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MINIMAL SURFACES THAT GENERALIZE THE ENNEPER'S SURFACE

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Abstract. In this article we give the construction of some minimal surfaces in the Euclidean spaces starting from the Enneper's surface.

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1. Introduction

We start with the well-known Enneper's surface which is defined in the 3-dimensional Cartesian by the application $x : \mathbb{R}^2 \to \mathbb{R}^3$, $x(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$. The point (u,v) is mapped to $(f(u,v), g(u,v), h(u,v))$, which is a point on Enneper's surface [2]. This surface has self-intersections and it is one of the first examples of minimal surface. The fact that Enneper's surface is minimal is a consequence of the following theorem:

Theorem 1.1. [1] Consider the Enneper's surface $x(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$. Then:

a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2,\ F = 0. \tag{1.1}$$

b) The coefficients of the second fundamental form are

$$e = 2,\ g = -2,\ f = 0. \tag{1.2}$$

c) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2},\ k_2 = -\frac{2}{(1 + u^2 + v^2)^2}. \tag{1.3}$$

It follows that the mean curvature $H = \frac{Eg + eG - 2Ff}{EG - F^2}$ is zero, which means the Enneper's surface is minimal.

If we look more carefully to the formula of the mean curvature ,H , of the Enneper's surface we observe that $H = \frac{E(e+g) - 2f \cdot 0}{EG - F^2} = 0$, because $e + g = 0$. Starting with this observation we want to find out new examples of minimal surface in $\mathbb{R}^3$, for which the first and the second fundamental forms are in some sense similar to the first and the second fundamental forms of the Enneper's surface. That means if we consider a surface in $\mathbb{R}^3$, $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ such that the first fundamental form satisfies $E = G$, $F = 0$ and the second fundamental form satisfies $e + g = 0$, with f arbitrary, this surface will be minimal.

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2. Main results

We consider an arbitrary surface in $\mathbb{R}^3$, $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, where the components of the surface are $\varphi(u,v) = (f(u,v), g(u,v), h(u,v))$, $f, g, h : \mathbb{R}^2 \to \mathbb{R}$. We denote by $\varphi'_u := (f'_u, g'_u, h'_u)$, $\varphi'_v := (f'_v, g'_v, h'_v)$, $\varphi''_{u^2} := (f''_{u^2}, g''_{u^2}, h''_{u^2})$, $\varphi''_{v^2} := (f''_{v^2}, g''_{v^2}, h''_{v^2})$, $\varphi''_{uv} := (f''_{uv}, g''_{uv}, h''_{uv})$.

We want to find φ different from the Enneper's surface such that the first fundamental form of φ satisfies $E_0 = G_0$, $F_0 = 0$ and the second fundamental form of φ satisfies $e_0 + g_0 = 0$.

We know that the coefficients of the first fundamental form are given by the following formulas:

$$\begin{aligned} E_0 &= |\varphi'_u|^2 = (f'_u)^2 + (g'_u)^2 + (h'_u)^2, \\ G_0 &= |\varphi'_v|^2 = (f'_v)^2 + (g'_v)^2 + (h'_v)^2, \\ F_0 &= <\varphi'_u, \varphi'_v> = f'_u f'_v + g'_u g'_v + h'_u h'_v. \end{aligned} \tag{2.1}$$

So, $E_0 = G_0$ implies $(f'_u)^2 + (g'_u)^2 + (h'_u)^2 = (f'_v)^2 + (g'_v)^2 + (h'_v)^2$ and $F_0 = 0$ implies $f'_u f'_v + g'_u g'_v + h'_u h'_v = 0$.

If we consider $N = \frac{\varphi'_u \times \varphi'_v}{|\varphi'_u \times \varphi'_v|}$ the unit normal vector to the surface then the coefficients of the second fundamental form are given by the formulas:

$$\begin{aligned} e_0 &= <\varphi''_{u^2}, N>, \\ g_0 &= <\varphi''_{v^2}, N>, \\ f_0 &= <\varphi''_{uv}, N>. \end{aligned} \tag{2.2}$$

So, $e_0 + g_0 = <\varphi''_{u^2} + \varphi''_{v^2}, N>$ and a sufficient condition that $e_0 + g_0 = 0$ is that $\varphi''_{u^2} + \varphi''_{v^2} = (0,0,0)$, which implies $\triangle f = \triangle g = \triangle h = 0$, where $\triangle$ is the Laplace operator.

Hence, because we are looking for new surfaces which satisfy $E_0 = G_0$, $F_0 = 0$, and $e_0 + g_0 = 0$ it is sufficient to look for solution of the following system

$$\begin{aligned} &\triangle f = \triangle g = \triangle h = 0 \text{ (that means } f, g, h \text{ are harmonic)}, \\ &f'_u f'_v + g'_u g'_v + h'_u h'_v = 0, \\ &(f'_u)^2 + (g'_u)^2 + (h'_u)^2 = (f'_v)^2 + (g'_v)^2 + (h'_v)^2. \end{aligned} \tag{2.3}$$

Remark 2.1. Every solution of the system (2.3) will give us a minimal surface in $\mathbb{R}^3$.

In general, it is very difficult to give the general solution of the above system, but we are interested in some solution of it, to give examples of minimal surfaces. We keep in mind that we want to generalize the Enneper's surface, so we will consider that f, g, h satisfy the following conditions, because the components of the Enneper's surface satisfy these conditions:

$$f'_v = g'_u, \; f'_u + g'_v = 2, \; f''_{v^2} = h'_u, \; f''_{uv} = -h'_v. \tag{2.4}$$

Remark 2.2. We only have to consider f to be harmonic, because from (2.4) it follows that g and h are also harmonic.

Thus, from (2.4) the second equation of the system (2.3) becomes:

$$\begin{aligned} 0 &= f_u^{'}f_v^{'} + f_v{'}g_v^{'} + f_{v^2}^{''}(-f_{uv}^{''}) \\ &= f_v^{'}(f_u^{'} + g_v^{'}) - f_{v^2}^{''}f_{uv}^{''} \\ &= 2f_v^{'} - f_{v^2}^{''}f_{uv}^{''} \Rightarrow \\ 2f_v^{'} - f_{v^2}^{''}f_{uv}^{''} &= 0, \end{aligned} \tag{2.5}$$

and the third equation becomes:

$$\begin{aligned} (f_u^{'})^2 + (g_u^{'})^2 + (h_u^{'})^2 &= (f_v^{'})^2 + (g_v^{'})^2 + (h_v^{'})^2 \ \Rightarrow \\ (f_u^{'})^2 + (f_v^{'})^2 + (f_{v^2}^{''})^2 &= (f_v^{'})^2 + (2 - f_u^{'})^2 + (-f_{uv}^{''})^2 \ \Rightarrow \\ (f_u^{'})^2 + (f_v^{'})^2 + (f_{v^2}^{''})^2 &= (f_u^{'})^2 + (f_v^{'})^2 - 4f_u^{'} + 4 + (f_{uv}^{''})^2 \ \Rightarrow \\ 4 - 4f_u^{'} - (f_{v^2}^{''})^2 + (f_{uv}^{''})^2 &= 0. \end{aligned} \tag{2.6}$$

Now we reduced our system (2.3) to a particular one which depends only on f, and that is:

$$\begin{aligned} &\triangle f = 0 \text{ (that means } f \text{ is harmonic)}, \\ &2f_v^{'} - f_{v^2}^{''}f_{uv}^{''} = 0, \\ &4 - 4f_u^{'} - (f_{v^2}^{''})^2 + (f_{uv}^{''})^2 = 0. \end{aligned} \tag{2.7}$$

We observe that one particular solution of the second equation of the system (2.7) is $f_v^{'} = (au+b)(\frac{2}{a}v+c)$, where $a \neq 0, b, c \in \mathbb{R}$. We will see that this solution will lead to some generalization of the Enneper's surface.

So, we have that $f(u,v) = (au+b)(\frac{1}{a}v^2 + cv + d) + \psi(u)$, where $\psi : \mathbb{R} \to \mathbb{R}$ is some function. Hence

$$\begin{aligned} f_u^{'} &= \psi^{'}(u) + a(\tfrac{1}{a}v^2 + cv + d), \\ f_{v^2}^{''} &= \tfrac{2}{a}(au+b) = 2u + 2\tfrac{b}{a}, \\ f_{uv}^{''} &= a(\tfrac{2}{a}v + c) = 2v + ac, \end{aligned} \tag{2.8}$$

and if we introduce them in the third equation of the system (2.7) we obtain

$$\begin{aligned} 4f_u^{'} &= 4 - (f_{v^2}^{''})^2 + (f_{uv}^{''})^2 = 4 - (2u + 2\tfrac{b}{a})^2 + (2v + ac)^2 \ \Rightarrow \\ f_u^{'} &= 1 - u^2 - 2\tfrac{b}{a}u - \tfrac{b^2}{a^2} + v^2 + acv + \tfrac{a^2c^2}{4}. \end{aligned} \tag{2.9}$$

But $f_u^{'} = \psi^{'}(u) + v^2 + acv + ad,$ and then

$$\begin{aligned} \psi^{'}(u) &= 1 - u^2 - 2\tfrac{b}{a}u - \tfrac{b^2}{a^2} + \tfrac{a^2c^2}{4} - ad \ \Rightarrow \\ \psi(u) &= u - \tfrac{u^3}{3} - \tfrac{b}{a}u^2 + (-\tfrac{b^2}{a^2} + \tfrac{a^2c^2}{4} - ad)u + e \ \Rightarrow \\ f(u,v) &= (u - \tfrac{u^3}{3} + uv^2) \\ &\quad + [-\tfrac{b}{a}u^2 + acuv + \tfrac{b}{a}v^2 + (-\tfrac{b^2}{a^2} + \tfrac{a^2c^2}{4})u + bcv + bd + e], \end{aligned} \tag{2.10}$$

where $a \neq 0, b, c, d, e \in \mathbb{R}$.

Using (2.4) we obtain that

$$f_v^{'} = g_u^{'} = (au+b)(\tfrac{2}{a}v+c) \ \Rightarrow$$
$$g(u,v) = (\tfrac{a}{2}u^2+bu+k)(\tfrac{2}{a}v+c)+\theta(v),$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a function. But,

$$\begin{aligned} g_v^{'} &= 2 - f_u^{'} \\ &= 2-(1-u^2-2\tfrac{b}{a}u-\tfrac{b^2}{a^2}+v^2+acv+\tfrac{a^2c^2}{4}) \\ &= \theta^{'}(v)+u^2+2\tfrac{b}{a}u+\tfrac{2k}{a} \ \Rightarrow \\ \theta^{'}(v) &= -v^2-acv+(1+\tfrac{b^2}{a^2}-\tfrac{a^2c^2}{4}-\tfrac{2k}{a}) \ \Rightarrow \\ \theta(v) &= -\tfrac{v^3}{3}-\tfrac{ac}{2}v^2+(1+\tfrac{b^2}{a^2}-\tfrac{a^2c^2}{4}-\tfrac{2k}{a})v+l \Rightarrow \\ g(u,v) &= (v-\tfrac{v^3}{3}+vu^2)+[-\tfrac{ac}{2}v^2+2\tfrac{b}{a}uv \\ &\quad +\tfrac{ac}{2}u^2+(\tfrac{b^2}{a^2}-\tfrac{a^2c^2}{4})v+bcu+kc+l], \end{aligned} \tag{2.11}$$

where $k, l \in \mathbb{R}$.

Using again (2.4) we get that $h_u^{'} = f_{v^2}^{''} = 2u+2\frac{b}{a} \Rightarrow h(u,v) = u^2+2\frac{b}{a}u+\phi(v)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a function and $h_v^{'} = -f_{uv}^{''} = -(2v+ac)$.

So, we have that

$$\begin{aligned} \phi^{'}(v) = -2v-ac \Rightarrow \phi(v) = -v^2-acv+j \ \Rightarrow \\ h(u,v) = (u^2-v^2)+2\tfrac{b}{a}u-acv+j, \text{ with } j \in \mathbb{R}. \end{aligned} \tag{2.12}$$

With loss of generality we can consider the parameters $l = k = j = e = d = 0$, and obtain the following minimal surfaces $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(u,v) = (f(u,v), g(u,v), h(u,v))$, where f, g, h are given by:

$$\begin{aligned} f(u,v) &= (u-\tfrac{u^3}{3}+uv^2)+ \\ &\quad [-\tfrac{b}{a}u^2+acuv+\tfrac{b}{a}v^2+(-\tfrac{b^2}{a^2}+\tfrac{a^2c^2}{4})u+bcv], \\ g(u,v) &= (v-\tfrac{v^3}{3}+vu^2)+ \\ &\quad [-\tfrac{ac}{2}v^2+2\tfrac{b}{a}uv+\tfrac{ac}{2}u^2+(\tfrac{b^2}{a^2}-\tfrac{a^2c^2}{4})v+bcu], \\ h(u,v) &= (u^2-v^2)+2\tfrac{b}{a}u-acv, \end{aligned} \tag{2.13}$$

where $a \neq 0$, $b, c \in \mathbb{R}$.

If we denote $\alpha = \frac{b}{a}$ and $\beta = ac$ then $\alpha\beta = bc$ and we can consider a family of minimal surfaces which are only indexed by two real parameters α, β. Hence, if we consider the following three functions $f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f_{\alpha,\beta}(u,v) &= (u-\tfrac{u^3}{3}+uv^2)+ \\ &\quad [-\alpha u^2+\beta uv+\alpha v^2+(-\alpha^2+\tfrac{\beta^2}{4})u+\alpha\beta v], \\ g_{\alpha,\beta}(u,v) &= (v-\tfrac{v^3}{3}+vu^2)+ \\ &\quad [-\tfrac{\beta}{2}v^2+2\alpha uv+\tfrac{\beta}{2}u^2+(\alpha^2-\tfrac{\beta^2}{4})v+\alpha\beta u], \\ h_{\alpha,\beta}(u,v) &= (u^2-v^2)+[2\alpha u-\beta v], \end{aligned} \tag{2.14}$$

where $\alpha, \beta \in \mathbb{R}$, then the following surface given by the parametrization $\varphi_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi_{\alpha,\beta}(u,v) = (f_{\alpha,\beta}(u,v),\, g_{\alpha,\beta}(u,v),\, h_{\alpha,\beta}(u,v))$ is a minimal surface in $\mathbb{R}^3$ for all α, β real numbers.

Now we can state the following theorem:

Theorem 2.1. We consider the following family of surfaces in $\mathbb{R}^3$ parametrized by $\varphi_{\alpha,\beta}(u,v) = (f_{\alpha,\beta}(u,v),\ g_{\alpha,\beta}(u,v),\ h_{\alpha,\beta}(u,v))$, $\alpha,\beta \in \mathbb{R}$, where $f_{\alpha,\beta}(u,v)$, $g_{\alpha,\beta}(u,v)$, $h_{\alpha,\beta}(u,v)$ are defined in (2.14). Then the followings are true:

a) $\varphi_{\alpha,\beta}$ is minimal for all $\alpha,\beta \in \mathbb{R}$;

b) The coefficients of the first fundamental form are $E_{\alpha,\beta} = G_{\alpha,\beta} = [1 + (u+\alpha)^2 + (v+\frac{\beta}{2})^2]^2$; $F_{\alpha,\beta} = 0$;

c) The coefficients of the second fundamental form are $e_{\alpha,\beta} = 2$, $g_{\alpha,\beta} = -2$, $f_{\alpha,\beta} = 0$;

d) The Gaussian curvature is $K_{\alpha,\beta} = -\frac{4}{[1+(u+\alpha)^2+(v+\frac{\beta}{2})^2]^4}$;

e) The principal curvatures are

$$k^1_{\alpha,\beta} = \frac{2}{[1+(u+\alpha)^2+(v+\frac{\beta}{2})^2]^2},\ k^2_{\alpha,\beta} = -\frac{2}{[1+(u+\alpha)^2+(v+\frac{\beta}{2})^2]^2}.$$

Proof. a) It follows from the above construction of $\varphi_{\alpha,\beta}$. We construct $\varphi_{\alpha,\beta}$ such that $\varphi_{\alpha,\beta}$ is minimal.

b) We will put $a = u+\alpha$ and $b = v + \frac{\beta}{2}$. We have that:

$$\begin{aligned}
(\varphi_{\alpha,\beta})'_u &= (1-u^2+v^2-2\alpha u+\beta v-\alpha^2+\tfrac{\beta^2}{4},\\
&\qquad 2uv+2\alpha v+\beta u+\alpha\beta, 2u+2\alpha)\\
&= (1-a^2+b^2, 2ab, 2a),\\
(\varphi_{\alpha,\beta})'_v &= (2uv+\beta u+2\alpha v+\alpha\beta,\\
&\qquad 1-v^2+u^2-\beta v+2\alpha u+\alpha^2-\tfrac{\beta^2}{4}, -2v-\beta)\\
&= (2ab, 1+a^2-b^2, -2b),\\
(\varphi_{\alpha,\beta})''_{u^2} &= (-2a, 2b, 2),\\
(\varphi_{\alpha,\beta})''_{v^2} &= (2a, -2b, -2),\\
(\varphi_{\alpha,\beta})''_{uv} &= (2b, 2a, 0).
\end{aligned} \tag{2.15}$$

We compute now the first fundamental form:

$$\begin{aligned}
E_{\alpha,\beta} &= |(\varphi_{\alpha,\beta})'_u|^2\\
&= (1-a^2+b^2)^2+4a^2b^2+4a^2\\
&= [1+a^2+b^2]^2\\
&= [1+(u+\alpha)^2+(v+\tfrac{\beta}{2})^2]^2,\\
G_{\alpha,\beta} &= |(\varphi_{\alpha,\beta})'_v|^2\\
&= 4a^2b^2+(1+a^2-b^2)^2+4b^2\\
&= [1+a^2+b^2]^2\\
&= [1+(u+\alpha)^2+(v+\tfrac{\beta}{2})^2]^2,\\
F_{\alpha,\beta} &= \langle(\varphi_{\alpha,\beta})'_u, (\varphi_{\alpha,\beta})'_v\rangle\\
&= 2ab-2a^3b+2ab^3+2ab+2a^3b-2ab^3-4ab\\
&= 0.
\end{aligned} \tag{2.16}$$

c) We compute the unit normal vector $N_{\alpha,\beta} = \frac{(\varphi_{\alpha,\beta})'_u \times (\varphi_{\alpha,\beta})'_v}{|(\varphi_{\alpha,\beta})'_u \times (\varphi_{\alpha,\beta})'_v|}$ and we get $N_{\alpha,\beta} = \frac{1}{1+a^2+b^2}(-2a, 2b, 1-a^2-b^2)$.

We compute now the second fundamental form:

$$
\begin{array}{rcl}
e_{\alpha,\beta} & = & \langle(\varphi_{\alpha,\beta})''_{u^2}, N_{\alpha,\beta}\rangle \\
& = & \frac{1}{1+a^2+b^2}(4a^2+4b^2+2-2a^2-2b^2) \\
& = & 2, \\
g_{\alpha,\beta} & = & \langle(\varphi_{\alpha,\beta})''_{v^2}, N_{\alpha,\beta}\rangle \\
& = & \frac{1}{1+a^2+b^2}(-4a^2-4b^2-2+2a^2+2b^2) \\
& = & -2, \\
f_{\alpha,\beta} & = & \langle(\varphi_{\alpha,\beta})''_{uv}, N_{\alpha,\beta}\rangle \\
& = & \frac{1}{1+a^2+b^2}(-4ab+4ab) \\
& = & 0,
\end{array}
\tag{2.17}
$$

d) The Gaussian curvature is

$$
K_{\alpha,\beta} = \frac{e_{\alpha,\beta}g_{\alpha,\beta} - f^2_{\alpha,\beta}}{E_{\alpha,\beta}G_{\alpha,\beta} - F^2_{\alpha,\beta}} = -\frac{e^2_{\alpha,\beta}}{E^2_{\alpha,\beta}} = -\frac{4}{[1+(u+\alpha)^2+(v+\frac{\beta}{2})^2]^4}. \tag{2.18}
$$

e) Principal curvatures $k^1_{\alpha,\beta}, k^2_{\alpha,\beta}$ are the solution of the equation

$$
\begin{array}{c}
X^2 - 2HX + K = 0 \;\Rightarrow\; X^2 - \frac{4}{[1+(u+\alpha)^2+(v+\frac{\beta}{2})^2]^4} = 0 \;\Rightarrow \\
k^1_{\alpha,\beta} = \frac{2}{[1+(u+\alpha)^2+(v+\frac{\beta}{2})^2]^2},\; k^2_{\alpha,\beta} = -\frac{2}{[1+(u+\alpha)^2+(v+\frac{\beta}{2})^2]^2}.
\end{array}
\tag{2.19}
$$

□

Remark 2.3. We observe that the Enneper's surface is given by $\varphi_{0,0}$, so it is a particular case of $\varphi_{\alpha,\beta}$, . Thus the family $(\varphi_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}}$ generalizes in some sense the Enneper's surface.

Remark 2.4. We gave only a particular solution of the system (2.3). It will be interesting to find out more solutions of the system (2.3).

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