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ON N(k)−QUASI EINSTEIN MANIFOLD

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Abstract. In the present paper we have studied an $N(k)$-quasi Einstein manifold satisfying $R(\xi, X).\tilde{P}$, where $\tilde{P}$ is the pseudo-projective curvature tensor. Among others, it is shown that if quasi-Einstein manifold with constant associated scalars is Ricci symmetric then the generator of the manifold is a Killing vector field.

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1. Introduction

A quasi-Einstein manifold is a simple and natural generalization of the Einstein manifold. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold [2] if the Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), X, Y \in TM \tag{1.1}$$

or equivalently, its Ricci operator Q satisfies

$$Q = aI + bn \otimes \xi \tag{1.2}$$

for some smooth functions a and $b \neq 0$, where η is a non-zero 1-form such that,

$$g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1 \tag{1.3}$$

for the associated vector field ξ. The scalars a and b are called associated scalars, η associated 1-form and ξ the generator of the manifold. An n-dimensional manifold of this kind is denoted by the symbol $(QE)_n$. It is obvious that if $b = 0$ and $a = \frac{r}{n}$ then this reduces to the well-known Einstein manifold. This justifies the name 'Quasi-Einstein Manifold', given to this type of manifolds. In an n-dimensional quasi-Einstein manifold the Ricci tensor has precisely two

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distinct eigenvalues a and $a+b$, where a is of multiplicity of $(n-1)$ and $a+b$ is simple. A proper η-Einstein contact metric manifold ([1],[3]) is a natural example of a quasi-Einstein manifold.

In 2007, M.M. Tripathi and J.S. Kim [9] studied a quasi-Einstein manifold whose generator ξ belongs to the k-nullity distribution $N(k)$ and called such a manifold as *$N(k)$-quasi Einstein manifold*. In [9], the authors have proved that conformally flat quasi-Einstein manifolds are certain $N(k)$-quasi Einstein manifolds. The derivation conditions $R(\xi,X).R=0$ and $R(\xi,X).S=0$ have also been studied in [8], where R and S denote the curvature and Ricci tensor, respectively. Cihan Özgür and M.M. Tripathi [5] continued the study of the $N(k)$-quasi Einstein manifold. In [5], the derivation conditions $Z(\xi,X).R=0$ and $Z(\xi,X).Z=0$ on $N(k)$-quasi Einstein manifold were studied, where Z is the concircular curvature tensor. Moreover, in [5], for an $N(k)$-quasi Einstein manifold it was proved that $k=\frac{a+b}{n-1}$. C. Özgür [4], in 2008, studied the condition $R.P=0$ for an $N(k)$-quasi Einstein manifold, where P denotes the projective curvature tensor and some physical examples of $N(k)$-quasi Einstein manifolds are given. Again, in 2008, C. Özgür and Sibel Sular [6], studied $N(k)$-quasi Einstein manifold satisfying $R(\xi,X).C=0$ and $R(\xi,X).\tilde{C}=0$, where C and $\tilde{C}$ represent the Weyl conformal curvature tensor and the quasi-conformal curvature tensor, respectively. This paper is a continuation of previous studies.

The paper is organized as follows: After introduction in Section 2, we give the brief account of $N(k)$-quasi Einstein manifold. In Section 3, we study $N(k)$-quasi Einstein manifold satisfying $R(\xi,X).\tilde{P}=0$ and Section 4 deals with a Ricci symmetric quasi-Einstein manifold with constant associated scalars. It is shown that the generator of such manifold is a Killing vector field.

2. $N(k)$-quasi Einstein manifold

The k-nullity distribution $N(k)$ of a Riemannian manifold M^n is defined by [8]

$$N(k): p \longrightarrow N_p(k) = \{Z \in T_pM | R(X,Y,Z) = k(g(Y,Z)X - g(X,Z)Y\}$$

for all $X,Y \in TM$, where k is some smooth function. If the generator ξ of the quasi-Einstein manifold M^n belongs to the k-nullity distribution $N(k)$ for some smooth function k, then M^n is called $N(k)$-quasi Einstein manifold [9]. On $N(k)$-quasi Einstein manifold, we have [9]

$$R(Y,Z)\xi = k(\eta(Z)Y - \eta(Y)Z). \tag{2.1}$$

The above equation is equivalent to

$$R(\xi,Y)Z = k(g(Y,Z)\xi - \eta(Z)Y). \tag{2.2}$$

In particular, the above two equations imply that

$$\eta(R(Y,Z)\xi) = 0. \tag{2.3}$$

Moreover, it is known [5] that

Lemma 2.1. *In an n-dimensional $N(k)$-quasi Einstein manifold, it follows that*

$$k = \frac{a+b}{n-1}. \tag{2.4}$$

3. $N(k)$-quasi Einstein manifold satisfying $R(\xi, X).\tilde{P} = 0$.

In 2002, B. Prasad [7] introduced the notion of a pseudo-projective curvature tensor. The pseudo-projective curvature tensor $\tilde{P}$ on a manifold M^n of dimension n is defined as follows.

$$\begin{aligned}\tilde{P}(X,Y)Z &= \alpha R(X,Y)Z + \beta[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}[\frac{\alpha}{n-1} + \beta][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{3.1}$$

where α and β are the constants such that $\alpha, \beta \neq 0$, R is the curvature tensor and S is the Ricci tensor. It is obvious that if $\alpha = 1$ and $\beta = -\frac{1}{n-1}$, then the pseudo-projective curvature tensor reduces to a projective curvature tensor.

Let, $N(k)$-quasi Einstein manifold satisfy the condition

$$R(\xi, Y).\tilde{P} = 0. \tag{3.2}$$

This implies

$$\begin{aligned}0 &= R(\xi,Y)\tilde{P}(U,V)Z - \tilde{P}(R(\xi,Y)U,V)Z \\ &- \tilde{P}(U,R(\xi,Y)V)Z - \tilde{P}(U,V)R(\xi,Y)Z.\end{aligned} \tag{3.3}$$

Taking inner product of the equation (3.3) with ξ, we get

$$\begin{aligned}0 &= g(R(\xi,Y)\tilde{P}(U,V)Z,\xi) - g(\tilde{P}(R(\xi,Y)U,V)Z,\xi) \\ &- g(\tilde{P}(U,R(\xi,Y)V)Z,\xi) - g(\tilde{P}(U,V)R(\xi,Y)Z,\xi).\end{aligned}$$

By virtue of (2.2), the above equation gives

$$\begin{aligned}0 &= k[\acute{\tilde{P}}(U,V,Z,Y) - \eta(\tilde{P}(U,V)Z)\eta(Y) \\ &- g(Y,U)\eta(\tilde{P}(\xi,V)Z) + \eta(U)\eta(\tilde{P}(Y,V)Z) \\ &- g(Y,V)\eta(\tilde{P}(U,\xi)Z) + \eta(V)\eta(\tilde{P}(U,Y)Z) \\ &- g(Y,Z)\eta(\tilde{P}(U,V)\xi) + \eta(Z)\eta(\tilde{P}(U,V)Y)],\end{aligned} \tag{3.4}$$

where $\acute{\tilde{P}}(U,V,Z,Y) = g(\tilde{P}(U,V)Z,Y)$.

Now, from (1.1), (2.1), (3.1), we have

$$\eta(\tilde{P}(X,Y)Z) = \lambda[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{3.5}$$

where $\lambda = [\alpha k - \frac{r}{n}(\frac{\alpha}{n-1} + \beta) - \beta a]$; which, in view of Lemma 2.1, reduces to $\lambda = \frac{b(\alpha-\beta)}{n}$. From (3.6), it follows that

$$\eta(\tilde{P}(X,Y)\xi) = o, \tag{3.6}$$

$$\eta(\tilde{P}(\xi, Y)Z) = \lambda[g(Y, Z) - \eta(Y)\eta(Z)] \tag{3.7}$$

and

$$\eta(\tilde{P}(X, \xi)Z) = \lambda[\eta(X)\eta(Z) - g(X, Z)]. \tag{3.8}$$

Using (3.6), (3.7), (3.8) and (3.9) in (3.5), we obtain

$$0 = k[\acute{\tilde{P}}(U, V, Z, Y) - \lambda(g(Y, U)g(V, Z) - g(Y, V)g(U, Z)], \tag{3.9}$$

which, due to the equation (3.1), yields

$$\begin{aligned} 0 &= k[\alpha\acute{R}(X, Y, Z, W) + \beta\{S(Y, Z)g(X, W) - S(X, Z)g(Y, W)\} \\ &- \{\frac{r}{n}(\frac{\alpha}{n-1} + \beta) + \lambda\}(g(Y, Z)g(X, W) - g(X, Z)g(Y, W))]. \end{aligned} \tag{3.10}$$

Contracting above equation (3.11) over X and W, we get

$$0 = k[S(Y, Z) - \mu g(Y, Z)], \tag{3.11}$$

where $\mu = \frac{1}{\alpha+(n-1)\beta}[\lambda(n-1) + \frac{r}{n}\{\alpha + (n-1)\beta\}]$. Since the manifold under consideration is not an Einstein manifold, therefore it follows that $k = 0$.

Conversely, if $k = 0$, then in view of equation (2.2), we have $R(\xi, X) = 0$, which gives $R(\xi, X).\tilde{P} = 0$. Thus, we have the following theorem

Theorem 3.1. *In an $N(k)$-quasi Einstein manifold, $R(\xi, X).\tilde{P} = 0$ holds if and only if $k = 0$.*

4. Ricci-symmetric quasi-Einstein manifold

In this section we consider a quasi-Einstein manifold, whose associated scalars a and b are constant.

Definition 4.1. A Riemannian manifold M^n is called a Ricci-symmetric manifold if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = 0, \tag{4.1}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g.

Definition 4.2. The Ricci tensor of Riemannian manifold is said to be cyclic parallel if

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{4.2}$$

Let M^n be a quasi-Einstein manifold, whose associated scalars are constant, then by differentiating (1.1) covariantly with respect to Levi-Civita connection we get

$$(\nabla_X S)(Y, Z) = b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)]. \tag{4.3}$$

If Ricci tensor of M^n is symmetric, then the equation (4.3) implies that

$$b((\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y)) = 0,$$

which on putting $Z = \xi$ gives,

$$(\nabla_X\eta)(Y) = 0 \quad \text{as} \quad b \neq 0. \tag{4.4}$$

Putting Y=X in equation (4.4), we find

$$(\nabla_X\eta)(X) = 0$$

or equivalently

$$g(\nabla_X\xi, X) = 0,$$

and from (4.4), we also have

$$(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 0. \tag{4.5}$$

Therefore, we have the following two theorems.

Theorem 4.1. *If the quasi-Einstein manifold M^n with constant associated scalars is Ricci symmetric, then its generator ξ satisfies $g(\nabla_X\xi, X) = 0$.*

Theorem 4.2. *If the quasi-Einstein manifold M^n with constant associated scalars is Ricci symmetric, then its generator ξ is a Killing vector field.*

Next, from (4.3), we get

$$\begin{aligned}\sigma_{(X,Y,Z)}(\nabla_X S)(Y,Z) = b[&(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y)\\ &+ (\nabla_Y\eta)(Z)\eta(X) + (\nabla_Y\eta)(X)\eta(Z)\\ &+ (\nabla_Z\eta)(X)\eta(Y) + (\nabla_X\eta)(Y)\eta(X)],\end{aligned} \tag{4.6}$$

where $\sigma_{(X,Y,Z)}$ denotes a cyclic sum with respect to X, Y and Z.

i.e. $\sigma_{(X,Y,Z)}(\nabla_X S)(Y,Z) = (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)$.

If a generator of the quasi-Einstein manifold is a Killing vector, then we have the equation (4.5), which on using in (4.6), gives

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

Thus, we may have the following theorem:

Theorem 4.3. *If the generator of the quasi-Einstein manifold M^n with constant associated scalars is Killing, then its Ricci tensor is cyclic parallel.*

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