Novi Sad J. Math.
Vol. 40, No. 2, 2010, 29-36

ON THE UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract. In this paper we introduce an integral operator and derive some criteria for univalence of this integral operator for analytic functions in the open unit disk.

AMS Mathematics Subject Classification (2000): 30C45

Key words and phrases: Integral operator, univalence, starlike

1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane, and let $\mathcal{A}$ be the class of functions which are analytic in the unit disk normalized with $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{P}$ the class of the functions p which are analytic in $\mathcal{U}$, $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, for all $z \in \mathcal{U}$. Let S be the subclass of $\mathcal{A}$, consisting of all univalent functions f in $\mathcal{U}$, and we consider S^* the subclass of S, consisting of all starlike functions f in $\mathcal{U}$.

In this work we introduce a new integral operator, which is defined by

$$B_{\alpha,\beta}(z) = \left\{ \beta \int_0^z (g(u))^{\alpha} h(u) u^{\beta-\alpha-1} du \right\}^{\frac{1}{\beta}}, \tag{1.1}$$

for some α, β be complex numbers, $\beta \neq 0$, $g \in \mathcal{A}$ and $h \in \mathcal{P}$.

For $\beta = a + bi$, $a > 0$, $b \in \mathbb{R}$, $\alpha = a$, $g \in \mathcal{A}$ and $h \in \mathcal{P}$ from (1.1) we have the integral operator

$$B_{a,b}(z) = \left\{ (a+bi) \int_0^z g^a(u) h(u) u^{ib-1} du \right\}^{\frac{1}{a+bi}}. \tag{1.2}$$

In the particular case $g \in S^*$, $B_{a,b}$ is the Bazilevič integral operator [1].

From (1.1) for $\alpha = \beta = \frac{1}{\gamma}, \gamma$ be a complex number, let $\gamma \neq 0$ and $h(z) = 1$ for all $z \in U$, we obtain the integral operator

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$$J_{\gamma}(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{-1} g^{\frac{1}{\gamma}}(u)\, du \right\}^{\gamma}. \tag{1.3}$$

Miller and Mocanu [4], have observed that the integral operator J_γ is in the class S for $f \in S^*$ and $\gamma > 0$.

For $\beta = 1$, let α be a complex number and $h(z) = 1$ for all $z \in \mathcal{U}$, from (1.1), we have the Kim-Merkes integral operator [2],

$$K_{\alpha}(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{\alpha} du. \tag{1.4}$$

In the present paper, we consider some sufficient conditions for the integral operator $B_{\alpha,\beta}$ to be in the class S.

2. Preliminary results

We need the following theorems.

Theorem 2.1 ([6]). *Let α be a complex number, Re $\alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \le 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then for any complex number β, $\operatorname{Re}\beta \ge \operatorname{Re}\alpha$, the function

$$F_{\beta}(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{\frac{1}{\beta}} \tag{2.2}$$

is in the class S.

Theorem 2.2 (Schwarz [3]). *Let f be a regular function in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply greater than or equal to m, then*

$$|f(z)| \le \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R, \tag{2.3}$$

the equality (in the inequality (2.3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is a constant.

Theorem 2.3 ([5])**.** *If the function $g(z)$ is regular in $\mathcal{U}$ and $|g(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in U$ and $z \in \mathcal{U}$, the following inequalities hold:*

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)} g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right| \tag{2.4}$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \tag{2.5}$$

the equalities hold only in the case $g(z) = \frac{\varepsilon(z+u)}{1+\overline{u}z}$, where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 2.4 ([5])**.** *For $z = 0$, from inequality* (2.4) *we have*

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)} g(\xi)} \right| < |\xi| \tag{2.6}$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|\xi|}. \tag{2.7}$$

Considering $g(0) = a$ and $\xi = z$,

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|}, \tag{2.8}$$

for all $z \in \mathcal{U}$.

3. Main results

Theorem 3.1. *Let α be a complex number, $\operatorname{Re}\alpha > 0$, M_1, M_2 real positive numbers, the functions $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + \ldots$ and $h \in \mathcal{P}$.*

If

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq M_1, \quad (z \in \mathcal{U}), \tag{3.1}$$

$$\left| \frac{zh'(z)}{h(z)} \right| \leq M_2, \quad (z \in \mathcal{U}) \tag{3.2}$$

and

$$|\alpha|\, M_1 + M_2 \leq \operatorname{Re}\alpha, \tag{3.3}$$

then for every complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the integral operator $B_{\alpha,\beta}$, given by (1.1) *, is in the class S.*

Proof. We observe that

$$B_{\alpha,\beta}(z)=\left\{\beta\int\limits_0^z u^{\beta-1}\left(\frac{g(u)}{u}\right)^{\alpha}h(u)\,du\right\}^{\frac{1}{\beta}}. \tag{3.4}$$

Let us define the function

$$f(z)=\int\limits_0^z\left(\frac{g(u)}{u}\right)^{\alpha}h(u)\,du,\quad (z\in\mathcal{U}), \tag{3.5}$$

for $g\in\mathcal{A}$ and $h\in\mathcal{P}$. The function f is regular in $\mathcal{U}$ and $f(0)=f'(0)-1=0$. We have

$$\frac{zf''(z)}{f'(z)}=\alpha\left(\frac{zg'(z)}{g(z)}-1\right)+\frac{zh'(z)}{h(z)},\quad (z\in\mathcal{U}). \tag{3.6}$$

From (3.1), (3.2) and (3.6) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\le\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(|\alpha|M_1+M_2\right),\quad (z\in\mathcal{U}), \tag{3.7}$$

and by (3.3), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\le 1, \tag{3.8}$$

for all $z\in\mathcal{U}$.

From (3.5), we obtain $f'(z)=\left(\frac{g(z)}{z}\right)^{\alpha}h(z)$ and using (3.8), by Theorem 2.1. it results that the integral operator $B_{\alpha,\beta}$, given by (1.1), is in the class S. □

Theorem 3.2. *Let α be a complex number, $\operatorname{Re}\alpha>0$, M_1, M_2 real positive numbers, $M_1\in(0,1)$, the functions $h\in\mathcal{P}$, $h'(0)=0$ and $g\in\mathcal{A}$, $g(z)=z+a_2z^2+\ldots$*

If

$$\left|\frac{zg'(z)-g(z)}{zg(z)}\right|<M_1,\quad (z\in\mathcal{U}), \tag{3.9}$$

$$\left|\frac{h'(z)}{h(z)}\right|<M_2,\quad (z\in\mathcal{U}) \tag{3.10}$$

and

$$\frac{M_2}{1-M_1} < |\alpha| \leq \frac{1}{\max\limits_{|z|<1}\left[\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\,|z|\,\frac{|z|+|a_2|}{1+|a_2|\cdot|z|}\right]}\quad, \tag{3.11}$$

then for every complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, *the integral operator* $B_{\alpha,\beta}$, *defined by* (1.1), *is in the class* S.

Proof. The integral operator $B_{\alpha,\beta}$ is of the form (3.4).

We define the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha} h(u)\,du, \quad (z \in \mathcal{U}), \tag{3.12}$$

for $g \in \mathcal{A}$ and $h \in \mathcal{P}$.

We consider the function

$$k(z) = \frac{1}{|\alpha|}\frac{f''(z)}{f'(z)}, \tag{3.13}$$

for all $z \in \mathcal{U}$. We have

$$\frac{1}{|\alpha|}\left|\frac{f''(z)}{f'(z)}\right| \leq \left|\frac{zg'(z)-g(z)}{zg(z)}\right| + \frac{1}{|\alpha|}\left|\frac{h'(z)}{h(z)}\right|, \quad (z \in \mathcal{U}). \tag{3.14}$$

From (3.11) we have $|\alpha| > \frac{M_2}{1-M_1}$, $M_1 \in (0,1)$ and by (3.9), (3.10), (3.11) we obtain $|k(z)| < 1$, for all $z \in \mathcal{U}$.

We have $k(0) = \frac{\alpha}{|\alpha|}a_2$ and using Remark 2.4, we get

$$|k(z)| \leq \frac{|z|+|a_2|}{1+|a_2|\,|z|}, \quad (z \in \mathcal{U}). \tag{3.15}$$

Let us consider the function

$$Q(x) = \frac{1-x^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}x\frac{x+|a_2|}{1+|a_2|\,x}, \quad (x = |z|;\ z \in \mathcal{U}).$$

Because $Q\left(\frac{1}{2}\right) > 0$ it results that $\max\limits_{x\in[0,1]} Q(x) > 0$.

Using this result and from (3.13), (3.15), we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq |\alpha| \max_{|z|<1}\left[\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\,|z|\,\frac{|z|+|a_2|}{1+|a_2|\cdot|z|}\right], \tag{3.16}$$

for all $z \in \mathcal{U}$.

From (3.11) and (3.16) we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.17}$$

Consequently, in view of Theorem 2.1, we obtain that the integral operator $B_{\alpha,\beta}$, is in the class S. □

4. Corollaries

Corollary 4.1. *Let $a+bi$ be a complex number, $a>0$, M_1, M_2 real positive numbers, $M_1 \in (0,1)$, the functions $g \in \mathcal{A}$ and $h \in \mathcal{P}$.*

If

$$\left|\frac{zg'(z)}{g(z)}-1\right| \leq M_1, \quad (z \in \mathcal{U}), \tag{4.1}$$

$$\left|\frac{zh'(z)}{h(z)}\right| \leq M_2, \quad (z \in \mathcal{U}) \tag{4.2}$$

and

$$a \geq \frac{M_2}{1-M_1}\ , \tag{4.3}$$

then the integral operator $B_{a,b}$, given by (1.2), is in the class S.

Proof. We take, in Theorem 3.1, $\beta = a+bi$, $a>0$, $b \in \mathbb{R}$, $\alpha = a$ and obtain Corollary 4.1. □

Corollary 4.2. *Let γ be a complex number, $\gamma \neq 0$, $\operatorname{Re}\frac{1}{\gamma} > 0$ and the function $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + ...$*

If

$$\left|\frac{zg'(z)}{g(z)}-1\right| \leq |\gamma| \operatorname{Re}\frac{1}{\gamma}, \quad (z \in \mathcal{U}), \tag{4.4}$$

then the integral operator J_γ, given by (1.3), is in the class S.

Proof. For $\alpha = \beta = \frac{1}{\gamma}$ and $h(z) = 1$ for all $z \in \mathcal{U}$, from Theorem 3.1 we have Corollary 4.2. □

Corollary 4.3. *Let α be a complex number $0 < \operatorname{Re}\alpha \leq 1$ and the function $g \in A,\ g(z) = z + a_2 z^2 + ...$*

If

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq \frac{\operatorname{Re}\alpha}{|\alpha|}, \quad (z \in \mathcal{U}), \tag{4.5}$$

then the integral operator K_α, defined by (1.4), is in the class S.

Proof. For $\beta = 1$, and $h(z) = 1$ for all $z \in \mathcal{U}$, from Theorem 3.1 we obtain that K_α belongs to the class S. □

Corollary 4.4. *Let $a + bi$ be a complex number, $a > 0,\ M_1,\ M_2$ real positive numbers, $M_1 \in (0,1)$, the functions $h \in \mathcal{P},\ h'(0) = 0$ and $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + ...$*

If

$$\left|\frac{zg'(z) - g(z)}{zg(z)}\right| < M_1, \quad (z \in \mathcal{U}), \tag{4.6}$$

$$\left|\frac{h'(z)}{h(z)}\right| < M_2, \quad (z \in \mathcal{U}) \tag{4.7}$$

and

$$\frac{M_2}{1 - M_1} < a \leq \frac{1}{\max\limits_{|z|<1}\left[\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\,|z|\,\frac{|z|+|a_2|}{1+|a_2|\cdot|z|}\right]}\ , \tag{4.8}$$

then the integral operator $B_{a,b}$, defined by (1.2), *is in the class S.*

Proof. For $\beta = a + bi,\ a > 0,\ b \in \mathbb{R},\ \alpha = a$ from Theorem 3.2 we obtain Corollary 4.4. □

Corollary 4.5. *Let γ be a complex number, $\gamma \neq 0,\ \operatorname{Re}\frac{1}{\gamma} > 0$ and the function $g \in A,\ g(z) = z + a_2 z^2 + ...$*

If

$$\left|\frac{zg'(z) - g(z)}{zg(z)}\right| < 1, \quad (z \in \mathcal{U}) \tag{4.9}$$

and

$$|\gamma| \geq \max_{|z|<1}\left[\frac{1 - |z|^{2\operatorname{Re}\frac{1}{\gamma}}}{\operatorname{Re}\frac{1}{\gamma}}\,|z|\,\frac{|z| + |a_2|}{1 + |a_2|\cdot|z|}\right], \tag{4.10}$$

then the integral operator J_γ, given by (1.3), *is in the class S.*

Proof. From Theorem 3.2 for $\alpha = \beta = \frac{1}{\gamma}$ and $h(z) = 1$, for all $z \in \mathcal{U}$ we obtain Corollary 4.5. □

Corollary 4.6. *Let α be a complex number, $0 < \operatorname{Re}\alpha \leq 1$, the function $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + ...$*

If

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| < 1, \quad (z \in \mathcal{U}) \tag{4.11}$$

and

$$|\alpha| \leq \frac{1}{\max\limits_{|z|\leq 1} \left[\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} |z| \frac{|z|+|a_2|}{1+|a_2|\cdot|z|} \right]} \quad , \tag{4.12}$$

then the integral operator K_α, given by (1.4), *is in the class S.*

Proof. For $\beta = 1$ and $h(z) = 1$, for all $z \in \mathcal{U}$, from Theorem 3.2 we obtain that the integral operator K_α, is in the class S. □

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Received by the editors June 10, 2009