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ON LALLEMENT'S LEMMA[1]

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Abstract. Idempotent-consistent semigroups are defined by the property that each idempotent in a homomorphic image of a semigroup has an idempotent pre-image. In a manner this property is an another formulation for a well known Lallement's lemma. In this paper on an arbitrary semigroup we introduce a system of congruence relations and using them we give a new version of the proof of Lallement's lemma.

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1. Introduction and preliminaries

It is known that Lallement's lemma does not hold true in arbitrary semigroups. In fact, this lemma fails to hold in the semigroup of all positive integers under addition, since it does not have any idempotent element but the entire semigroup can be mapped onto a trivial semigroup, which of course is an idempotent.

The Lallement's lemma for regular semigroups says that if ρ is a congruence on a regular semigroup S and $a\rho$ is an idempotent in the quotient S/ρ then $a\,\rho\,e$ for some idempotent $e \in S$. We can formulate this property in terms of homomorphic images. The property featured in the conclusion of the lemma therefore has merited a name of its own and so we say that a congruence relation ξ on a semigroup S is *idempotent-consistent* (or *idempotent-surjective*) if for every idempotent class $a\xi$ of S/ξ there exists $e \in E(S)$ such that $a\xi e$. This property is in the conclusion of the well known Lallement's lemma. A semigroup is *idempotent-consistent* if all of its congruences enjoy this property. These notions were explored by P. M. Higgins [11], [12], P. M. Edwards [9], P. M. Edwards, P. M. Higgins and S. J. L. Kopamu [10], S. Bogdanović [3], and H. Mitsch [16].

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By $\mathbf{Z}^+$ we denote the set of all positive integers. By S^1 we denote a semigroup S with identity 1. By $E(S)$ we denote the set of all idempotents of a semigroup S. A semigroup which all its elements are idempotents is a *band*.

The notion of regularity was introduced by J. von Neumann [18], and it play the important role in a ring and a semigroup theory. An element a of a semigroup S is *regular* if there exists $x \in S$ such that $a = axa$. A semigroup S is *regular* if all its elements are regular.

As a generalization of regularity R. Arens and I. Kaplansky in [1] were introduced the notion of π-regularity. An element $a \in S$ is π*-regular* if there exists $n \in \mathbf{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$. A semigroup S is π*-regular* if all its elements are π-regular, i.e. if some power of every its element is regular.

The class of regular semigroups certainly does not exhaust the class of idempotent-consistent semigroups as it is a simple matter to check that every periodic semigroup, or more generally every (completely) π-regular, is idempotent-consistent. A generalization of Lallement's lemma that includes all the cases mentioned so far was provided by P. M. Edwards [9], where it was shown that the class of idempotent-consistent semigroups includes all π-regular semigroups. For the related result see also [4].

Although the class of π-regular semigroups does not contain all idempotent-consistent semigroups it was shown however in [17] by P. Murty, V. Ramana and K. Sudharani, that any idempotent-consistent and weakly commutative semigroup is also π-regular. A semigroup S is *weakly commutative* if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in bSa$.

The converse implication does not generally hold however in that not all idempotent-consistent semigroups are π-regular. This was first shown by S. J. L. Kopamu [15] through the introduction of the class of structurally regular semigroups which are defined using a special family of congruences. Some characterizations of semigroups, by congruences which are more general then ones introduced by S. J. L. Kopamu in [14], are considered by S. Bogdanović, Ž. Popović and M. Ćirić in [6] and [7]. S. J. L. Kopamu proved that Lallement's lemma holds for the class of all structurally regular semigroups.

Here, on an arbitrary semigroup, for $m, n \in \mathbf{Z}^+$, we define a family of congruence relations $\overline{\tau}_{(m,n)}$ and using them we prove: if the quotient semigroup $S/\overline{\tau}_{(m,n)}$ is π-regular, then a semigroup S is idempotent-consistent. This statement is one new version of Lallement's lemma. Also, we describe the structure of semigroups in which the relation $\overline{\tau}_{(m,n)}$ is a band congruence. The results presented in this paper are generalizations of results obtained by above mentioned authors.

Let ϱ be an arbitrary relation on a semigroup S. Then the *radical* $R(\varrho)$ of ϱ is a relation on S defined by:

$$(a, b) \in R(\varrho) \Leftrightarrow (\exists p, q \in \mathbf{Z}^+)\ (a^p, b^q) \in \varrho.$$

The radical $R(\varrho)$ was introduced by L. N. Shevrin in [19].

An equivalence relation ξ is a *left* (*right*) *congruence* if for all $a, b \in S$, $a\,\xi\, b$ implies $ca\,\xi\, cb$ ($ac\,\xi\, bc$). An equivalence ξ is a congruence if it is both left and right congruence. A congruence relation ξ is a band congruence on S if S/ξ is a band, i.e. if $a\,\xi\,a^2$, for all $a \in S$.

Let ξ be an equivalence on a semigroup S. By $\xi^\flat$ we define the largest congruence relation on S contained in ξ. It is well-known that

$$\xi^\flat = \{(a,b) \in S \times S \,|\, (\forall x, y \in S^1)\ (xay, xby) \in \xi\}.$$

Let $k \in \mathbf{Z}^+$ be a fixed integer. On a semigroup S we define the following relation by

$$(a,b) \in \eta_k \ \Leftrightarrow\ a^k = b^k.$$

It is easy to verify that η_k is an equivalence relation on a semigroup S. This relation was introduced by S. Bogdanović, Ž. Popović and M. Ćirić in [7].

For undefined notions and notations we refer [4], [5], [11] and [13].

The following two lemmas will be used to establish our main theorems.

Lemma 1. *Let ξ be a congruence relation on a semigroup S. Then $R(\xi) = \xi$ if and only if ξ is a band congruence on S.*

This lemma was proved by S. Bogdanović, Ž. Popović and M. Ćirić in [7].

Let ξ be a congruence relation on a semigroup S. An element $a \in S$ is ξ-*regular* if there exists $b \in S$ such that $a\xi = (aba)\xi$. A semigroup S is ξ-*regular* if all its elements are ξ-regular, i.e. if S/ξ is a regular semigroup. An element $b \in S$ such that $a\xi = (aba)\xi$ and $b\xi = (bab)\xi$ is a ξ-*inverse* of the element a.

Lemma 2. *For any ξ-regular element of a semigroup S there exists a ξ-inverse element.*

Proof. Let $a, b \in S$ such that $a\xi = (aba)\xi$, then it is easily to verify that

$$(a\xi)\,(bab)\xi\,(a\xi) = a\xi \text{ and } (bab)\xi\,(a\xi)\,(bab)\xi = (bab)\xi.$$

Thus $a\xi$ and $(bab)\xi$ are mutually inverses. □

2. The system of $\overline{\tau}_{(m,n)}$ congruences

Further, by previously defined relations on a semigroup S we define the following relations:

$$(a,b) \in \tau \ \Leftrightarrow\ (\exists k \in \mathbf{Z}^+)\ (a,b) \in \eta_k;$$

$$(a,b) \in \tau^\flat \ \Leftrightarrow\ (\forall x, y \in S^1)\ (xay, xby) \in \tau.$$

It is easy to verify that the relation τ is an equivalence on a semigroup S.

Let $m, n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\overline{\tau}_{(m,n)}$ by

$$(a,b) \in \overline{\tau}_{(m,n)} \ \Leftrightarrow\ (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \tau.$$

The main characteristic of previous defined relation gives the following theorem.

Theorem 1. *Let S be a semigroup and let $m, n \in \mathbf{Z}^+$. Then $\overline{\tau}_{(m,n)}$ is a congruence relation on S.*

Proof. It is clear that $\overline{\tau}_{(m,n)}$ is reflexive and symmetric. Assume $a, b, c \in S$ such that $a\,\overline{\tau}_{(m,n)}\,b$ and $b\,\overline{\tau}_{(m,n)}\,c$. Then for every $x \in S^m$ and $y \in S^n$ there exist $k, l \in \mathbf{Z}^+$ such that

$$(xay)^k = (xby)^k \quad \text{and} \quad (xby)^l = (xcy)^l$$

whence

$$(xay)^{kl} = (xby)^{kl} = (xby)^{lk} = (xcy)^{lk}.$$

So, we have that $xay\,\eta_{lk}\,xcy$, i.e. $xay\,\tau\,xcy$. Thus $\overline{\tau}_{(m,n)}$ is transitive.

Also, it is easy to prove that $\overline{\tau}_{(m,n)}$ is a compatible relation on S. Therefore $\overline{\tau}_{(m,n)}$ is a congruence on S. □

If instead of τ we assume the equality relation, then we obtain relations which discussed by S. J. L. Kopamu in [14] and [15].

Remark 1. *Let μ be an equivalence relation on a semigroup S and let $m, n \in \mathbf{Z}^+$. Then a relation $\overline{\mu}_{(m,n)}$ defined on S by*

$$(a, b) \in \overline{\mu}_{(m,n)} \;\Leftrightarrow\; (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \mu$$

is a congruence relation on S. But, there exists a relation μ which is not equivalence, for example $\mu = \longrightarrow$, for which the relation $\overline{\mu}_{(m,n)}$ is a congruence on S.

The complete description of the $\overline{\mu}_{(m,n)}$ congruence, for $\mu = \longrightarrow$, was given by S. Bogdanović, Ž. Popović and M. Ćirić in [8].

If the $\overline{\tau}_{(m,n)}$ relation is a band congruence, then the following two statements hold.

Theorem 2. *Let $m, n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\overline{\tau}_{(m,n)}$ is a band congruence on S;*
(ii) *$(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\tau\,xa^2y$;*
(iii) *$R(\overline{\tau}_{(m,n)}) = \overline{\tau}_{(m,n)}$.*

Proof. (i)⇔(ii) This equivalence follows immediately.

(i)⇔(iii) This equivalence immediately follows by Lemma 1. □

Proposition 1. *Let $m, n \in \mathbf{Z}^+$. If $\overline{\tau}_{(m,n)}$ is a band congruence on a semigroup S, then $\tau \subseteq \overline{\tau}_{(m,n)}$.*

Proof. Since $\overline{\tau}_{(m,n)}$ is a band congruence on S, then $xay\,\tau\,xa^iy$, for every $i \in \mathbf{Z}^+$ and for all $x \in S^m$, $y \in S^n$, $a \in S$. Assume $a, b \in S$ such that $a\,\tau\,b$. Then $a^k = b^k$, for some $k \in \mathbf{Z}^+$. Thus for every $x \in S^m$, $y \in S^n$ and $k \in \mathbf{Z}^+$ we have that

$$xay\,\tau\,xa^ky = xb^ky\,\tau\,xby.$$

Since τ is transitive, then $a\,\overline{\tau}_{(m,n)}\,b$. Therefore $\tau \subseteq \overline{\tau}_{(m,n)}$. □

3. The proof of Lallement's lemma

Before we present the main result of this paper, we give the following helpful lemma.

Lemma 3. *Let $m,n \in \mathbf{Z}^+$. An element $a \in S$ is $\overline{\tau}_{(m,n)}$-regular if and only if a has a $\overline{\tau}_{(m,n)}$-inverse element.*

Proof. Let $a \in S$ is $\overline{\tau}_{(m,n)}$-regular. Then $a\overline{\tau}_{(m,n)}axa$, for some $x \in S$, i.e. $(uav)^p = (uaxav)^p$, for every $u \in S^m$ and every $v \in S^n$ and some $p \in \mathbf{Z}^+$. Put $x' = xax$. Since $xav \in S^{n+2} \subseteq S^n$ then we have that $(uax'av)^q = (uaxaxav)^q = (uaxav)^q$, for some $q \in \mathbf{Z}^+$. Hence,

$$(uax'av)^{qp} = ((uax'av)^q)^p = ((uaxav)^q)^p = ((uaxav)^p)^q = ((uav)^p)^q = (uav)^{pq}.$$

Thus, $a\overline{\tau}_{(m,n)}ax'a$. Since $ux \in S^{m+1} \subseteq S^m$ and $xaxv \in S^{n+3} \subseteq S^n$ we have that $(ux'ax'v)^k = (uxaxaxaxv)^k = (uxaxaxv)^k$, for some $k \in \mathbf{Z}^+$. Also, since $ux \in S^m$ and $xv \in S^n$ we have and $(uxaxaxv)^t = (uxaxv)^t = (ux'v)^t$, for some $t \in \mathbf{Z}^+$. Hence,

$$(ux'ax'v)^{kt} = ((ux'ax'v)^k)^t = ((uxaxaxv)^k)^t =$$
$$= ((uxaxaxv)^t)^k = ((ux'v)^t)^k = (ux'v)^{tk}.$$

Thus, $x'ax'\overline{\tau}_{(m,n)}x'$. Therefore, x' is a $\overline{\tau}_{(m,n)}$-inverse of a.

The converse follows immediately. □

By the following theorem we give a new result of the type of Lallement's lemma. This theorem is a generalization of results obtained by P. M. Edwards, P. M. Higgins and S. J. L. Kopamu [10].

Theorem 3. *Let $m,n \in \mathbf{Z}^+$. Let ϕ be a homomorphism from a semigroup S onto T and let $S/\overline{\tau}_{(m,n)}$ be a π-regular semigroup. Then for every $f \in E(T)$ there exists $e \in E(S)$ such that $e\phi = f$.*

Proof. Since ϕ is surjective, then there exists $a \in S$ such that $a\phi = f$. Assume $a^{2(mn)} \in S$, then by Lemma 3 we have that

$$\text{(1)}\quad a^{2(mn)i}\overline{\tau}_{(m,n)} = (a^{2(mn)i}xa^{2(mn)i})\overline{\tau}_{(m,n)}, \quad x\overline{\tau}_{(m,n)} = (xa^{2(mn)i}x)\overline{\tau}_{(m,n)},$$

for some $x \in S$ and $i \in \mathbf{Z}^+$, whence

$$\begin{aligned} ((a^{(mn)i}xa^{(mn)i})^j)^2 &= ((a^{(mn)i}xa^{(mn)i})^2)^j = (a^{(mn)i}(xa^{2(mn)i}x)a^{(mn)i})^j \\ &= (a^{(ni)m}(xa^{2(mn)i}x)a^{(mi)n})^j = (a^{(ni)m}xa^{(mi)n})^j \\ &= (a^{(mn)i}xa^{(mn)i})^j \in E(S), \end{aligned}$$

for some $j \in \mathbf{Z}^+$. Let $e = (a^{(mn)i}xa^{(mn)i})^j$, then

$$\begin{aligned} e\phi &= ((a^{(mn)i}xa^{(mn)i})^j)\phi = ((a^{(mn)i}\phi)(x\phi)(a^{(mn)i}\phi))^j \\ &= ((a\phi)^{(mn)i}(x\phi)(a\phi)^{(mn)i})^j \\ &= ((a\phi)^{3(mn)i}(x\phi)(a\phi)^{3(mn)i})^j, \qquad (\text{since } (a\phi)^2 = a\phi = f = f^2) \\ &= ((a^{3(mn)i}\phi)(x\phi)(a^{3(mn)i}\phi))^j = ((a^{3(mn)i}xa^{3(mn)i})^j)\phi \\ &= ((a^{(mn)i}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^j)\phi. \end{aligned}$$

By (1) there exists $k \in \mathbf{Z}^+$ such that

$$(a^{(mn)i}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^k = (a^{(ni)m}(a^{2(mn)i}xa^{2(mn)i})a^{(mi)n})^k =$$

$$(a^{(ni)m}a^{2(mn)i}a^{(mi)n})^k = a^{4(mn)ik}.$$

Finally,

$$\begin{aligned}(e\phi)^k &= (((a^{(mn)i}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^j)\phi)^k \\ &= (((a^{i(mn)}(a^{2(mn)i}xa^{2(mn)i})a^{(mn)i})^k)\phi)^j = ((a^{4(mn)ik})\phi)^j \\ &= (a^{4(mn)ikj})\phi = (a\phi)^{4(mn)ikj} = f^{4(mn)ikj} = f.\end{aligned}$$

Therefore, $e\phi = f$. □

The proof of the following corollary immediately follows by the previous theorem.

Corollary 1. *Let $m, n \in \mathbf{Z}^+$. Every semigroup S for which $S/\overline{\tau}_{(m,n)}$ is π-regular is idempotent-consistent.*

The relation $\overline{\tau}_{(1,1)}$ we simply denote by $\overline{\tau}$. On a semigroup S this relation is defined by

$$(a,b) \in \overline{\tau} \iff (\forall x, y \in S)\ (xay, xby) \in \tau.$$

By Theorem 1 it is evident that:

Corollary 2. *Let S be an arbitrary semigroup, then $\overline{\tau}$ is a congruence relation on S.*

For $m = 1$ and $n = 1$ by previously obtained results we give the following corollaries which refer to the relation $\overline{\tau}$.

Corollary 3. *An element $a \in S$ is $\overline{\tau}$-regular if and only if a has a $\overline{\tau}$-inverse element.*

Corollary 4. *Let ϕ be a homomorphism from a semigroup S onto T and let $S/\overline{\tau}$ be a π-regular semigroup. Then for every $f \in E(T)$ there exists $e \in E(S)$ such that $e\phi = f$.*

Corollary 5. *Every semigroup S for which $S/\overline{\tau}$ is π-regular is idempotent-consistent.*

References

[1] Arens, R., Kaplansky, I., Topological representation of algebras. Trans. Amer. Math. Soc. 63 (1948), 457–481.

[2] Bogdanović, S., Bands of periodic power joined semigroups. Math. Sem. Notes Kobe Univ. 10 (1982), 667–670.

[3] S. Bogdanovic, Right π-inverse semigroups. Zb. rad. PMF Novi Sad 14 (2) (1984), 725–727.

[4] Bogdanović, S., Semigroups with a system of subsemigroups. Inst. of Math. Novi Sad, 1985.

[5] Bogdanović, S., Ćirić, M., Semigroups. Prosveta, Niš, 1993 [Serbian].

[6] Bogdanović, S., Popović, Ž., Ćirić, M., Bands of k-Archimedean semigroups. Semigroup Forum **80** (2010), 426–439.

[7] Bogdanović, S., Popović, Ž., Ćirić, M., Bands of η-simple semigroups (submitted for publication).

[8] Bogdanović, S., Popović, Ž., Ćirić, M., Semilattices of Archimedean semigroups viewed from certain congruence relations (submitted for publication).

[9] Edwards, P. M. Eventually regular semigroups, Bull. Austral. Math. Soc. 28 (1983), 23–38.

[10] Edwards, P. M., Higgins, P. M., Kopamu, S. J. L., Remarks concerning Lallement's Lemma and coextensions of eventually regular semigroups. Acta Sci. Math. (Szeged) 67 (2001), 105–120.

[11] Higgins, P. M., Techniques in Semigroup Theory. Oxford University Press, New York, 1992.

[12] Higgins, P. M., The convrese of Lallement's lemma. In: Semigroups and Applications (St Andrews, 1997), pp. 78–86, World Scientific, River Edge, NJ, 1998.

[13] Howie, J. M., An Introduction to Semigroup Theory. Academic Press, New York, 1976.

[14] Kopamu, S. J. L., On semigroup species. Comm. Algebra 23 (1995), 5513–5537.

[15] Kopamu, S. J. L., The concept of structural regularity. Portugaliae Math. 53 (4) (1996), 435–456.

[16] Mitsch, H., On the Lemma of Lallement. Comm. Algebra 24 (1996), 3085–3098.

[17] Murty, P., Ramana, V., Sudharani, K., On eventually regular semigroups (submitted for publication).

[18] von Neumann, J., On regular rings. Proc. Nat. Acad. Sci. USA 22 (1936), 707–713.

[19] Shevrin, L. N., Theory of epigroups I. Mat. Sb. 185 (1994), 129–160 [Russian].

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