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SOME FORCING RELATED CONVERGENCE STRUCTURES ON COMPLETE BOOLEAN ALGEBRAS

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Abstract. Let convergences $\lambda_i : \mathbb{B}^\omega \to P(\mathbb{B})$, $i \leq 4$, on a complete Boolean algebra $\mathbb{B}$ be defined in the following way. For a sequence $x = \langle x_n : n \in \omega \rangle$ in $\mathbb{B}$ and the corresponding $\mathbb{B}$-name for a subset of ω, $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$, let

$$\lambda_i(x) = \begin{cases} \{\|\tau_x \text{ is infinite}\|\} & \text{if } b_i(x) = 1_{\mathbb{B}}, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $b_1(x) = \|\tau_x \text{ is finite or cofinite}\|$, $b_2(x) = \|\tau_x \text{ is not unsupported}\|$, $b_3(x) = \|\tau_x \text{ is not a splitting real}\|$ and $b_4(x) = 1_{\mathbb{B}}$. Then λ_1 is the algebraic convergence generating the sequential topology on $\mathbb{B}$, while the convergences λ_2, λ_3 and λ_4, although different on each Boolean algebra producing splitting reals, generate the same topological convergence - a generalization of the convergence on the Aleksandrov cube, considered in [18].

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1. Introduction

In this paper we compare four convergence structures defined on complete Boolean algebras in terms of set-theoretic forcing. One is the algebraic convergence [20], [2] related to the von Neumann and the Maharam problem and generalizing the convergence on the Cantor cube. Another one is a generalization of the convergence on the Aleksandrov cube considered in [18].

In order to make the paper self-contained, in the first part of the paper we collect the relevant facts concerning convergence structures. Some of them are folklore, some scattered in the literature and, for more specific ones a reference is given, whenever it was available to the authors.

Our notation is mainly standard. So, ω denotes the set of natural numbers and Y^X denotes the set of all functions $f : X \to Y$. By $\omega^{\uparrow\omega}$ we denote the set of all strictly increasing functions from ω into ω. A **sequence** in a set X

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is each function $x : \omega \to X$. Then instead of $x(n)$ we usually write x_n and also $x = \langle x_n : n \in \omega \rangle$. If $x_n = a$, for each $n \in \omega$, the corresponding **constant sequence** will be denoted by $\langle a \rangle$. If $f \in \omega^{\uparrow\omega}$ then the sequence $y = x \circ f$ is said to be a **subsequence** of the sequence x, and we write $y \prec x$.

2. The topology induced by a convergence

If $\langle X, \mathcal{O} \rangle$ is a topological space, a point $a \in X$ is said to be a **limit point of a sequence** $x \in X^\omega$ (we will write: $x \to_{\mathcal{O}} a$) iff each neighborhood U of a contains all but finitely many members of the sequence. A space $\langle X, \mathcal{O} \rangle$ is called **sequential** iff a set $A \subset X$ is closed whenever it contains each limit of each sequence in A.

If X is a non-empty set, each mapping $\lambda : X^\omega \to P(X)$ will be called a **convergence** on X and the mapping $u_\lambda : P(X) \to P(X)$, defined by $u_\lambda(A) = \bigcup_{x \in A^\omega} \lambda(x)$, will be called the **operator of sequential closure** determined by λ. If $\lambda_1 : X^\omega \to P(X)$ is another convergence on X, then we will write $\lambda \leq \lambda_1$ iff $\lambda(x) \subset \lambda_1(x)$, for each sequence $x \in X^\omega$. Clearly, $\leq$ is a partial order on the set $\mathrm{Conv}(X) = \{\lambda : \lambda \text{ is a convergence on } X\}$.

Natural examples of these notions appear in general topology: if $\langle X, \mathcal{O} \rangle$ is a topological space, then the operator $\lim_{\mathcal{O}} : X^\omega \to P(X)$ defined by $\lim_{\mathcal{O}}(x) = \{a \in X : x \to_{\mathcal{O}} a\}$ is **the convergence on X determined by the topology $\mathcal{O}$**. In addition, we have the following fact (see [4]).

Fact 2.1. Let $\langle X, \mathcal{O} \rangle$ be a topological space and $\lambda = \lim_{\mathcal{O}}$. Then

(a) The operator λ satisfies the following conditions:
(L1) $\forall a \in X \;\; a \in \lambda(\langle a \rangle)$;
(L2) $\forall x \in X^\omega \; \forall y \prec x \; \lambda(x) \subset \lambda(y)$;
(L3) $\forall x \in X^\omega \; \forall a \in X \; ((\forall y \prec x \; \exists z \prec y \; a \in \lambda(z)) \Rightarrow a \in \lambda(x))$.

(b) For each subset A of X we have $A \subset u_\lambda(A) \subset \overline{A}$ and, consequently, if A is a closed set, then $u_\lambda(A) = A$.

(c) The space $\langle X, \mathcal{O} \rangle$ is sequential iff: $A \subset X$ is closed iff $u_\lambda(A) = A$.

(d) If $\mathcal{O}_1$ is another topology on X, then $\mathcal{O} \subset \mathcal{O}_1$ implies $\lim_{\mathcal{O}_1} \leq \lim_{\mathcal{O}}$.

(e) If $\mathcal{O}$ and $\mathcal{O}_1$ are sequential topologies and $\lim_{\mathcal{O}} = \lim_{\mathcal{O}_1}$, then $\mathcal{O} = \mathcal{O}_1$.

A convergence $\lambda : X^\omega \to P(X)$ is called a **topological convergence** iff there is a topology $\mathcal{O}$ on X such that $\lambda = \lim_{\mathcal{O}}$ [3]. Such a topology must not be unique as the following example shows.

Example 2.2. An infinite family of topologies having the same convergence of sequences. On the real line, the discrete and co-countable topology determine the same convergence of sequences: only almost-constant sequences converge. By Fact 2.1(d) the same holds for each topology between these two topologies.

[3] The problem of characterization of topological convergences was considered by Fréchet [6, 7], Urysohn [21] and, for the single-valued convergences, solved by Kisyński [13]. Concerning the multivalued convergences, several conditions for a convergence to be topological were obtained by many authors (see the papers of Antosik [1], Kamiński [10, 11, 12], Ferens, Kamiński and Kliś [5] and Koutník [14].)

Clearly, if $\lambda : X^\omega \to P(X)$ is a topological convergence, then Fact 2.1 holds for each topology $\mathcal{O}$ on X such that $\lambda = \lim_{\mathcal{O}}$.

If a convergence $\lambda : X^\omega \to P(X)$ is not topological, it can be extended to a topological one, namely there is a topology $\mathcal{O}$ on X such that $\lambda \leq \lim_{\mathcal{O}}$, that is

$$\forall x \in X^\omega \ \ \lambda(x) \subset \lim_{\mathcal{O}}(x). \tag{1}$$

Clearly, the antidiscrete topology $\mathcal{O}_{\mathrm{ad}}$ on X satisfies (1), because $\lim_{\mathcal{O}_{\mathrm{ad}}}(x) = X$, for each sequence x in X. By Fact 2.1(d), finer topologies produce smaller limits and, in fact, it is known that there is the maximal topology on X satisfying (1). This topology is described in the following theorem. Parts (b), (c) and (d) can be found in [11].

Theorem 2.3. Let $\lambda : X^\omega \to P(X)$ be a convergence on a non-empty set X. Then

(a) There is the maximal topology $\mathcal{O}_\lambda$ on X satisfying (1);
(b) $\mathcal{O}_\lambda = \{O \subset X : \forall x \in X^\omega \ (O \cap \lambda(x) \neq \emptyset \Rightarrow \exists n_0 \in \omega \ \forall n \geq n_0 \ x_n \in O)\}$;
(c) $\langle X, \mathcal{O}_\lambda \rangle$ is a sequential space;
(d) $\mathcal{O}_\lambda = \{X \setminus F : F \subset X \wedge u_\lambda(F) = F\}$, if λ satisfies (L1) and (L2);
(e) $\lim_{\mathcal{O}_\lambda} = \min\{\lambda' \in \mathrm{Conv}(X) : \lambda' \text{ is topological and } \lambda \leq \lambda'\}$;
(f) $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} = \mathcal{O}_\lambda$;
(g) If $\lambda_1 : X^\omega \to P(X)$ and $\lambda_1 \leq \lambda$, then $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda_1}$.

Proof. (a) Let Ω_λ be the set of all topologies $\mathcal{O}$ on X satisfying (1) and let $\mathcal{O}_\lambda$ be the topology on X generated by the subbase $\bigcup \Omega_\lambda$. It remains to be proved that $\mathcal{O}_\lambda \in \Omega_\lambda$. Let $x \in X^\omega$ and $a \in \lambda(x)$. If U is an open neighborhood of the point a in the space $\langle X, \mathcal{O}_\lambda \rangle$, then there is a finite subset $\{O_1, \ldots, O_k\}$ of $\bigcup \Omega_\lambda$ such that $a \in \bigcap_{i=1}^k O_i \subset U$. For $i \leq k$ let $\mathcal{O}_i$ be an element of Ω_λ such that $O_i \in \mathcal{O}_i$. Since $a \in \lambda(x) \subset \lim_{\mathcal{O}_i}(x)$, there is $n_i \in \omega$ such that $x_n \in O_i$, for each $n \geq n_i$. Thus, if $m = \max\{n_1, \ldots, n_k\}$, then $x_n \in \bigcap_{i=1}^k O_i \subset U$, for all $n \geq m$. So $a \in \lim_{\mathcal{O}_\lambda}(x)$ and we are done.

(b) Let $\mathcal{T}_\lambda$ denote the given family of subsets of X. First we prove that $\mathcal{T}_\lambda$ is a topology on X. Clearly $\emptyset, X \in \mathcal{T}_\lambda$.

Let $O_1, O_2 \in \mathcal{T}_\lambda$, and let x be a sequence in X. If $(O_1 \cap O_2) \cap \lambda(x) \neq \emptyset$, then there exist n_0^1 and n_0^2 such that for all $n \geq n_0^1$ we have $x_n \in O_1$, and for all $n \geq n_0^2$ we have $x_n \in O_2$. Therefore, for $n_0 = \max\{n_0^1, n_0^2\}$ and each $n \geq n_0$ we have that $x_n \in O_1 \cap O_2$, which proves that $O_1 \cap O_2 \in \mathcal{T}_\lambda$.

Let $O_i \in \mathcal{T}_\lambda$, $i \in I$. If $\bigcup_{i \in I} O_i \cap \lambda(x) \neq \emptyset$, then there exists i_0 such that $O_{i_0} \cap \lambda(x) \neq \emptyset$. Therefore, there exists $n_0^{i_0}$ such that for all $n \geq n_0^{i_0}$ we have that $x_n \in O_{i_0} \subset \bigcup_{i \in I} O_i$, which implies that $\bigcup_{i \in I} O_i \in \mathcal{T}_\lambda$.

Now we prove that $\mathcal{T}_\lambda$ satisfies (1). Let x be a sequence in X, $a \in \lambda(x)$ and $a \in O \in \mathcal{T}_\lambda$. Since $O \cap \lambda(x) \neq \emptyset$, there exists an n_0 such that $x_n \in O$, for $n \geq n_0$, thus, $a \in \lim_{\mathcal{T}_\lambda}(x)$.

Since the topology $\mathcal{T}_\lambda$ satisfies (1), by the maximality of $\mathcal{O}_\lambda$ we have $\mathcal{T}_\lambda \subset \mathcal{O}_\lambda$. Let us prove that each $O \in \mathcal{O}_\lambda$ belongs to $\mathcal{T}_\lambda$. Let $x \in X^\omega$ and $O \cap \lambda(x) \neq \emptyset$. By (1), for an $a \in O \cap \lambda(x)$ we have $a \in \lim_{\mathcal{O}_\lambda}(x)$. Therefore, there is n_0 such that $x_n \in O$, for each $n \geq n_0$, hence $O \in \mathcal{T}_\lambda$ indeed.

(c) Using (b), we show that the space $\langle X, \mathcal{T}_\lambda \rangle$ is sequential. Let $A \subset X$ and $\lim_{\mathcal{T}_\lambda}(y) \subset A$, for each sequence y in A. Suppose that $X \setminus A \notin \mathcal{T}_\lambda$. Then there are $x \in X^\omega$ and $b \in \lambda(x) \setminus A$ such that $x_n \in A$, for infinitely many $n \in \omega$ and, hence, x has a subsequence $y \in A^\omega$. Since $b \in \lambda(x)$, by (1) we have $b \in \lim_{\mathcal{T}_\lambda}(x)$, and, since $\lim_{\mathcal{T}_\lambda}$ satisfies (L2), we have $b \in \lim_{\mathcal{T}_\lambda}(y) \subset A$. A contradiction. Thus $X \setminus A \in \mathcal{T}_\lambda$, that is A is a closed set in the space $\langle X, \mathcal{T}_\lambda \rangle$.

(d) First we prove

Claim 1. If λ satisfies conditions (L1) and (L2) then
(i) $u_\lambda(\emptyset) = \emptyset$;
(ii) $A \subset u_\lambda(A)$;
(iii) $A \subset B \Rightarrow u_\lambda(A) \subset u_\lambda(B)$;
(iv) $u_\lambda(A \cup B) = u_\lambda(A) \cup u_\lambda(B)$.

Proof of Claim 1. The statements (i) and (iii) are obvious, (ii) follows from (L1) and (iii) implies $u_\lambda(A) \cup u_\lambda(B) \subset u_\lambda(A \cup B)$. If $a \in u_\lambda(A \cup B)$, then $a \in \lambda(x)$ for some $x \in (A \cup B)^\omega$. Clearly, there is a subsequence y of x such that $y \in A^\omega$ or $y \in B^\omega$ and, by (L2), we have $a \in \lambda(y)$. Thus $a \in u_\lambda(A)$ or $a \in u_\lambda(B)$ and (iv) is proved.

Let us prove that the family $\mathcal{F} = \{F \subset X : u_\lambda(F) = F\}$ satisfies the axioms for closed sets. By (i), (ii) and (iv) of Claim 1 we have $\emptyset, X \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite unions. If $F_i \in \mathcal{F}$, $i \in I$, then, by (ii), $\bigcap_{i \in I} F_i \subset u_\lambda(\bigcap_{i \in I} F_i)$. By (iii), for each $j \in I$ we have $u_\lambda(\bigcap_{i \in I} F_i) \subset u_\lambda(F_j) = F_j$, thus $u_\lambda(\bigcap_{i \in I} F_i) \subset \bigcap_{i \in I} F_i$, so $\bigcap_{i \in I} F_i \in \mathcal{F}$.

For a proof that $\mathcal{O} = \{X \setminus F : F \subset X \wedge u_\lambda(F) = F\} \subset \mathcal{O}_\lambda$ it is sufficient to show that $\mathcal{O}$ satisfies (1). So for $x \in X^\omega$ and $a \in \lambda(x)$ we show that $a \in \lim_{\mathcal{O}}(x)$. Let $a \in O \in \mathcal{O}$. Then $O = X \setminus F$ for some $F \subset X$ satisfying $u_\lambda(F) = F$. Suppose $x_n \in F$ for infinitely many $n \in \omega$. Then there is a subsequence y of x such that $y \in F^\omega$ and by (L2), $a \in \lambda(y) \subset u_\lambda(F) = F$ which is not true. Thus there is $n_0 \in \omega$ such that $x_n \in O$ for all $n \geq n_0$. Consequently, $a \in \lim_{\mathcal{O}}(x)$.

In order to prove that $\mathcal{O}_\lambda \subset \mathcal{O}$ we take $O \in \mathcal{O}_\lambda$ and show that $X \setminus O \in \mathcal{F}$ or, equivalently, $u_\lambda(X \setminus O) \cap O = \emptyset$. Suppose there is $a \in u_\lambda(X \setminus O) \cap O$. Then there is $x \in (X \setminus O)^\omega$ such that $a \in \lambda(x)$. Since $\mathcal{O}_\lambda$ satisfies (1) we have $a \in \lim_{\mathcal{O}_\lambda}(x)$. So $a \in O$ implies there is $n_0 \in \omega$ such that $x_n \in O$ for all $n \geq n_0$ which is impossible because $x \in (X \setminus O)^\omega$. Thus $X \setminus O \in \mathcal{F}$ that is $O \in \mathcal{O}$.

(e) Clearly $\lim_{\mathcal{O}_\lambda}$ is a topological convergence and, by (1), $\lambda \leq \lim_{\mathcal{O}_\lambda}$. If $\lambda' = \lim_{\mathcal{O}'}$ and $\lambda \leq \lambda'$, then, by the maximality of $\mathcal{O}_\lambda$, we have $\mathcal{O}' \subset \mathcal{O}_\lambda$ which, by Fact 2.1(d), implies $\lim_{\mathcal{O}_\lambda} \leq \lim_{\mathcal{O}'} = \lambda$.

(f) Applying (a) to the convergence $\lim_{\mathcal{O}_\lambda}$ we conclude that for each topology $\mathcal{O}$ on X satisfying $\lim_{\mathcal{O}_\lambda} \leq \lim_{\mathcal{O}}$ we have $\mathcal{O} \subset \mathcal{O}_{\lim_{\mathcal{O}_\lambda}}$ so, for $\mathcal{O} = \mathcal{O}_\lambda$ we obtain $\mathcal{O}_\lambda \subset \mathcal{O}_{\lim_{\mathcal{O}_\lambda}}$. On the other hand, since $\lambda \leq \lim_{\mathcal{O}_\lambda}$, we have $\Omega_{\lim_{\mathcal{O}_\lambda}} \subset \Omega_\lambda$ and since $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} \in \Omega_{\lim_{\mathcal{O}_\lambda}}$, we have $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} \in \Omega_\lambda$, which implies $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} \subset \mathcal{O}_\lambda$.

(g) Using notation of (a) we have $\Omega_\lambda \subset \Omega_{\lambda_1}$. Since $\mathcal{O}_\lambda \in \Omega_\lambda$ we have $\mathcal{O}_\lambda \in \Omega_{\lambda_1}$ thus $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda_1}$ by the maximality of $\mathcal{O}_{\lambda_1}$. □

If $\lambda : X^\omega \to P(X)$ is a convergence, then the topological convergence $\lim_{\mathcal{O}_\lambda}$ corresponding to the topology $\mathcal{O}_\lambda$ provided by Theorem 2.3 will be called the **a**

posteriori convergence determined by λ. It is natural to ask when does the equality $\lambda = \lim_{\mathcal{O}_\lambda}$ hold?

Theorem 2.4. A convergence $\lambda : X^\omega \to P(X)$ is topological iff $\lambda = \lim_{\mathcal{O}_\lambda}$.

Proof. The implication "$\Leftarrow$" is trivial. Let $\lambda = \lim_{\mathcal{O}}$ for some topology $\mathcal{O}$ on X. Since $\mathcal{O}_\lambda$ satisfies (1) we have $\lambda \leq \lim_{\mathcal{O}_\lambda}$. Since $\lambda \leq \lim_{\mathcal{O}}$, by the maximality of $\mathcal{O}_\lambda$ we have $\mathcal{O} \subset \mathcal{O}_\lambda$, which by Fact 2.1(d) implies $\lim_{\mathcal{O}_\lambda} \leq \lim_{\mathcal{O}} = \lambda$. □

Remark 2.5. If $\langle X, \mathcal{O}\rangle$ is a topological space and $\lim_{\mathcal{O}}$ the corresponding convergence, then the maximal topology $\mathcal{O}_{\lim_{\mathcal{O}}}$ provided by Theorem 2.3 can be finer than $\mathcal{O}$. (For example, if $\mathcal{O}$ is the co-countable topology on $\mathbb{R}$, then $\mathcal{O}_{\lim_{\mathcal{O}}}$ will be the discrete topology, see Example 2.2). But, by Theorem 2.4 we have $\lim_{\mathcal{O}} = \lim_{\mathcal{O}_{\lim_{\mathcal{O}}}}$, namely these two topologies have the same convergence of sequences.

Theorem 2.6. A space $\langle X, \mathcal{O}\rangle$ is sequential iff $\mathcal{O} = \mathcal{O}_\lambda$, where $\lambda = \lim_{\mathcal{O}}$. Consequently, in the set of topologies on X having the same convergence of sequences, λ, $\mathcal{O}_\lambda$ is the unique sequential topology.

Proof. The implication "$\Leftarrow$" follows from Theorem 2.3(c). Let $\mathcal{O}$ be a sequential topology. By Theorem 2.4 we have $\lim_{\mathcal{O}} = \lim_{\mathcal{O}_\lambda}$ and since $\mathcal{O}_\lambda$ is a sequential topology as well, by Fact 2.1(e) we have $\mathcal{O} = \mathcal{O}_\lambda$. □

A convergence $\lambda : X^\omega \to P(X)$ such that $|\lambda(x)| \leq 1$, for each $x \in X^\omega$ will be called a **single-valued convergence**. (Somewhere such convergences are called Hausdorff, but the topology generated by them must not be Hausdorff, see [4, 1.6.E] or [2].) By Fact 2.1(a) and the following theorem of Kisyński [13] (see also [4, 1.7.18–20]), a single-valued convergence λ is topological iff it satisfies conditions (L1)-(L3).

Theorem 2.7. Let λ be a single-valued convergence on X satisfying (L1)-(L3). Then $\mathcal{U}_\lambda = \{X \setminus F : F \subset X \wedge u_\lambda(F) = F\}$ is a sequential T_1 topology on X and $\lambda = \lim_{\mathcal{U}_\lambda}$.

By Theorem 2.3(d), the topology $\mathcal{U}_\lambda$ from the previous theorem is equal to $\mathcal{O}_\lambda$.

If $\lambda : X^\omega \to P(X)$ is a multi-valued convergence, then conditions (L1)-(L3) are not sufficient for λ to be a topological convergence or, equivalently, for the equality $\lambda = \lim_{\mathcal{O}_\lambda}$ (see Theorem 2.4). The following example showing this can be found in [8].

Example 2.8. A convergence satisfying (L1)-(L3) which is not a topological convergence. Let $X = \{1, 2, 3\}$ and, for a sequence $x = \langle x_n : n \in \omega\rangle \in X^\omega$ let $r(x) = \{k \in X : x_n = k$ for infinitely many $n \in \omega\}$. It is easy to check that the convergence $\lambda : X^\omega \to P(X)$ defined by

$$\lambda(x) = \begin{cases} \{1,2\} & \text{if } r(x) = \{1\}, \\ \{2,3\} & \text{if } r(x) = \{2\}, \\ \{3\} & \text{if } r(x) = \{3\}, \\ \{2\} & \text{if } r(x) = \{1,2\}, \\ \emptyset & \text{if } r(x) = \{1,3\}, \\ \{3\} & \text{if } r(x) = \{2,3\}, \\ \emptyset & \text{if } r(x) = \{1,2,3\}, \end{cases}$$

satisfies conditions (L1), (L2) and (L3) and we reconstruct the topology O_λ. By Theorem 2.3(d), $\mathcal{F}_\lambda = \{F \subset X : u_\lambda(F) = F\}$ is the corresponding family of closed sets. So, if $1 \in F \in \mathcal{F}_\lambda$, then $\lambda(\langle 1 \rangle) = \{1,2\} \subset u_\lambda(F) = F$ thus $2 \in F$. Consequently, $\{1\}, \{1,3\} \notin \mathcal{F}_\lambda$. Similarly $2 \in F \in \mathcal{F}_\lambda$ implies $3 \in F$ and hence $\{2\}, \{1,2\} \notin \mathcal{F}_\lambda$. Since $u_\lambda(\{3\}) = \bigcup_{x \in \{3\}^\omega} \lambda(x) = \lambda(\langle 3 \rangle) = \{3\}$ we have $\{3\} \in \mathcal{F}_\lambda$ and since $u_\lambda(\{2,3\}) = \bigcup_{x \in \{2,3\}^\omega} \lambda(x) = \{2,3\}$ we have $\{2,3\} \in \mathcal{F}_\lambda$. Thus $O_\lambda = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$.

Finally, since X is the only neighborhood of the point 3, we have $3 \in \lim_{\mathcal{O}_\lambda}(\langle 1 \rangle)$ although $3 \notin \lambda(\langle 1 \rangle)$, which implies that $\lambda \neq \lim_{\mathcal{O}_\lambda}$.

If a convergence $\lambda : X^\omega \to P(X)$ satisfies conditions (L1) and (L2), then the closure operator in the space $\langle X, \mathcal{O}_\lambda \rangle$ can be described in the following way.

Theorem 2.9. Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying (L1) and (L2) and let the mappings $u^\alpha : P(X) \to P(X)$, $\alpha \leq \omega_1$, be defined by recursion in the following way: for $A \subset X$

$u^0(A) = A$,

$u^{\alpha+1}(A) = u_\lambda(u^\alpha(A))$ and

$u^\gamma(A) = \bigcup_{\alpha<\gamma} u^\alpha(A)$, for limit $\gamma \leq \omega_1$.

Then u^{ω_1} is the closure operator in the space $\langle X, \mathcal{O}_\lambda \rangle$.

Proof. By Theorem 2.3(d), a set $F \subset X$ is closed in the space $\langle X, \mathcal{O}_\lambda \rangle$ iff $u_\lambda(F) = F$. Hence we show that for each $A \subset X$

(i) $A \subset u^{\omega_1}(A)$,

(ii) $u_\lambda(u^{\omega_1}(A)) = u^{\omega_1}(A)$,

(iii) $A \subset F = u_\lambda(F) \Rightarrow u^{\omega_1}(A) \subset F$.

By (L1) we have $A \subset u_\lambda(A)$, so for each $A \subset X$ and each $\alpha, \beta \leq \omega_1$ we have

$$\alpha < \beta \Rightarrow u^\alpha(A) \subset u^\beta(A). \tag{2}$$

Clearly, (i) is true. In (ii) we prove "$\subset$" only. Let $x = \langle x_n \rangle \in (u^{\omega_1}(A))^\omega$ and $a \in \lambda(x)$. For $n \in \omega$ we have $x_n \in \bigcup_{\alpha<\omega_1} u^\alpha(A)$, thus there is $\alpha_n < \omega_1$ such that $x_n \in u^{\alpha_n}(A)$. Let $\alpha < \omega_1$ where $\alpha_n < \alpha$, for all $n \in \omega$. Then by (2) $x \in (u^\alpha(A))^\omega$ and consequently $a \in u_\lambda(u^\alpha(A)) = u^{\alpha+1}(A) \subset u^{\omega_1}(A)$.

For a proof of (iii) we suppose $A \subset F = u_\lambda(F)$ and using induction we show that

$$\forall \alpha \leq \omega_1 \;\; u^\alpha(A) \subset F. \tag{3}$$

Clearly $u^0(A) \subset F$. Let $\alpha \leq \omega_1$ and $u^\beta(A) \subset F$, for all $\beta < \alpha$. If α is a limit ordinal, then clearly $u^\alpha(A) \subset F$. If $\alpha = \beta + 1$, then by the induction hypothesis $u^\beta(A) \subset F$, hence $u^\alpha(A) = u_\lambda(u^\beta(A)) \subset u_\lambda(F) = F$ so $u^\alpha(A) \subset F$ again and (3) is proved, which implies that $u^{\omega_1}(A) \subset F$. □

3. The closure of λ under (L1)-(L3)

By Theorems 2.3(d) and 2.9, if a convergence λ satisfies conditions (L1) and (L2), we obtain additional information about the topology $\mathcal{O}_\lambda$ and the a posteriori convergence $\lim_{\mathcal{O}_\lambda}$. If, in addition, λ is a single-valued convergence satisfying (L3), it is a topological convergence, that is $\lambda = \lim_{\mathcal{O}_\lambda}$, and $\mathcal{O}_\lambda$ is described in Theorem 2.7. So, if λ does not satisfy conditions (L1)-(L3), it is useful to find a new convergence producing the same topology and satisfying conditions (L1)-(L3), which can be written in the following form:

(L1) $\forall a \in X \;\; a \in \lambda(\langle a \rangle)$,
(L2) $\forall x \in X^\omega \; \forall f \in \omega^{\uparrow\omega} \; \lambda(x) \subset \lambda(x \circ f)$,
(L3) $\forall x \in X^\omega \; \forall a \in X \; ((\forall f \in \omega^{\uparrow\omega} \; \exists g \in \omega^{\uparrow\omega} \; a \in \lambda(x \circ f \circ g)) \Rightarrow a \in \lambda(x))$.

Theorem 3.1. Let $\lambda : X^\omega \to P(X)$ be a convergence. Then

(a) The convergence $\lambda' : X^\omega \to P(X)$ defined by

$$\lambda'(x) = \begin{cases} \lambda(x) \cup \{a\} & \text{if } x = \langle a \rangle, \text{ for some } a \in X, \\ \lambda(x) & \text{otherwise,} \end{cases}$$

is the minimal convergence satisfying (L1) and $\lambda \leq \lambda'$;

(b) If λ satisfies (L1), then the convergence $\bar{\lambda} : X^\omega \to P(X)$ defined by

$$\bar{\lambda}(y) = \bigcup\nolimits_{x \in X^\omega, f \in \omega^{\uparrow\omega}, y = x \circ f} \lambda(x)$$

is the minimal convergence satisfying (L1), (L2) and $\lambda \leq \bar{\lambda}$;

(c) If λ satisfies (L1) and (L2), the convergence $\lambda^* : X^\omega \to P(X)$ defined by

$$\lambda^*(y) = \bigcap\nolimits_{f \in \omega^{\uparrow\omega}} \bigcup\nolimits_{g \in \omega^{\uparrow\omega}} \lambda(y \circ f \circ g) \tag{4}$$

is the minimal convergence satisfying (L1)-(L3) and $\lambda \leq \lambda^*$;

(d) If λ is an arbitrary convergence, then λ'^{-*} is the minimal convergence $\geq \lambda$ satisfying (L1), (L2) and (L3) and we have $\lambda \leq \lambda' \leq \lambda'^{-} \leq \lambda'^{-*} \leq \lim_{\mathcal{O}_\lambda}$ and $\mathcal{O}_\lambda = \mathcal{O}_{\lambda'} = \mathcal{O}_{\lambda'^{-}} = \mathcal{O}_{\lambda'^{-*}}$.

Proof. (a) is evident.

(b) If $y \in X^\omega$, then $y = y \circ \mathrm{id}_\omega$, where $\mathrm{id}_\omega : \omega \to \omega$ is the identity mapping, so, by the definition of $\bar{\lambda}$, we have $\lambda(y) \subset \bar{\lambda}(y)$. Thus $\lambda \leq \bar{\lambda}$.

Since λ satisfies (L1) and $\lambda \leq \bar{\lambda}$, for each $a \in X$ we have $a \in \lambda(\langle a \rangle) \subset \bar{\lambda}(\langle a \rangle)$ thus $\bar{\lambda}$ satisfies (L1). In order to prove (L2) for $\bar{\lambda}$ we take $y \in X^\omega$ and $g \in \omega^{\uparrow\omega}$ and show that

$$\bar{\lambda}(y) \subset \bar{\lambda}(y \circ g). \tag{5}$$

Let $a \in \bar{\lambda}(y)$. Then there are $x \in X^\omega$ and $f \in \omega^{\uparrow\omega}$ such that $y = x \circ f$ and $a \in \lambda(x)$. But then $y \circ g = x \circ f \circ g$ and $f \circ g \in \omega^{\uparrow\omega}$, so $\lambda(x) \subset \bar{\lambda}(y \circ g)$ hence $a \in \bar{\lambda}(y \circ g)$ and (5) is proved.

For a proof of the minimality of $\bar{\lambda}$ suppose that $\lambda_1 : X^\omega \to P(X)$ satisfies (L1), (L2) and $\lambda \leq \lambda_1$. We prove that $\bar{\lambda} \leq \lambda_1$. Let $y \in X^\omega$ and $a \in \bar{\lambda}(y)$. Then there are $x \in X^\omega$ and $f \in \omega^{\uparrow\omega}$ such that $y = x \circ f$ and $a \in \lambda(x)$. Since $\lambda \leq \lambda_1$ we have $a \in \lambda_1(x)$, and, since λ_1 fulfills (L2), we have $\lambda_1(x) \subset \lambda_1(x \circ f) = \lambda_1(y)$ so $a \in \lambda_1(y)$. Thus $\bar{\lambda}(y) \subset \lambda_1(y)$ for all $y \in X^\omega$, that is $\bar{\lambda} \leq \lambda_1$.

(c) Let $a \in \lambda(y)$. If $f \in \omega^{\uparrow\omega}$, then for $g = \mathrm{id}_\omega$ we have $g \in \omega^{\uparrow\omega}$ and $\lambda(y \circ f \circ g) = \lambda(y \circ f)$. Since λ satisfies (L2) there holds $\lambda(y) \subset \lambda(y \circ f)$ and hence $a \in \lambda(y \circ f \circ g)$. Thus $a \in \lambda^*(y)$ and $\lambda \leq \lambda^*$ is proved.

Since λ satisfies (L1) and $\lambda \leq \lambda^*$, for each $a \in X$ we have $a \in \lambda(\langle a \rangle) \subset \lambda^*(\langle a \rangle)$ thus λ^* satisfies (L1).

In order to prove that λ^* satisfies (L2) we take $y \in X^\omega$, $a \in \lambda^*(y)$ and $h \in \omega^{\uparrow\omega}$ and show that $a \in \lambda^*(y \circ h)$ that is

$$\forall \varphi \in \omega^{\uparrow\omega} \;\; \exists g \in \omega^{\uparrow\omega} \;\; a \in \lambda(y \circ h \circ \varphi \circ g). \tag{6}$$

Since $a \in \lambda^*(y)$ there holds

$$\forall f \in \omega^{\uparrow\omega} \;\; \exists g \in \omega^{\uparrow\omega} \;\; a \in \lambda(y \circ f \circ g). \tag{7}$$

So, if $\varphi \in \omega^{\uparrow\omega}$, then $h \circ \varphi \in \omega^{\uparrow\omega}$ and by (7) for $f = h \circ \varphi$ there is $g \in \omega^{\uparrow\omega}$ such that $a \in \lambda(y \circ f \circ g) = \lambda(y \circ h \circ \varphi \circ g)$ and (6) is proved.

For a proof that λ^* satisfies (L3) we take $y \in X^\omega$, $a \in X$ and suppose that $\forall f \in \omega^{\uparrow\omega} \; \exists g \in \omega^{\uparrow\omega} \; a \in \lambda^*(y \circ f \circ g)$ or equivalently,

$$\forall f \in \omega^{\uparrow\omega} \;\; \exists g \in \omega^{\uparrow\omega} \;\; \forall F \in \omega^{\uparrow\omega} \;\; \exists G \in \omega^{\uparrow\omega} \;\; a \in \lambda(y \circ f \circ g \circ F \circ G). \tag{8}$$

We have to prove that $a \in \lambda^*(y)$, that is

$$\forall \varphi \in \omega^{\uparrow\omega} \;\; \exists \psi \in \omega^{\uparrow\omega} \;\; a \in \lambda(y \circ \varphi \circ \psi). \tag{9}$$

So, let $\varphi \in \omega^{\uparrow\omega}$. By (8), for $f = \varphi$ there is $g \in \omega^{\uparrow\omega}$ such that $\forall F \in \omega^{\uparrow\omega} \; \exists G \in \omega^{\uparrow\omega} \; a \in \lambda(y \circ \varphi \circ g \circ F \circ G)$ so in particular, for $F = \mathrm{id}_\omega$ there exists $G \in \omega^{\uparrow\omega}$ such that $a \in \lambda(y \circ \varphi \circ g \circ G)$. Clearly $\psi = g \circ G \in \omega^{\uparrow\omega}$ and $a \in \lambda(y \circ \varphi \circ \psi)$, which proves (9).

For a proof of the minimality of λ^* suppose that $\lambda_1 : X^\omega \to P(X)$ satisfies (L1)-(L3) and $\lambda \leq \lambda_1$. We prove that $\lambda^* \leq \lambda_1$. Let $y \in X^\omega$ and $a \in \lambda^*(y)$. Then $\forall f \in \omega^{\uparrow\omega} \; \exists g \in \omega^{\uparrow\omega} \; a \in \lambda(y \circ f \circ g)$. Since $\lambda \leq \lambda_1$ we have $\lambda(y \circ f \circ g) \subset \lambda_1(y \circ f \circ g)$ so $\forall f \in \omega^{\uparrow\omega} \; \exists g \in \omega^{\uparrow\omega} \; a \in \lambda_1(y \circ f \circ g)$. But, since λ_1 fulfills (L3), this implies $a \in \lambda_1(y)$. Thus $\lambda^*(y) \subset \lambda_1(y)$ for all $y \in X^\omega$, that is $\lambda^* \leq \lambda_1$.

(d) By Fact 2.1(a), the convergence $\lim_{\mathcal{O}_\lambda}$ satisfies conditions (L1), (L2) and (L3). So, since $\lambda \leq \lim_{\mathcal{O}_\lambda}$ and $\lim_{\mathcal{O}_\lambda}$ satisfies (L1), we have $\lambda' \leq \lim_{\mathcal{O}_\lambda}$. Since $\lim_{\mathcal{O}_\lambda}$ satisfies (L1) and (L2), by (b) we have $\lambda'^- \leq \lim_{\mathcal{O}_\lambda}$. Finally, by (c) we have $\lambda'^{-*} \leq \lim_{\mathcal{O}_\lambda}$. Thus $\lambda \leq \lambda' \leq \lambda'^- \leq \lambda'^{-*} \leq \lim_{\mathcal{O}_\lambda}$ which by Theorem 2.3(g) implies $\mathcal{O}_\lambda \supset \mathcal{O}_{\lambda'} \supset \mathcal{O}_{\lambda'^-} \supset \mathcal{O}_{\lambda'^{-*}} \supset \mathcal{O}_{\lim_{\mathcal{O}_\lambda}}$. But, by Theorem 2.3(g), we have $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} = \mathcal{O}_\lambda$ which gives the desired equality. $\Box$

4. Weakly-topological convergences

A convergence $\lambda : X^\omega \to P(X)$ will be called **weakly-topological** iff it satisfies conditions (L1) and (L2) and λ^* is a topological convergence.

Theorem 4.1. For a convergence $\lambda : X^\omega \to P(X)$ satisfying (L1) and (L2) the following conditions are equivalent:

(a) λ is a weakly topological convergence,

(b) $\lambda^* = \lim_{\mathcal{O}_{\lambda^*}}$,

(c) $\lambda^* = \lim_{\mathcal{O}_\lambda}$, that is for each $x \in X^\omega$ and $a \in X$

$$a \in \lim_{\mathcal{O}_\lambda}(x) \Leftrightarrow \forall y \prec x \ \ \exists z \prec y \ \ a \in \lambda(z).$$

Proof. (a) $\Leftrightarrow$ (b) is Theorem 2.4 and (b) $\Leftrightarrow$ (c) follows from Theorem 3.1, because $\lambda = \lambda' = \lambda'^-$. □

For a single-valued convergence λ conditions (L1) and (L2) imply that λ is a weakly-topological convergence. Namely, we have

Theorem 4.2. Let $\lambda : X^\omega \to P(X)$ be a single-valued convergence satisfying (L1) and (L2). Then

(a) λ^* is a single-valued convergence;

(b) $\lambda^* = \lim_{\mathcal{O}_\lambda}$, that is λ is a weakly-topological convergence.

Proof. (a) Let $x \in X^\omega$ and $a, b \in \lambda^*(x)$. Since $\mathrm{id}_\omega \in \omega^{\uparrow\omega}$, by (4), there exists $g_a \in \omega^{\uparrow\omega}$ such that $a \in \lambda(x \circ \mathrm{id}_\omega \circ g_a) = \lambda(x \circ g_a)$. Also, by (4), there exists g_b such that $b \in \lambda(x \circ g_a \circ g_b)$. Since $x \circ g_a \circ g_b \prec x \circ g_a$ and λ satisfies (L2) we have $a \in \lambda(x \circ g_a \circ g_b)$, so $|\lambda(x \circ g_a \circ g_b)| \leq 1$ implies $a = b$.

(b) By Theorem 3.1(d) we have $\mathcal{O}_\lambda = \mathcal{O}_{\lambda^*}$. By (a) and since λ^* satisfies (L1)-(L3), by Theorem 2.7 λ^* is a topological convergence, so, by Theorem 2.4, $\lambda^* = \lim_{\mathcal{O}_{\lambda^*}}$, that is $\lambda^* = \lim_{\mathcal{O}_\lambda}$. □

Example 4.3. A convergence satisfying (L1)-(L3) which is not weakly topological. The convergence λ defined in Example 2.8 satisfies (L1)-(L3) and, by Theorem 3.1(c), we have $\lambda^* = \lambda$. But λ is not a topological convergence.

5. Fréchet spaces and condition (L4)

A topological space $\langle X, \mathcal{O}\rangle$ is called a **Fréchet space** iff the closure of a set is equal to its sequential closure (i.e. $\overline{A} = \{a \in X : \exists x \in A^\omega \ a \in \lim_{\mathcal{O}}(x)\}$, for each $A \subset X$). It is known that each Fréchet space is sequential and that there is a Hausdorff sequential space which is not Fréchet (see [4, 1.6.19]).

Although each convergence of sequences produces a sequential space, for being Fréchet additional conditions are necessary [4].

[4] According to the results of Fréchet [6, 7], Urysohn [21] and Kisyński [13], a single-valued convergence is topological and produces a Fréchet topology iff it satisfies conditions (L1)-(L4). For multivalued convergence see the paper [8] of Gutierres and Hofmann.

Fact 5.1. (a) If $\langle X, \mathcal{O} \rangle$ is a Fréchet space and $\lambda = \lim_{\mathcal{O}}$, then

(L4) For each double sequence $\langle x_i^n : n, i \in \omega \rangle$ in X, each sequence $\langle x^n : n \in \omega \rangle$ in X and each $a \in X$ such that $x^n \in \lambda(\langle x_i^n : i \in \omega \rangle)$, for each $n \in \omega$ and $a \in \lambda(\langle x^n : n \in \omega \rangle)$ there is a sequence y in the set $\{x_i^n : n, i \in \omega\}$ such that $a \in \lambda(y)$.

(b) If $\lambda : X^\omega \to P(X)$ is a topological convergence such that $\langle X, \mathcal{O}_\lambda \rangle$ is a Fréchet space, then λ satisfies (L4).

Proof. (a) follows from the fact that each limit of a sequence in a set belongs to its closure. (b) follows from (a) and Theorem 2.4. □

Using Theorem 2.9 we obtain the following equivalents of condition (L4).

Theorem 5.2. Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying (L1) and (L2). Then the following conditions are equivalent

(a) $u_\lambda^2 = u_\lambda$;
(b) $u^{\omega_1} = u_\lambda$;
(c) λ satisfies (L4).

Proof. (a)⇒(b) Let $u_\lambda^2 = u_\lambda$ and $A \subset X$. Using induction it is easy to prove that $u^\alpha(A) = u_\lambda(A)$ for each $\alpha \in [1, \omega_1]$. Thus $u^{\omega_1}(A) = u_\lambda(A)$.

(b)⇒(c). Suppose that $u^{\omega_1} = u_\lambda$. Let $A = \{x_i^n : n, i \in \omega\} \subset X$ and $x = \langle x^n : n \in \omega \rangle$, where $x^n \in \lambda(\langle x_i^n : i \in \omega \rangle)$, $n \in \omega$, and let $a \in \lambda(x)$. Then $x^n \in u_\lambda(A)$, $n \in \omega$, thus $x \in u_\lambda(A)^\omega$ so $a \in \lambda(x) \subset u_\lambda(u_\lambda(A)) = u^{\omega_1}(u^{\omega_1}(A)) = u^{\omega_1}(A) = u_\lambda(A)$. Thus there is $y \in A^\omega$ such that $a \in \lambda(y)$.

(c)⇒(a). Suppose λ satisfies (L4). For $A \subset X$ we prove $u_\lambda(u_\lambda(A)) \subset u_\lambda(A)$. Let $a \in u_\lambda(u_\lambda(A))$. Then there is $x = \langle x^n : n \in \omega \rangle \in u_\lambda(A)^\omega$ such that $a \in \lambda(x)$. For each $n \in \omega$ we have $x^n \in u_\lambda(A)$ hence there is $\langle x_i^n : i \in \omega \rangle \in A^\omega$ such that $x^n \in \lambda(\langle x_i^n : i \in \omega \rangle)$. By (L4) there is $y \in \{x_i^n : n, i \in \omega\}^\omega \subset A^\omega$ such that $a \in \lambda(y)$ so, since $y \in A^\omega$, we have $a \in u_\lambda(A)$. □

Theorem 5.3. Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying (L1) and (L2). Then

(a) λ satisfies (L4) ⇒ $\langle X, \mathcal{O}_\lambda \rangle$ is a Fréchet space.
(b) λ satisfies (L4) ⇔ $\langle X, \mathcal{O}_\lambda \rangle$ is a Fréchet space, if λ is weakly-topological.

Proof. (a) Let $A \subset X$ and let $b \in \overline{A}$. By Theorems 2.9 and 5.2 we have $\overline{A} = u^{\omega_1}(A) = u_\lambda(A)$ so $b \in u_\lambda(A)$ and, hence, there is a sequence x in A such that $b \in \lambda(x) \subset \lim_{\mathcal{O}_\lambda}(x)$.

(b) Suppose that λ is a weakly-topological convergence and $\langle X, \mathcal{O}_\lambda \rangle$ a Fréchet space. Let $A = \{x_i^n : n, i \in \omega\} \subset X$ and $x = \langle x^n : n \in \omega \rangle$, where $x^n \in \lambda(\langle x_i^n : i \in \omega \rangle)$, $n \in \omega$, and let $a \in \lambda(x)$. Then, since $\lambda \leq \lim_{\mathcal{O}_\lambda}$ we have $x^n \in \lim_{\mathcal{O}_\lambda}(\langle x_i^n : i \in \omega \rangle)$, $n \in \omega$, and $a \in \lim_{\mathcal{O}_\lambda}(x)$. By Fact 5.1 there is a sequence y in A such that $a \in \lim_{\mathcal{O}_\lambda}(y)$. Since the convergence λ is weakly topological, by Theorem 4.1 there is $z \prec y$ such that $a \in \lambda(z)$. Clearly, z is a sequence in A. □

The following example shows that the converse of (a) of the previous theorem is not true.

Example 5.4. $\langle X, \mathcal{O}_\lambda \rangle$ is a Fréchet space, although λ satisfies (L1)-(L3) and does not satisfy (L4). Let $X = \{1,2,3\}$ and let λ be the convergence considered in Example 2.8. Since $\langle X, \mathcal{O}_\lambda \rangle$ is a first countable space, it is a Frechét space. But

$$u_\lambda(\{1\}) = \bigcup_{x \in \{1\}^\omega} \lambda(x) = \lambda(\langle 1 \rangle) = \{1,2\},$$

$$u_\lambda(u_\lambda(\{1\})) = \bigcup_{x \in \{1,2\}^\omega} \lambda(x) = \{1,2\} \cup \{2,3\} \cup \{2\} = \{1,2,3\}$$

and, hence, $u_\lambda^2 \neq u_\lambda$, so, by Theorem 5.2, λ does not satisfy (L4).

6. Forcing, sequences and reals

The assertions contained in the rest of the paper are mainly proved by the method of **forcing**. Roughly speaking, the forcing construction has the following steps. First, for a convenient complete Boolean algebra $\mathbb{B}$ belonging to the model V of ZFC in which we work (the **ground model**), the class $V^{\mathbb{B}}$ of $\mathbb{B}$**-names** (i.e. special $\mathbb{B}$-valued functions) is constructed by recursion. Second, for each ZFC formula $\varphi(v_0, \ldots, v_n)$ and arbitrary names $\tau_0, \ldots, \tau_n$ the **Boolean value** $\|\varphi(\tau_0, \ldots, \tau_n)\|$ is defined by recursion. Finally, if $G \subset \mathbb{B}$ is a $\mathbb{B}$**-generic filter over** V (i.e. G intersects all dense subsets of $\mathbb{B}^+$ belonging to V) then for each name τ the G**-evaluation** of τ, denoted by τ_G is defined by $\tau_G = \{\sigma_G : \sigma \in \mathrm{dom}(\tau) \wedge \tau(\sigma) \in G\}$ and $V_{\mathbb{B}}[G] = \{\tau_G : \tau \in V^{\mathbb{B}}\}$ is the corresponding **generic extension** of V, the minimal model of ZFC such that $V \subset V_{\mathbb{B}}[G] \ni G$. The properties of $V_{\mathbb{B}}[G]$ are controlled by the choice of $\mathbb{B}$ and G and by the **forcing relation** $\Vdash$ defined by

$$b \Vdash \varphi(\tau) \overset{def}{\Longleftrightarrow} \forall G \in \mathcal{G}_V^{\mathbb{B}} \, (b \in G \Rightarrow V_{\mathbb{B}}[G] \models \varphi(\tau_G)) \, .$$

(Here "$G \in \mathcal{G}_V^{\mathbb{B}}$" will be an abbreviation for "G is a $\mathbb{B}$-generic filter over V".) If $A \in V$, then there is a $\mathbb{B}$-name $\check{A} = \{\langle a, 1 \rangle : a \in A\}$ such that $(\check{A})_G = A$, in each extension $V_{\mathbb{B}}[G]$. A proof of the following statement can be found in [9].

Fact 6.1. If φ and ψ are ZFC formulas and $A \in V$, then
(a) $\|\varphi \wedge \psi\| = \|\varphi\| \wedge \|\psi\|$;
(b) $\|\neg\varphi\| = \|\varphi\|'$;
(c) $\|\forall x \, \varphi(x)\| = \bigwedge_{\tau \in V^{\mathbb{B}}} \|\varphi(\tau)\|$;
(d) $\|\forall x \in \check{A} \, \varphi(x)\| = \bigwedge_{a \in A} \|\varphi(\check{a})\|$;
(e) $b \Vdash \varphi$ if and only if $b \leq \|\varphi\|$;
(f) $1 \Vdash \varphi \Rightarrow \psi$ if and only if $\|\varphi\| \leq \|\psi\|$;
(g) If ZFC $\vdash \varphi(x)$, then $1 \Vdash \varphi(\tau)$, for each $\tau \in V^{\mathbb{B}}$;
(h) If $V_{\mathbb{B}}[G] \models \varphi$, then there is $b \in G$ such that $b \Vdash \varphi$;
(i) If $1 \Vdash \exists x \, \varphi(x)$, then $1 \Vdash \varphi(\tau)$, for some $\tau \in V^{\mathbb{B}}$ (The Maximum Principle).

Subsets of ω are called **reals** and can be coded by convenient names. Namely, if $x = \langle x_n : n \in \omega \rangle$ is a sequence in $\mathbb{B}$, then $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ is a $\mathbb{B}$-name, $1 \Vdash \tau_x \subset \check{\omega}$ and $\|\check{n} \in \tau_x\| = x_n$, for each $n \in \omega$. On the other hand, if $r \in P(\omega) \cap V_{\mathbb{B}}[G]$, then $r = \tau_G$ for some $\tau \in V^{\mathbb{B}}$ and there is $b \in G$ such that

$b \Vdash \tau \subset \check{\omega}$. If we define $x_n = \|\check{n} \in \tau\|$, $n \in \omega$, then $b \Vdash \tau = \tau_x$, so each real belonging to $V_{\mathbb{B}}[G]$ can be represented by a **nice name** of the form τ_x.

A real $r \in [\omega]^\omega \cap V_{\mathbb{B}}[G]$ will be called: **new** iff $r \notin V$; **dependent** iff there is $A \in [\omega]^\omega \cap V$ such that $A \subset r$ or $A \subset \omega \setminus r$; **independent** or a **splitting real** iff it is not dependent [16]; **supported** iff there is $A \in [\omega]^\omega \cap V$ such that $A \subset r$; **unsupported** iff it is not supported [15].

Theorem 6.2. Let $x = \langle x_n : n \in \omega \rangle$ be a sequence in a complete Boolean algebra $\mathbb{B}$, $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ and B an infinite subset of ω. Then

(a) $\|\tau_x = \check{\omega}\| = \bigwedge_{n\in\omega} x_n$;

(b) $\|\tau_x$ is cofinite $\| = \bigvee_{k\in\omega} \bigwedge_{n\geq k} x_n$ $(= \liminf x)$;

(c) $\|\tau_x$ is supported $\| = \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} x_n$;

(d) $\|\tau_x$ is infinite $\| = \bigwedge_{k\in\omega} \bigvee_{n\geq k} x_n$ $(= \limsup x)$;

(e) $\|\check{B} \subset^* \tau_x\| = \bigvee_{k\in\omega} \bigwedge_{n\in B\setminus k} x_n$ $(= \liminf_{n\in B} x_n)$;

(f) $\||\tau_x \cap \check{B}| = \check{\omega}\| = \bigwedge_{k\in\omega} \bigvee_{n\in B\setminus k} x_n$ $(= \limsup_{n\in B} x_n)$;

(g) $\|\tau_x = \check{\omega}\| \leq \|\tau_x$ is cofinite$\| \leq \|\tau_x$ is old infinite$\| \leq \|\tau_x$ is supported$\| \leq \|\tau_x$ is infinite dependent$\| \leq \|\tau_x$ is infinite$\|$;

(h) $\|\tau_x$ is cofinite$\| \leq \|\check{B} \subset^* \tau_x\| \leq \||\tau_x \cap \check{B}| = \check{\omega}\| \leq \|\tau_x$ is infinite$\|$.

Proof. (c) By Fact 6.1, $\|\tau_x$ is supported$\| = \|\exists A \in ([\omega]^\omega)^{V\check{}}\ \forall n \in A\ n \in \tau_x\| = \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} \|\check{n} \in \tau_x\| = \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} x_n$. The proof of the rest is similar.

(g) Clearly, $X = \omega \Rightarrow X$ is cofinite $\Rightarrow X$ is old infinite $\Rightarrow X$ is supported $\Rightarrow X$ is infinite dependent $\Rightarrow X$ is infinite. Now we apply Fact 6.1(f) and (g).

(h) X is cofinite $\Rightarrow B \subset^* X \Rightarrow X \cap B$ is infinite $\Rightarrow X$ is infinite. □

Lemma 6.3. If $x = \langle x_n : n \in \omega \rangle$ is a sequence in a c.B.a. $\mathbb{B}$ and $f \in \omega^{\uparrow\omega}$, then $y = x \circ f$ is a subsequence of x and for the $\mathbb{B}$-names τ_x and $\tau_{x\circ f}$ we have

(a) $1 \Vdash \tau_{x\circ f} = f^{-1}[\tau_x]$;

(b) $\limsup x \circ f = \||f[\omega]\check{} \cap \tau_x| = \check{\omega}\|$;

(c) $\liminf x \circ f = \|f[\omega]\check{} \subset^* \tau_x\|$;

(d) $\liminf x \leq \liminf y \leq \limsup y \leq \limsup x$.

If by x' we denote the sequence $\langle x'_n : n \in \omega \rangle$, then

(e) $1 \Vdash \tau_{x'} = \check{\omega} \setminus \tau_x$.

Proof. (a) Suppose $G \in \mathcal{G}_V^{\mathbb{B}}$. Then $n \in (\tau_{x\circ f})_G$ iff $x_{f(n)} \in G$ iff $f(n) \in (\tau_x)_G$ iff $n \in f^{-1}[(\tau_x)_G]$.

(b) By (a) and Theorem 6.2(d) we have $\limsup x \circ f = \||\tau_{x\circ f}| = \check{\omega}\| = \||f^{-1}[\tau_x]| = \check{\omega}\| = \||f[\omega]\check{} \cap \tau_x| = \check{\omega}\|$, since f is an injection.

The proof of (c) is similar and (d) follows from (b) and (c).

(e) is true since $n \in (\tau_{x'})_G$ iff $x'_n \in G$ iff $x_n \notin G$ iff $n \in \omega \setminus (\tau_x)_G$. □

We will use the following well-known fact (see [9]).

Fact 6.4. A c.B.a. $\mathbb{B}$ does not add new reals by forcing iff $\mathbb{B}$ is $(\omega, 2)$-distributive.

7. Convergence structures on Boolean algebras

If $\mathbb{B}$ is a Boolean algebra and $A \subset \mathbb{B}$ let $A\uparrow = \{b \in \mathbb{B} : \exists a \in A\, a \leq b\}$. We will say that a set A is **upward closed** iff $A = A\uparrow$. For simplicity, for a sequence $x = \langle x_n : n \in \omega \rangle$ in $\mathbb{B}$ we introduce the following notation:

$$\begin{aligned} v_1(x) &= \|\tau_x \text{ is cofinite}\| = \liminf x, \\ v_2(x) &= \|\tau_x \text{ is supported}\|, \\ v_3(x) &= \|\tau_x \text{ is infinite dependent}\|, \\ v_4(x) &= \|\tau_x \text{ is infinite}\| = \limsup x, \end{aligned}$$

where $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ is the $\mathbb{B}$-name for a real corresponding to x. By Theorem 6.2 we have

$$v_1(x) \leq v_2(x) \leq v_3(x) \leq v_4(x), \tag{10}$$

and we define convergences $\lambda_i : \mathbb{B}^\omega \to P(\mathbb{B})$, $i \leq 4$, on $\mathbb{B}$ by

$$\lambda_i(x) = \begin{cases} \{v_4(x)\} & \text{if } v_i(x) = v_4(x) \\ \emptyset & \text{if } v_i(x) < v_4(x) \end{cases} \tag{11}$$

Using (10), (11) and Theorem 2.3(g) we easily prove

Theorem 7.1. (a) $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$;
(b) $\mathcal{O}_{\lambda_4} \subset \mathcal{O}_{\lambda_3} \subset \mathcal{O}_{\lambda_2} \subset \mathcal{O}_{\lambda_1}$.

The convergence λ_1

First we give a forcing characterization of this convergence.

Theorem 7.2. If $\mathbb{B}$ is a complete Boolean algebra, then for each sequence x in $\mathbb{B}$

$$\lambda_1(x) = \begin{cases} \{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is finite or cofinite}, \\ \emptyset & \text{otherwise}. \end{cases} \tag{12}$$

Proof. By the definition of λ_1 and Fact 6.1 we have

$$\begin{aligned} \lambda_1(x) \neq \emptyset &\Leftrightarrow \|\tau_x \text{is cofinite}\| = \|\tau_x \text{ is infinite}\| \\ &\Leftrightarrow \|\tau_x \text{is infinite}\| \leq \|\tau_x \text{ is cofinite}\| \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is infinite} \Rightarrow \tau_x \text{ is cofinite} \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is finite or cofinite}. \end{aligned}$$

□

The convergence λ_1 is the well known **algebraic convergence**, related to the von Neumann - Maharam problem (see [20]) and generates the **sequential topology** on $\mathbb{B}$, usually denoted by τ_s (see [2]). It is known that

- λ_1 satisfies (L1) and (L2), it is single-valued and, by Theorem 4.2, weakly-topological.
- λ_1 is a topological convergence iff it satisfies (L3) (see Theorem 2.7) iff the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive (see [17]).
- λ_1 generates a Fréchet topology iff the algebra $\mathbb{B}$ is weakly-distributive and $\mathfrak{b}$-cc, where $\mathfrak{b}$ is the bounding number (see [3]).
- $\lim_{\mathcal{O}_{\lambda_1}} = a \Rightarrow a = a_x = b_x$ (see [17]), where

$$\begin{aligned} a_x &= \bigwedge_{A\in[\omega]^\omega} \bigvee_{B\in[A]^\omega} \bigwedge_{n\in B} x_n, \\ b_x &= \bigvee_{A\in[\omega]^\omega} \bigwedge_{B\in[A]^\omega} \bigvee_{n\in B} x_n. \end{aligned}$$

- $\lim_{\mathcal{O}_{\lambda_1}} = a \Leftrightarrow a = a_x = b_x$, in Boolean algebras satisfying condition ($\hbar$) (see [17]) given by

$$\forall x \in \mathbb{B}^\omega \ \exists y \prec x \ \forall z \prec y \ \ \limsup z = \limsup y$$

  More about condition ($\hbar$) (implied by the ccc) can be found in [19].

The convergence λ_4

Since $\lambda_4(x) = \{\limsup x\}$, for each sequence x in $\mathbb{B}$, the convergence λ_4 satisfies condition (L1).

Example 7.3. λ_4 does not satisfy (L2). For the sequence $x = \langle 0, 1, 0, 1, \ldots\rangle$ we have $\lambda_4(x) = \{1\}$ but for its subsequence $y = \langle 0, 0, 0, \ldots\rangle$ we have $\lambda_4(y) = \{0\} \not\ni 1$.

Theorem 7.4. The closure of the convergence λ_4 under (L2) is given by

$$\bar{\lambda}_4(y) = \{\limsup y\}\uparrow .$$

Proof. By Theorem 3.1, we prove that, for each sequence $y = \langle y_n : n \in \omega\rangle$ in $\mathbb{B}$

$$\bigcup_{x\in\mathbb{B}^\omega,\ f\in\omega^{\uparrow\omega},\ y=x\circ f} \lambda_4(x) = \{\limsup y\}\uparrow .$$

($\subset$) Suppose that $x \in \mathbb{B}^\omega$, $f \in \omega^{\uparrow\omega}$, $y = x \circ f$ and $b \in \lambda_4(x)$, that is, $b = \limsup x$. Since $y \prec x$, by Lemma 6.3(d) we have $\limsup y \leq \limsup x = b$, which implies $b \in \{\limsup y\}\uparrow$.

($\supset$) Let $b \geq \limsup y$. Let $x = \langle y_0, b, y_1, b, y_2, \ldots\rangle$ and $f, g \in \omega^{\uparrow\omega}$, where $f(k) = 2k$ and $g(k) = 2k + 1$, for $k \in \omega$. Then $y = x \circ f$ and $z = x \circ g = \langle b\rangle$. By Theorem 6.2(d) and Lemma 6.3(b) we have

$$\begin{aligned} \limsup x &= \| |\tau_x| = \check{\omega} \| = \| |\tau_x \cap \check{f}[\omega]| = \check{\omega} \| \vee \| |\tau_x \cap \check{g}[\omega]| = \check{\omega} \| \\ &= \| |\tau_y| = \check{\omega} \| \vee \| |\tau_z| = \check{\omega} \| = \| |\tau_y| = \check{\omega} \| \vee b = \limsup y \vee b = b \end{aligned}$$

and, hence, $b \in \lambda_4(x)$. Thus $b \in \bigcup_{x\in\mathbb{B}^\omega,\ f\in\omega^{\uparrow\omega},\ y=x\circ f} \lambda_4(x)$. $\square$

The convergence $\bar\lambda_4$, generalizing the convergence on the Aleksandrov cube, was investigated in [18]. In particular, it is shown that

- $\bar\lambda_4$ is a topological convergence iff $\mathbb{B}$ is $(\omega, 2)$-distributive,
- $\bar\lambda_4$ is a weakly topological convergence if $\mathbb{B}$ satisfies condition $(\hbar)$ or if $\bar\lambda_4$ satisfies (L4),
- $\mathcal{O}_{\lambda_4}$ is a T_0 connected compact topology on $\mathbb{B}$.
- $\mathcal{O}_{\lambda_4}$ and its dual generate the sequential topology, when $\mathbb{B}$ is a Maharam algebra.

The convergences λ_2 and λ_3

First we give a forcing characterization of these convergences.

Theorem 7.5. If $\mathbb{B}$ is a complete Boolean algebra, then for each sequence x in $\mathbb{B}$

$$(13)\qquad \lambda_2(x) = \begin{cases} \{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is finite or supported,} \\ \emptyset & \text{otherwise.} \end{cases}$$

$$(14)\qquad \lambda_3(x) = \begin{cases} \{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is not splitting,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. By the definition of λ_2 and λ_3 and Fact 6.1 we have

$$\begin{aligned}
\lambda_2(x) \neq \emptyset &\Leftrightarrow \|\tau_x \text{is supported}\| = \|\tau_x \text{ is infinite}\| \\
&\Leftrightarrow \|\tau_x \text{is infinite}\| \leq \|\tau_x \text{ is supported}\| \\
&\Leftrightarrow 1 \Vdash \tau_x \text{ is infinite} \Rightarrow \tau_x \text{ is supported} \\
&\Leftrightarrow 1 \Vdash \tau_x \text{ is finite or supported;} \\
\lambda_3(x) \neq \emptyset &\Leftrightarrow \|\tau_x \text{is infinite dependent}\| = \|\tau_x \text{ is infinite}\| \\
&\Leftrightarrow \|\tau_x \text{is infinite}\| \leq \|\tau_x \text{ is infinite dependent }\| \\
&\Leftrightarrow 1 \Vdash \tau_x \text{ is infinite} \Rightarrow \tau_x \text{ is infinite dependent} \\
&\Leftrightarrow 1 \Vdash \tau_x \text{ is finite} \vee \tau_x \text{ is infinite dependent} \\
&\Leftrightarrow 1 \Vdash \tau_x \text{ is not splitting.}
\end{aligned}$$

$\square$

Theorem 7.6. For each complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent:

(a) $\mathbb{B}$ is $(\omega, 2)$-distributive;
(b) $1 \Vdash \forall r \subset \check\omega\ (r \text{ is supported})$;
(c) $\lambda_2 = \lambda_4$;
(d) $\lambda_2 = \lambda_3$.

Proof. (a) $\Leftrightarrow$ (b). This is a well known fact (see [15]).

(b) $\Rightarrow$ (c). Suppose that (b) holds and $\lambda_2 < \lambda_4$. Then there is a sequence x in $\mathbb{B}$ such that $\lambda_2(x) = \emptyset$ and, by Theorem 7.5, $b \Vdash$ "τ_x is unsupported", for some $b \in \mathbb{B}^+$. A contradiction.

(c) $\Rightarrow$ (d). This follows from Theorem 7.1(a).

(d) $\Rightarrow$ (b). Let $\lambda_2 = \lambda_3$. Suppose that there is an extension $V_{\mathbb{B}}[G]$ and $X \in V_{\mathbb{B}}[G] \cap P(\omega)$ such that X is unsupported. Then there is a $\mathbb{B}$-name σ such that $X = \sigma_G$ and

$$1 \Vdash \sigma \subset \check{\omega}. \tag{15}$$

For the function $f : \omega \to \omega$, defined by $f(k) = 2k$, we have

$$1 \Vdash \check{f}[\sigma] \cap \{1, 3, 5, \dots\}\check{} = \emptyset. \tag{16}$$

Since $f \in V$, the set $f[\sigma_G]$ is unsupported and, by Fact 6.1(h), there is $b \in G$ such that

$$b \Vdash \check{f}[\sigma] \text{ is unsupported.} \tag{17}$$

For $n \in \omega$, let us define $x_n = \|\check{n} \in \check{f}[\sigma]\|$, let $x = \langle x_n : n \in \omega\rangle$ and $\tau_x = \{\langle \check{n}, x_n\rangle : n \in \omega\}$. By (15) we have $1 \Vdash \check{f}[\sigma] \subset \check{\omega}$ and, hence,

$$1 \Vdash \check{f}[\sigma] = \tau_x. \tag{18}$$

By (16) and (18), $1 \Vdash$ "τ_x is not splitting", which implies $\lambda_3(x) \neq \emptyset$. By (17) and (18) we have $b \Vdash$ "τ_x is unsupported" and, hence, $\lambda_2(x) = \emptyset$. Thus $\lambda_2 \neq \lambda_3$, a contradiction. □

Theorem 7.7. For each complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent:

(a) Forcing by $\mathbb{B}$ does not produce splitting reals;

(b) $\lambda_3 = \lambda_4$.

Proof. By Theorem 7.5, $\lambda_3 = \lambda_4$ iff $1 \Vdash$ "τ_x is not splitting", for each sequence x in $\mathbb{B}$. Since each real produced by forcing is coded by a nice name determined by a sequence in $\mathbb{B}$, the proof is over. □

Concerning the inequalities $\lambda_2 \leq \lambda_3 \leq \lambda_4$ we note that, by Theorem 7.6, $\lambda_2 = \lambda_3 < \lambda_4$ is impossible. In the following examples we show that, up to this restriction, everything is possible.

Example 7.8. $\lambda_2 = \lambda_3 = \lambda_4$. This holds in each $(\omega, 2)$-distributive and, in particular, in each atomic complete Boolean algebra.

Example 7.9. $\lambda_2 < \lambda_3 = \lambda_4$. This holds in each complete Boolean algebra which produces new reals, but does not produce splitting reals, for example in r.o.$(\mathbb{P})$, where $\mathbb{P}$ is the Sacks or the Miller forcing.

Example 7.10. $\lambda_2 < \lambda_3 < \lambda_4$. This holds in each complete Boolean algebra which produces splitting reals, for example in r.o.($\mathbb{P}$), where $\mathbb{P}$ is the Cohen or the random forcing.

The convergence λ_2 satisfies condition (L1) because for a constant sequence $\langle b : n \in \omega \rangle$ in $\mathbb{B}$ we have $\| \tau_{\langle b \rangle}$ is infinite$\| = \| \tau_{\langle b \rangle}$ is supported$\| = b$, thus $\lambda_2(\langle b \rangle) = \{b\}$. Since $\lambda_2 \leq \lambda_3$, the convergence λ_3 satisfies (L1) as well.

Example 7.11. λ_2 does not satisfy (L2). Namely, if $x = \langle 0, 1, 0, 1, \ldots \rangle$, then $1 \Vdash \tau_x = \{1, 3, 5, \ldots\}\check{}$ and we have $\| |\tau_x| = \check{\omega} \| = \| \tau_x$ is supported$\| = 1$, so $\lambda_2(x) = \{1\}$. But, for $y = \langle 0, 0, 0, \ldots \rangle \prec x$ we have $\lambda_2(y) = \{0\} \not\ni 1$.

Theorem 7.12. The closure of the convergence λ_2 under (L2) is given by

$$\bar{\lambda}_2(y) = \{\limsup y\} \uparrow .$$

Proof. ($\subset$) Since $\lambda_2 \leq \lambda_4 \leq \bar{\lambda}_4$, by Theorems 3.1 and 7.4 we have $\bar{\lambda}_2(y) \subset \bar{\lambda}_4(y) = \{\limsup y\} \uparrow$.

($\supset$) Let $b \in \{\limsup y\} \uparrow$. By Theorem 3.1, we prove that

$$b \in \bigcup_{x \in \mathbb{B}^\omega,\ f \in \omega^{\uparrow \omega},\ y = x \circ f} \lambda_2(x).$$

For $x = \langle y_0, b, y_1, b, y_2, \ldots \rangle$ and $f \in \omega^{\uparrow \omega}$, defined by $f(k) = 2k$ we have $y = x \circ f$. By Theorem 7.4, $\| |\tau_x| = \check{\omega} \| = b$. Since $b \Vdash \check{N} \subset \tau_x$, where N is the set of odd numbers, we have $b \Vdash \tau_x$ is supported, so $b \leq \| \tau_x$ is supported$\|$. Thus $b \leq \| \tau_x$ is supported$\| \leq \| |\tau_x| = \check{\omega} \| = b$ and, hence, $b \in \lambda_2(x)$. □

Theorem 7.13. (a) $\bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4$;

(b) $\mathcal{O}_{\lambda_2} = \mathcal{O}_{\lambda_3} = \mathcal{O}_{\lambda_4}$.

Proof. (a) By Theorem 7.1(a) we have $\lambda_2 \leq \lambda_3 \leq \lambda_4$ and, by Theorem 3.1(b), $\bar{\lambda}_2 \leq \bar{\lambda}_3 \leq \bar{\lambda}_4$. By Theorems 7.4 and 7.12 we have $\bar{\lambda}_2 = \bar{\lambda}_4$ and (a) is proved.

(b) follows from (a) and Theorem 3.1(d). □

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