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HOW TO DEVELOP FOURTH AND SEVENTH ORDER ITERATIVE METHODS?

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Abstract. The work contributes two new iterative methods of convergence orders four and seven for solving nonlinear equations. During each iteration, the fourth order method requests three functional evaluations while the seventh order method requests four functional evaluations. Computational results demonstrate that the methods are efficient and exhibit equal or better performance as compared with other well known methods and the classical Newton method.

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1. Introduction

Many problems in science and engineering require solving the nonlinear equation

$$f(x) = 0, \tag{1}$$

[1–13]. One of the best known and probably the most used method for solving the preceding equation is the Newton's method. The classical Newton method is given as follows (**NM**):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3, \ldots, \quad \text{and} \quad |f'(x_n)| \neq 0. \tag{2}$$

The Newton's method converges quadratically [1–13]. There exist numerous modifications of the Newton's method which improve the convergence rate (see [1–13] and references therein). In this work, we develop two new iterative methods of fourth and seventh orders. The fourth order iterative method require three functional evaluations during each iteration, while the seventh order method needs four functional evaluations during each iteration. To construct the methods, we express the derivative at the next step as a linear combination of derivatives at the previous steps, and slopes. The next section presents our contribution.

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2. Construction of new iterative methods

Consider the double Newton method

$$\left\{ \begin{array}{lcl} y_n & = & x_n - f(x_n)/f'(x_n), \\ x_{n+1} & = & y_n - f(y_n)/f'(y_n). \end{array} \right. \tag{3}$$

It is known that the double Newton method converges with fourth order. According to the Kung and Traub conjecture [7] an optimal iterative method without memory based on n evaluations could achieve an optimal convergence order of 2^{n-1}. Since the double Newton method converges with fourth order and it requires four evaluations during each step. Therefore for the double Newton method to be optimal it must require only three function evaluations. Our aim is to develop an approximation for $f'(y_n)$ in terms of $f(x_n)$, $f'(x_n)$ and $f(y_n)$. Let us express $f'(y_n)$ as a linear combination of $f'(x_n)$ (slope at the point x_n) and $(f(y_n) - f(x_n))/(y_n - x_n)$ (slope of the line joining the points x_n and y_n)

$$f'(y_n) = \alpha\, f'(x_n) + (1-\alpha)\, \frac{f(y_n) - f(x_n)}{y_n - x_n}, \tag{4}$$

where α is a real number. Combining the preceding equation and the equation (3), we propose the following iterative method (**M-1**)

$$\left\{ \begin{array}{lcl} y_n & = & x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = & y_n - \dfrac{f(y_n)}{\left(\alpha\, f'(x_n) + (1-\alpha)\, \dfrac{f(y_n) - f(x_n)}{y_n - x_n}\right)}. \end{array} \right. \tag{5}$$

We prove the fourth order convergent behavior of the iterative families (5) through the following theorem.

Theorem 1. *Let γ be a simple zero of a sufficiently differentiable function $f\colon \mathbf{D} \subset \mathbf{R} \mapsto \mathbf{R}$ in an open interval $\mathbf{D}$. If x_0 is sufficiently close to γ, the convergence order of the method (5) is 4 if and only if $\alpha = -1$. The error equation for the method is given as*

$$e_{n+1} = -\frac{\left(c_3 c_1 - c_2^2\right) c_2}{c_1^3} e_n^4 + O\left(e_n^5\right).$$

Here, $e_n = x_n - \gamma$, $c_m = f^m(\gamma)/m!$ with $m \geq 1$.

Proof. The Taylor's expansion of $f(x)$ and $f'(x_n)$ around the solution γ is given as

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O\left(e_n^5\right), \tag{6}$$

$$f'(x_n) = c_1 + 2\, c_2 e_n + 3\, c_3 e_n^2 + 4\, c_4 e_n^3 + O\left(e_n^4\right). \tag{7}$$

Here, we have accounted for $f(\gamma) = 0$. Dividing the equations (6) and (7) we obtain

(8)
$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{c_2}{c_1}{e_n}^2 - 2\,\frac{c_3c_1 - {c_2}^2}{{c_1}^2}{e_n}^3 - \frac{3\,c_4{c_1}^2 - 7\,c_3c_2c_1 + 4\,{c_2}^3}{{c_1}^3}{e_n}^4 + O\left({e_n}^5\right).$$

By the Taylor's expansion of $f(y_n)$ around x_n and using the first step of the method (5), we get

$$f(y_n) = f(x_n) + f'(x_n)\left(-\frac{f(x_n)}{f'(x_n)}\right) + \frac{1}{2}f''(x_n)\left(-\frac{f(x_n)}{f'(x_n)}\right)^2 + \cdots, \tag{9}$$

the successive derivatives of $f(x_n)$ are obtained by differentiating (7) repeatedly. Substituting these derivatives and using the equations (8) in the former equation

$$f(y_n) = c_2e_n^2 + 2\,\frac{c_3c_1 - {c_2}^2}{c_1}e_n^3 + \frac{3\,c_4{c_1}^2 - 7\,c_2c_3c_1 + 5\,{c_2}^3}{{c_1}^2}e_n^4 + O\left(e_n^5\right). \tag{10}$$

Finally, substituting from the equations (6), (7) and (10) into the second step of the contributed method (5), we obtain the error equation for the method. Therefore, the contributed method (5) is fourth order convergent. This completes our proof. Note that the method (5) is the well-known Ostrowski's method [11]. Here, we have presented an alternative derivation of the Ostrowski's method [11]. $\square$

Similarly, to construct a higher order method, we consider a three-step method (**M-2**)

$$\begin{cases} y_n &= x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \dfrac{f(y_n)}{\left(-f'(x_n) + 2\dfrac{f(y_n) - f(x_n)}{y_n - x_n}\right)}, \\ x_{n+1} &= z_n - \dfrac{f(z_n)}{\mathcal{FD}(z_n)}, \end{cases} \tag{11}$$

where $\mathcal{FD}(z_n)$ is defined as a linear combination of the slopes of the three lines passing through the points x_n and y_n; y_n and z_n; x_n and z_n.

$$\mathcal{FD}(z_n) = \alpha_1\,\frac{f(y_n) - f(x_n)}{y_n - x_n} + \alpha_2\,\frac{f(z_n) - f(y_n)}{z_n - y_n} + (1 - \alpha_1 - \alpha_2)\,\frac{f(z_n) - f(x_n)}{z_n - x_n}.$$

Theorem 2. *Let γ be a simple zero of a sufficiently differentiable function $f\colon \mathbf{D} \subset \mathbf{R} \mapsto \mathbf{R}$ in an open interval $\mathbf{D}$. If x_0 is sufficiently close to γ, the convergence order of the method (11) is 7 if and only if $\alpha_1 = -1$ and $\alpha_2 = 1$. The error equation for the method (11) is given as*

$$e_{n+1} = \frac{\left(c_3c_1 - c_2^2\right)c_2^2c_3}{c_1^5}e_n^7 + O\left(e_n^8\right).$$

Proof. Substituting from the equations (6), (7), (8), (10) into the second step of the contributed method (11) yields

(12)
$$z_n = \gamma - \frac{\left(c_3 c_1 - c_2^2\right) c_2}{c_1^3} e_n^4 - 2\,\frac{c_2 c_4 {c_1}^2 + {c_3}^2 {c_1}^2 - 4\,c_3 c_1 {c_2}^2 + 2\,{c_2}^4}{{c_1}^4} {e_n}^5 + O\left({e_n}^6\right).$$

Here, we have used the first step of the method (11). To find a Taylor expansion $f(z_n)$, we consider the Taylor's series of $f(x)$ around y_n

$$f(z_n) = f(y_n) + f'(y_n)\,(z_n - y_n) + \frac{f''(y_n)}{2}(z_n - y_n)^2 + \cdots, \tag{13}$$

substituting from equation (10) and using the second step of the contributed method (11), we obtain

(14)
$$f(z_n) = -\frac{\left(c_3 c_1 - {c_2}^2\right) c_2}{{c_1}^2} {e_n}^4 - 2\,\frac{c_2 c_4 {c_1}^2 + {c_3}^2 {c_1}^2 - 4\,c_3 c_1 {c_2}^2 + 2\,{c_2}^4}{{c_1}^3} {e_n}^5 + O\left({e_n}^6\right).$$

Here, the higher order derivatives of $f(x)$ at the point y_n are obtained by differentiating the equation (10) with respect e_n. Finally, to obtain the error equation for the method (11), substituting from the equations (6), (10), (14) into the third step of the contributed method (11) yields the error equation

$$\begin{aligned}
e_{n+1} = & \frac{c_2^2[(-1+\alpha_2)c_1 c_3 + (1-\alpha_2){c_2}^2]e_n^5}{c_1^4} + \frac{c_2}{c_1^5}[2\,(1-\alpha_2){c_1}^2 c_2 c_4 \\
& + 3\,(1-\alpha_2){c_1}^2 c_3^2 + (\alpha_1 - 10 + 12\,\alpha_2 - {\alpha_2}^2) c_1 {c_2}^2 c_3 \\
& + (-7\,\alpha_2 - \alpha_1 + {\alpha_2}^2 + 5){c_2}^4]e_n^6 - \frac{1}{c_1^6}[3(1-\alpha_2){c_1}^3 {c_2}^2 c_5 \\
& + 10((1-\alpha_2)c_1^3 c_2 c_3 + (19\,\alpha_2 - 15 - 2\,\alpha_2^2 + 2\,\alpha_1)c_1^2 c_2^3)c_4 \\
& + 2(1-\alpha_2)c_1^3 c_3^3 + (5\,\alpha_1 - 30 + 38\,\alpha_2 - 4\,{\alpha_2}^2)c_1^2 c_2^2 c_3^2 \\
& + (-{\alpha_2}^3 - 15\,\alpha_1 + 15\,{\alpha_2}^2 - 71\,\alpha_2 + 45 + 2\,\alpha_2\alpha_1)c_1 c_2^4 c_3 \\
& + (-2\,\alpha_2\alpha_1 + 29\,\alpha_2 + 8\,\alpha_1 - 9\,{\alpha_2}^2 + {\alpha_2}^3 - 15){c_2}^6]e_n^7 + O(e_n^8),
\end{aligned}$$

which shows that the convergence order of the method (11) is 7 iff $\alpha_1 = -1$ and $\alpha_2 = 1$. This completes our proof. The first two steps of the derived method (11) formulates the well known Ostrowski's method [11]. For optimal methods, requiring evaluation of four functions and that comply with the Kung-Traub conjecture [7], we refer to [12, 15, 16, and references therein]. □

The Kung-Traub conjecture (still unproved) states that an optimal iterative method without memory based on n evaluations could achieve an optimal convergence order of 2^{n-1}. The method (5) requires three evaluations ($f(x_n)$, $f'(x_n)$ and $f(y_n)$) during each iteration. Therefore the method (5) is optimal in the sense of the King-Traub conjecture [7]. On the other hand, the method (11) requests four evaluations ($f(x_n)$, $f'(x_n)$, $f(y_n)$ and $f(z_n)$) during each iteration. Consequently according to the Kung-Traub conjecture for the method (11) to be optimal its convergence order must be eight.

3. Numerical examples

The convergence order ξ of an iterative method is defined as

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{\xi}} = c \neq 0,$$

and furthermore this leads to the following approximation of the computational order of convergence (COC)

$$\rho \approx \frac{\ln|(x_{n+1}-\gamma)/(x_n-\gamma)|}{\ln|(x_n-\gamma)/(x_{n-1}-\gamma)|}.$$

For convergence it is required: $|x_{n+1}-x_n| < \epsilon$ and $|f(x_n)| < \epsilon$. Here, $\epsilon = 10^{-320}$. We test the methods for the following functions

$f_0(x) = \sin(x) - x/100,\ \gamma = 0.0.$ $\quad f_1(x) = x^3 + 4\,x^2 - 10,\ \gamma \approx 1.365.$

$f_2(x) = \tan^{-1}(x)\,,\ \gamma = 0.0.$ $\quad f_3(x) = x^4 + \sin\big(\pi/x^2\big) - 5,\ \gamma = \sqrt{2}.$

$f_4(x) = \exp\big(-x^2 + x + 2\big) - 1,\ \gamma = -1.0.$ $\quad f_5(x) = 1/3x^4 - x^2 - 1/3x + 1,\ \gamma = 1.0.$

Computational results (number of functional evaluations, COC during the second last iterative step) for various methods are presented in Table 1. In the table, various methods are abbreviated as follows: **CM** the Chebyshev method [4, 14]; **HM** the Halley method [4, 14]; **EM** the Euler method also referred to as the Cauchy method [2, 4, 5, 7, 14]; **NM** the Newton iterative method [4, 14]; **RWB** method proposed by Ren et al. [7]; **NETA** method proposed by Neta et al. [6]; **CH** the method developed by Chun et al. [2]; **WKL** the method developed by Wang et. al. [10], and finally the contributed methods **M-1** and (**M-2**). Free parameters are randomly selected as: for the method **RWB** $a = b = c = 1$, in the method by Chun et al. (**CH**) $\beta = 1$, in the method **WKL** $\alpha = \beta = 1$ and in the method **NETA** $a = 10$.

$f(x)$	x_0	HM	CM	EM	NM	RWB	NETA	CH	WKL	M-1	M-2
$f_0(x)$	0,9	(33, 3)	(36, 3)	(33, 3)	(40, 2)	(44, 6)	(36, 6)	(36, 6)	(44, 6)	(30, 4)	**(28, 7)**
$f_1(x)$	1,0	(36, 3)	(39, 3)	(39, 3)	(20, 2)	(20, 3)	(20, 6)	(20, 6)	(20, 3)	(18, 4)	**(16, 7)**
$f_2(x)$	0,5	(21, 3)	(21, 3)	(21, 3)	(18, 2)	(20, 6)	**(16, 7)**	**(16, 7)**	(20, 6)	(18, 5)	**(16, 8)**
$f_3(x)$	0,85	div	(54, 3)	(24, 3)	**(20, 2)**	**(20, 6)**	**(20, 6)**	**(20, 6)**	**(20, 6)**	(21, 4)	**(20, 7)**
$f_4(x)$	−0,45	(24, 3)	(24, 3)	(21, 3)	(20, 2)	(20, 6)	(20, 6)	(20, 6)	(20, 6)	**(18, 4)**	(21, 7)
$f_5(x)$	0,5	(24, 3)	(27, 3)	(24, 3)	(26, 2)	(20, 6)	(20, 6)	(20, 6)	(20, 6)	(21, 4)	**(16, 7)**

Table 1: (number of functional evaluations, COC) for various iterative methods.

An optimal iterative method for solving nonlinear equations must require least number of function evaluations. In Table 1, the methods which require least number of function evaluations are marked in bold. We acknowledge, through Table 1 that the methods contributed in this article are equal to or better than the performance of the existing methods in the literature.

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