi(b)) \in \mathrm{E}(\mathsf{G}_2)$;
- $\mathrm{col}_1(a, b) = \mathrm{col}_2(\varphi(a), \varphi(b))$ for all $(a, b) \in \mathrm{E}(\mathsf{G}_1)$.

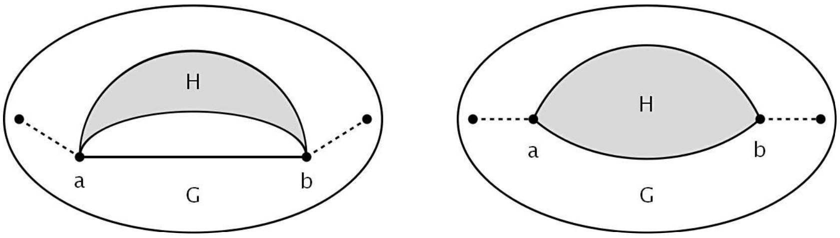

Figure 1: Gluing and replacing

Next, let $s, t \in \mathbf{T}$ be distinct lattice terms. For a fixed positive integer n the graph with

$$\mathrm{V}(\mathsf{G}) = \{x_0 = y_0, y_1, \dots, y_n = x_1\}$$

and

$$\mathrm{E}(\mathsf{G}) = \{(y_{i-1}, y_i) \mid i = 1, \dots, n\}$$

is called the $(s, t)^n$-arc, if the color of the edge (y_{i-1}, y_i) is s iff i is odd and t iff i is even. For a meet-irreducible lattice term $t = t_1 \vee \cdots \vee t_\ell \in \mathbf{M} \setminus \mathbf{V}$ we define a graph the so called t^n-*arc* as follows. See Figure 2 for an illustration.

1. If $\ell = 2$ then the t^n-arc is equal to the $(t_1, t_2)^n$-arc.
2. If $\ell > 2$ and for $t_0 = t_1 \vee \cdots \vee t_{\ell-1}$ the t_0^n-arc is defined then we take the $(t_0, t_\ell)^n$-arc and replace all t_0-colored edges by distinct t_0^n-arcs.

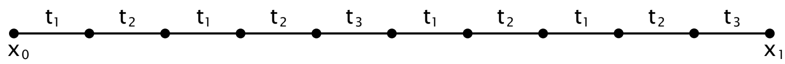

Figure 2: The $(t_1 \vee t_2 \vee t_3)^4$-arc

Now, for an arbitrary lattice term $t \in \mathbf{T}$ we define $\mathsf{G}^n(t)$ by induction on length(t).

1. If $t \in \mathbf{V}$ is a variable then $\mathsf{G}^n(t)$ consists of a single edge (x_0, t, x_1).
2. Let $t = t_1 \vee \cdots \vee t_\ell \in \mathbf{M} \setminus \mathbf{V}$ be a join-reducible lattice term. Then we take the t^n-arc and we replace all t_i-colored edges by distinct graphs isomorphic to $\mathsf{G}^n(t_i)$.

3. Let $t = t_1 \wedge \cdots \wedge t_\ell \in \mathbf{J} \setminus \mathbf{V}$ be a meet-reducible lattice term. First, we take a graph that consists of a single edge (x_0, t, x_1). Then for each lattice term $t_i \notin \mathbf{V}$ we glue distinct t_i^n-arcs to (x_0, x_1). Finally if s_{ij} is a joinand of t_i then we replace all s_{ij}-colored edges by distinct graphs isomorphic to $\mathsf{G}^n(s_{ij})$.

For a graph $\mathsf{G}^n(t)$, let the subgraphs occuring in the definition (isomorphic to $\mathsf{G}^n(t_i)$ or $\mathsf{G}^n(s_{ij})$) be called *complex subgraphs* of $\mathsf{G}^n(t)$. In case 3, the subgraph of $\mathsf{G}^n(t)$ that we get from the t_i^n-arc after the replacing will be called the t_i^n-arc of $\mathsf{G}^n(t)$.

Note that for each edge (a, s, b), we have $\mathsf{S}(a, s, b) \cong \mathsf{G}^n(s)$, and there exists exactly one isomorphism between $\mathsf{S}(a, s, b)$ and $\mathsf{G}^n(s)$.

Let $p, q \in \mathbf{T}$ be k-ary lattice terms. An edge (a, r, b) of $\mathsf{G} = \mathsf{G}^n(p)$ is called an α_i*-edge* if $r = \alpha_i$ or α_i is a meetand of r. Notice that α_i-colored edges are α_i-edges but not conversely. Let X denote the vertex set of G. The smallest member of $\mathrm{Eq}(X)$ generated by the relation $\{(a, b) \mid (a, b) \text{ is an } \alpha_i \text{ edge}\}$ will be denoted by $\alpha_i|_{\mathsf{G}}$. For a k-ary lattice term $q \in \mathbf{T}$ the equivalence $q(\alpha_1|_{\mathsf{G}}, \ldots, \alpha_k|_{\mathsf{G}}) \in \mathrm{Eq}(X)$ will be denoted by $q|_{\mathsf{G}}$. By an (undirected) $q|_{\mathsf{G}}$-path we mean an undirected path such that for every undirected edge (a, b) of the path $(a, b) \in q|_{\mathsf{G}}$. Similarly, for any equivalences $\vartheta_1, \ldots, \vartheta_\ell \in \mathrm{Eq}(X)$ by an (undirected) $\vartheta_1 \cup \cdots \cup \vartheta_\ell$-path we mean an undirected path such that for every undirected edge (a, b) of the path $(a, b) \in \vartheta_1 \cup \cdots \cup \vartheta_\ell$.

For $n = 4$, the following lemma is exactly Lemma 8 in Czédli [4].

Lemma 3. *Let $p, q \in \mathbf{T}$ be k-ary lattice terms. For a fixed integer $n \geq 4$ let $\mathsf{G} = \mathsf{G}^n(p)$ be the graph we defined earlier.*

1. *Let $(a, b) \in \mathrm{E}(\mathsf{G})$ be an edge and let $y_0, y_1 \in \mathrm{V}(\mathsf{S}(a, b))$ be vertices. Then*
$$(y_0, y_1) \in q|_{\mathsf{G}} \quad \textit{iff} \quad (y_0, y_1) \in q|_{\mathsf{S}(a,b)}.$$
2. *Let $y_0, y_1 \in \mathrm{V}(\mathsf{G})$ be vertices. Then $(y_0, y_1) \in q|_{\mathsf{G}}$ iff there is an (undirected) $q|_{\mathsf{G}}$-path from y_0 to y_1. In other words $q|_{\mathsf{G}}$ is the equivalence generated by $q|_{\mathsf{G}} \cap \mathrm{E}(\mathsf{G})$.*

Proof. Observe that each color of an arc occurs at least twice in the arc. This is what the proof in [4] relies on. Hence essentially the same proof works here. The details are left to the reader. □

Note that the previous lemma is not true for $n = 3$. Indeed, for $p = (\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_3)$ and $q = \alpha_1 \vee \alpha_3$ one can see that $(y_0, y_1) \in q|_{\mathsf{G}^3(p)}$ but $(y_0, y_1) \notin q|_{\mathsf{S}(y_0,y_1)}$, see Figure 3.

Let $t \in \mathbf{T}$ be a lattice term and X denote the vertex set of $\mathsf{G}^n(t)$. The sublattice of $\mathrm{Eq}(X)$ generated by $\{\alpha_1|_{\mathsf{G}^n(t)}, \ldots, \alpha_k|_{\mathsf{G}^n(t)}\}$ will be denoted by $\mathsf{L}^n(t)$. Note, that this definition is the same Czédli has given in [4] for $n = 4$, and the following Lemma shows that for $n \geq 4$ we obtain the same lattices. Also note, that at the end of the paper we will see that $\mathsf{L}^n(t)$ is a test lattice (in fact it is a t-lattice).

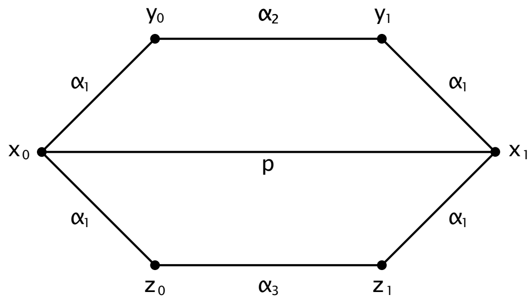

Figure 3: $\mathsf{G}^3(p)$, $p = (\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_3)$

Lemma 4. *For every k-ary lattice term $t \in \mathbf{T}$ and integers $n, m \geq 4$ we have $\mathsf{L}^n(t) \cong \mathsf{L}^m(t)$.*

Proof. Using Lemma 3, one can prove by induction on length(p)+length(t) that for every lattice term $p \in \mathbf{T}$

$$(1) \qquad (x_0, x_1) \in p|_{\mathsf{G}^n(t)} \quad \text{iff} \quad (x_0, x_1) \in p|_{\mathsf{G}^m(t)}.$$

Let $\varphi : \{\alpha_1|_{\mathsf{G}^n(t)}, \ldots, \alpha_k|_{\mathsf{G}^n(t)}\} \to \{\alpha_1|_{\mathsf{G}^m(t)}, \ldots, \alpha_k|_{\mathsf{G}^m(t)}\}$ be the map defined by $\alpha_i|_{\mathsf{G}^n(t)} \mapsto \alpha_i|_{\mathsf{G}^m(t)}$. We are going to prove that for all lattice terms $p, q \in \mathbf{T}$

$$(2) \qquad p|_{\mathsf{G}^n(t)} = q|_{\mathsf{G}^n(t)} \quad \text{iff} \quad p|_{\mathsf{G}^m(t)} = q|_{\mathsf{G}^m(t)}.$$

This will clearly imply that φ can be extended to an isomorphism from $\mathsf{L}^n(t)$ to $\mathsf{L}^m(t)$.

We prove (2) by induction on length(t). If $t \in \mathbf{V}$ is a variable then $\mathsf{G}^n(t) = \mathsf{G}^m(t)$ and (2) holds trivially. Hence we can assume that length$(t) > 1$.

It suffices to prove the "only if" part of (2). Let $p|_{\mathsf{G}^n(t)} = q|_{\mathsf{G}^n(t)}$ and let $(y_0, y_1) \in p|_{\mathsf{G}^m(t)}$. From Lemma 3, we know that there exists a $p|_{\mathsf{G}^m(t)}$-path from y_0 to y_1. Let $z_0 = y_0, z_1, \ldots, z_\ell = y_1$ be the vertices of this path. If y_0 and y_1 are not the two endpoints of $\mathsf{G}^m(t)$ then the subgraphs $\mathsf{S}(z_{i-1}, z_i)$ are smaller then $\mathsf{G}^m(t)$. From Lemma 3, we know that $(z_{i-1}, z_i) \in p|_{\mathsf{G}^m(t)}$ iff $(z_{i-1}, z_i) \in p|_{\mathsf{S}(z_{i-1}, z_i)}$ for all $i \in \{1, \ldots, \ell\}$. We can use the induction hypothesis to col(z_{i-1}, z_i) and we get $(z_{i-1}, z_i) \in q|_{\mathsf{S}(z_{i-1}, z_i)}$. From Lemma 3, we conclude that $(z_{i-1}, z_i) \in q|_{\mathsf{G}^m(t)}$ for all $i \in \{1, \ldots, \ell\}$. It implies that $(y_0, y_1) \in q|_{\mathsf{G}^m(t)}$.

If y_0 and y_1 are the endpoints of $\mathsf{G}^m(t)$ then we know from (1) that $(y_0, y_1) \in q|_{\mathsf{G}^m(t)}$. We got that $p|_{\mathsf{G}^m(t)} \leq q|_{\mathsf{G}^m(t)}$. Hence $p|_{\mathsf{G}^m(t)} = q|_{\mathsf{G}^m(t)}$ by the symmetry of p and q. □

Lemma 5 (Graph lemma)**.** *Let $p \in \mathbf{T}$ be a k-ary lattice term, let Y be a set and let $\vartheta_1, \ldots, \vartheta_k \in \mathrm{Eq}(Y)$ and $y_0, y_1 \in Y$. A mapping $\varphi : \mathrm{V}(\mathsf{G}^n(p)) \to Y$ is called a representation mapping if $\varphi(x_i) = y_i$ holds and for each edge $(a, \alpha_j, b) \in \mathrm{E}(\mathsf{G}^n(p))$ we have $(\varphi(a), \varphi(b)) \in \vartheta_j$. The following conditions are equivalent:*

1. $(y_0, y_1) \in p(\vartheta_1, \dots, \vartheta_k)$;
2. *there exists a natural number n_0 such that for each natural number $n \geq n_0$ there is a representation mapping.*

Each representation mapping φ satisfies that for all $q \in \mathbf{T}$ and all $a, b \in$ $\mathrm{V}(\mathsf{G}^n(p))$

(3) $$(a, b) \in q|_{\mathsf{G}^n(p)} \quad \textit{implies} \quad (\varphi(a), \varphi(b)) \in q(\vartheta_1, \dots, \vartheta_k).$$

Proof. The equivalence of 1 and 2 can be found in Czédli and Day [5]; it also follows immediately from the definition by induction on length(p). (3) follows from Lemma 3. □

Let $\mathcal{V}$ be the variety of lattices. Let $\mathcal{V}_{\mathrm{eq}}$ be the variety generated by the finite equivalence lattices and let $p \leq_{\mathrm{eq}} q$ denote that the lattice identity $p \leq q$ holds in $\mathcal{V}_{\mathrm{eq}}$.

Corollary 6. *Let $p, q \in \mathbf{T}$ be k-ary lattice terms. The following conditions are eqivalent (for every integer $n \geq n_0$):*

1. $p \leq_{\mathrm{eq}} q$;
2. $p(\alpha_1|_{\mathsf{G}^n(p)}, \dots, \alpha_k|_{\mathsf{G}^n(p)}) \leq q(\alpha_1|_{\mathsf{G}^n(p)}, \dots, \alpha_k|_{\mathsf{G}^n(p)})$ *in* $\mathsf{L}^n(p)$;
3. $(x_0, x_1) \in q(\alpha_1|_{\mathsf{G}^n(p)}, \dots, \alpha_k|_{\mathsf{G}^n(p)})$.

Proof. 1 implies 2 since $\mathsf{L}^n(p)$ is a sublattice of $\mathrm{Eq}(\mathrm{V}(\mathsf{G}^n(p)))$. The construction of $\mathsf{G}^n(p)$ implies that $(x_0, x_1) \in p|_{\mathsf{G}^n(p)}$, hence 2 implies 3. Now let us assume 3. It is enough to prove that $p \leq q$ in all finite equivalence lattices. Let Y be a finite set, let $\vartheta_1, \dots, \vartheta_k \in \mathrm{Eq}(Y)$ and let $(y_0, y_1) \in p(\vartheta_1, \dots, \vartheta_k)$. We apply Lemma 5 with the same notation: $(y_0, y_1) = (\varphi(x_0), \varphi(x_1)) \in q(\vartheta_1, \dots, \vartheta_k)$, which completes the proof. □

4. Lattice identities in free lattices

Now, we are in the position of proving that the word problem for $\mathcal{V}_{\mathrm{eq}}$ has exactly the same solution as it has for $\mathcal{V}$, see Whitman [12] or Freese, Ježek and Nation [7]. Theorem 1 is a consequence of the following lemma.

Lemma 7. *Let $p \in \mathbf{J}$ be a meet-reducible lattice term with meetands: $p_1, \dots, p_u$ and let $q \in \mathbf{M}$ be a join-reducible lattice term with joinands: $q_1, \dots, q_v$.*

1. *If $p \leq_{\mathrm{eq}} q$ then either $p_i \leq_{\mathrm{eq}} q$ for some $i \in \{1, \dots, u\}$ or $p \leq_{\mathrm{eq}} q_j$ for some $j \in \{1, \dots, v\}$.*
2. *If $\alpha \leq_{\mathrm{eq}} q$ for some variable $\alpha \in \mathbf{V}$ then $\alpha \leq_{\mathrm{eq}} q_j$ for some $j \in \{1, \dots, v\}$.*
3. *If $p \leq_{\mathrm{eq}} \beta$ for some variable $\beta \in \mathbf{V}$ then $p_i \leq_{\mathrm{eq}} \beta$ for some $i \in \{1, \dots, u\}$.*

Proof. 1 Let us assume $p \leq_{\text{eq}} q$. By Corollary 6 we know that $(x_0, x_1) \in q|_{\mathsf{G}^n(p)}$ (for every integer $n \geq 4$). Using Lemma 3 we get a $q_1|_{\mathsf{G}^n(p)} \cup \cdots \cup q_v|_{\mathsf{G}^n(p)}$-path from x_0 to x_1. Let $(z_0 = x_0, z_1, \ldots, z_\ell = x_1)$ be a minimal $q_1|_{\mathsf{G}^n(p)} \cup \cdots \cup q_v|_{\mathsf{G}^n(p)}$-path ($\ell$ is minimal).

If $\ell = 1$ then $(x_0, x_1) = (z_0, z_1) \in q_j|_{\mathsf{G}^n(p)}$ for some $j \in \{1, \ldots, v\}$ and from Corollary 6 we get $p \leq_{\text{eq}} q_j$.

Let us assume that $\ell > 1$. Since ℓ is minimal, $(z_0, z_1, \ldots, z_\ell)$ goes entirely in the p_i-arc for some $i \in \{1, \ldots, u\}$, and it connects all endpoints of the complex subgraphs on the p_i-arc. Hence for an arbitrary complex subgraph on the p_i-arc $\mathsf{S}(a, r_{ij}, b)$ we know that $(a, b) \in q|_{\mathsf{G}^n(p)}$. Using Lemma 3 we get $(a, b) \in q|_{\mathsf{S}(a, r_{ij}, b)}$. Since $\mathsf{S}(a, r_{ij}, b) \cong \mathsf{G}^n(r_{ij})$, we have $(x_0, x_1) \in q|_{\mathsf{G}^n(r_{ij})}$. Using Corollary 6 we get $r_{ij} \leq_{\text{eq}} q$. This argument works for all complex subgraphs of the p_i-arc, therefore $p_i \leq_{\text{eq}} q$.

2 and 3 are dual statement, hence it is enough to prove 2. By Corollary 6 it is sufficient to prove that $\alpha|_{\mathsf{G}^n(\alpha)} \leq q|_{\mathsf{G}^n(\alpha)}$ implies $\alpha|_{\mathsf{G}^n(\alpha)} \leq q_j|_{\mathsf{G}^n(\alpha)}$ for some $j \in \{1, \ldots, v\}$, which is trivial. □

Lemma 7 implies an algorithm that decides wether a lattice identity holds in $\mathcal{V}_{\text{eq}}$. We note that it is the same algorithm that Whitman gave for the word problem in $\mathcal{V}$, see Theorem 1 in [12], but we do not use this fact. All identities generated by the algorithm trivially hold in $\mathcal{V}$. Therefore $\mathcal{V}_{\text{eq}} = \mathcal{V}$. It proves Theorem 1. It also implies that we can replace $\leq_{\text{eq}}$ with $\leq_{\text{triv}}$ in Corollary 6 and in Lemma 7. Hence Theorem 2 is an easy consequence of Lemma 7.

References

[1] Czédli, G., On properties of rings that can be characterized by infinite lattice identities. Studia Sci. Math. Hungar. 16 (1981), 45–60.

[2] Czédli, G., Mal'cev conditions for Horn sentences with congruence permutability. Acta. Math. Hungar. 44 (1984), 115–124.

[3] Czédli, G., On the word problem of lattices with the help of graphs. Periodica Math. Hungar. 23 (1991), 49–58.

[4] Czédli, G., A visual approach to test lattices. Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 48 (2009), 33-52.

[5] Czédli, G., Day, A., Horn sentences with (W) and weak Mal'cev conditions. Algebra Univers. 19 (1984), 217–230.

[6] Day, A., Doubling constructions in lattice theory. Canad. J. Math. 44 (1992), 252–269.

[7] Freese, R., Ježek, J., Nation, J. B., Free Lattices. Amer. Math. Soc., Providence, RI, 1995.

[8] Jónsson, B., On the representation of lattices. Math. Scand. 1 (1953), 193–206.

[9] Grätzer, G., General Lattice Theory (2nd ed.). Birkhäuser Verlag, Basel, 1998.

[10] Pudlák, P., Tůma, J., Every finite lattice can be embedded in a finite partition lattice. Algebra Univers. 10 (1980), 74–95.

[11] Sachs, D., Identities in finite partition lattices. Proc. Amer. Math. Soc. 12 (1961), 944–945.

[12] Whitman, P. M., Free lattices. Ann. Math. 42 (1941), 325–330.

[13] Whitman, P. M., Lattices, equivalence relations, and subgroups. Bull. Amer. Math. Soc. 2 (1946), 507–522.

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