Novi Sad J. Math.
Vol. 40, No. 2, 2010, 7-16

ON COUNTABLE FAMILIES OF TOPOLOGIES ON A SET

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Abstract

Considering a countable number of topologies on a set X, we introduce the notion of $(\aleph_0)$topological spaces as a generalization of the notions of both bitopological spaces and (ω)topological spaces, and study some of their properties.

AMS Mathematics Subject Classification (2000): 54A10

Key words and phrases: bitopological space, (ω)topological space, $(\aleph_0)$topological space, $(\aleph_0)$open set, (m)Hausdorff, $(\aleph_0)$Hausdorff, $(\aleph_0)$regular, $(\aleph_0)$normal, completely $(\aleph_0)$normal, $(\aleph_0)$compact, $(\aleph_0)$paracompact, $(\beta\text{-}\aleph_0)$paracompact

1. Introduction

The purpose of this paper is to study a space equipped with countable number of topologies. The notion of bitopological spaces was introduced in Kelly [5] and the notion of (ω)topological spaces in Bose and Tiwari [1]. By generalizing both of these notions, we introduce here the notion of $(\aleph_0)$topological spaces (Definition 2.1). In the $(\aleph_0)$topological spaces, we define $(\aleph_0)$Hausdorffness, $(\aleph_0)$regularity, $(\aleph_0)$normality and $(\aleph_0)$compactness. We also define complete $(\aleph_0)$regularity and complete $(\aleph_0)$normality. We prove some results for these notions.

A set X equipped with an increasing sequence $\{\mathcal{T}_n\}$ of topologies is called a (ω)topological space. If $\mathcal{T}_n = \mathcal{T}_{n'}$ for all n, n', then the (ω)topological space X becomes a topological space. Thus the notion of (ω)topological spaces generalizes the notion of topological spaces. But it does not generalize the notion of bitopological spaces.

Fletcher et al. [4] attempted to define pairwise paracompactness in a bitopological space. But in the presence of pairwise Hausdorffness, the two topologies coincide and the resulting single topological space is paracompact whenever the bitopological space is pairwise paracompact. Later Datta [2] introduced a notion of pairwise paracompactness. In the $(\aleph_0)$topological spaces, we introduce $(\aleph_0)$paracompactness. A space which is $(\aleph_0)$Hausdorff and $(\aleph_0)$paracompact is

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$(\aleph_0)$regular. Raghavan and Reilly [8] introduced α-, β-, γ- and δ-pairwise paracompactness. They presented a δ-pairwise paracompactness version of Michael's characterization (Michael [7]) of regular paracompact spaces. Unfortunately, the proof of this result is not correct (Kovár [6]). We introduce here a $(\beta$-$\aleph_0)$paracompactness and prove the Michael's theorem for $(\beta$-$\aleph_0)$paracompactness of the space X when it is $(\aleph_0)$regular (Theorem 3.16). From this, we obtain an analogue of Michael's theorem for the β-pairwise paracompactness of a pairwise regular bitopological space, as a particular case.

2. Definitions

Let $\{\mathcal{P}_n\}$ be a sequence of topologies on a set X. The sequence $\{\mathcal{P}_n\}$ is said to satisfy the condition $(*)$ (resp. condition (a)) if for any positive integer m, the union (resp. intersection) of a finite number of sets $\in \bigcup_{n\neq m} \mathcal{P}_n$ is a set $\in \bigcup_{n\neq m} \mathcal{P}_n$.

We introduce the following definitions.

Definition 2.1. If $\{\mathcal{P}_n\}$ is a sequence of topologies on a set X satisfying the condition $(*)$, then the pair $(X, \{\mathcal{P}_n\})$ is called an $(\aleph_0)$*topological space*. A set $G \in \bigcup_n \mathcal{P}_n$ is called an $(\aleph_0)$*open set*.

Throughout the paper, R and N denote the set of real numbers and the set of positive integers respectively. Members of N are generally denoted by k, l, m, n etc. If $\mathcal{T}$ is a topology on a set X, then $(\mathcal{T})clA$ and $(\mathcal{T})intA$ denote the closure and interior respectively of $A \subset X$ with respect to $\mathcal{T}$. Unless otherwise mentioned, X denotes the $(\aleph_0)$topological space $(X, \{\mathcal{P}_n\})$. Elements of X are denoted by x, y etc. All the sets considered here are subsets of X.

Example 2.1. We enumerate all the rational numbers: $r_1, r_2, r_3, \ldots$. Let $\mathcal{Q}_n$ denote the particular point topology (Steen and Seebach, Jr. [9]) on R, where each nonempty set $\in \mathcal{Q}_n$ contains r_n.

Then $(R, \{\mathcal{Q}_n\})$ is an $(\aleph_0)$topological space, where the sequence $\{\mathcal{Q}_n\}$ does not satisfy the condition (a). In the $(\aleph_0)$topological space considered in Example 2.2, $\{\mathcal{P}_n\}$ satisfies the condition (a).

Definition 2.2. Suppose $Y \subset X$ and $\mathcal{P}_n \mid Y$ denotes the subspace topology on Y induced by $\mathcal{P}_n$. Then $(Y, \{\mathcal{P}_n \mid Y\})$ is called a *subspace* of $(X, \{\mathcal{P}_n\})$.

Definition 2.3. A function $f\colon X \to R$ is said to be $(\bigcup_{n\in M} \mathcal{P}_n)$*upper semi-continuous* (resp. $(\bigcup_{n\in M} \mathcal{P}_n)$*lower semi-continuous*) if for every $a \in R$, $f^{-1}((-\infty, a)) \in \bigcup_{n\in M} \mathcal{P}_n$ (resp. $f^{-1}((a, \infty)) \in \bigcup_{n\in M} \mathcal{P}_n$) where $M \subset N$.

Definition 2.4. X is said to be $(\aleph_0)$*compact* if every $(\aleph_0)$open cover $\mathcal{C}$ of X with $\mathcal{C} \cap \mathcal{P}_n \neq \emptyset$ for at least two values of n has a finite subcover. Let $M \subset N$. X is said to be (M)*compact* if every cover of X consisting of the sets $\in \bigcup_{n\in M} \mathcal{P}_n$ has a finite subcover.

Definition 2.5. An $(\aleph_0)$open cover $\{U_\alpha \mid U_\alpha \in \mathcal{P}_{n_\alpha}\}$ of X is said to be *shrinkable* if there is another $(\aleph_0)$open cover $\{V_\alpha \mid V_\alpha \in \mathcal{P}_{n_\alpha}\}$ of X with $(\mathcal{P}_n)clV_\alpha \subset U_\alpha$ for some $n \neq n_\alpha$.

Definition 2.6. A set D is called a $(\mathcal{P}_n)$*neighbourhood*, or in short a $(\mathcal{P}_n)$*nbd*, of a set A if $A \subset (\mathcal{P}_n)intD$.

Definition 2.7. X is said to be (m)*Hausdorff* if for any pair of distinct points $x, y \in X$, there exist $U \in \mathcal{P}_m$ and $V \in \bigcup_{n\neq m} \mathcal{P}_n$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. If X is (m)Hausdorff for all $m \in N$, then X is said to be an $(\aleph_0)$*Hausdorff space.*

The $(\aleph_0)$topological space $(R, \{\mathcal{Q}_n\})$ considered in Example 2.1, is not (n)Hausdorff for any n. Also for any n, the topological space $(R, \mathcal{Q}_n)$ is not Hausdorff and for any pair of integers m, n, the bitopological space $(R, \mathcal{Q}_m, \mathcal{Q}_n)$ is not pairwise Hausdorff.

Example 2.2. Let the sequence $\{\mathcal{P}_n\}$ of topologies on R be defined by

$$\mathcal{P}_1 = \mathcal{U},$$

$$\mathcal{P}_n = \{\emptyset\} \cup \{G \cup (n, \infty) \mid G \in \mathcal{U}\} \text{ for all } n > 1,$$

where $\mathcal{U}$ is the usual topology on R.

Then the $(\aleph_0)$topological space $(R, \{\mathcal{P}_n\})$ is (1)Hausdorff but it is not (n)Hausdorff for any $n \neq 1$, and so it is not $(\aleph_0)$Hausdorff.

Definition 2.8. X is said to be (m)*regular* if for any point $x \in X$ and any $(\mathcal{P}_n)$closed set A with $n \neq m$ and $x \notin A$, there exist $U \in \bigcup_{n\neq m} \mathcal{P}_n$ and $V \in \mathcal{P}_m$ such that $x \in U, A \subset V$, $U \cap V = \emptyset$. X is said to be $(\aleph_0)$*regular* if it is (m)regular for all $m \in N$.

It is easy to see that X is (m)regular iff for any point $x \in X$ and any $(\mathcal{P}_n)$open set G containing x with $n \neq m$, there exists a $U \in \bigcup_{n\neq m} \mathcal{P}_n$ containing x such that $(\mathcal{P}_m)clU \subset G$.

Definition 2.9. X is said to be *completely* (m)*regular* if for any point $x \in X$ and any $(\mathcal{P}_n)$closed set A with $n \neq m$ and $x \notin A$, there exists a function $f : X \to [0, 1]$ such that $f(x) = 0, f(y) = 1$ for all $y \in A$, f is $(\bigcup_{n\neq m} \mathcal{P}_n)$upper semi-continuous and $(\mathcal{P}_m)$lower semi-continuous. X is *completely* $(\aleph_0)$*regular* if it is completely (m)regular for all $m \in N$.

Definition 2.10. X is said to be (m)*normal* if given a $(\mathcal{P}_m)$closed set A and a $(\mathcal{P}_n)$closed set B with $n \neq m$ and $A \cap B = \emptyset$, there exist $U \in \bigcup_{n\neq m} \mathcal{P}_n$ and $V \in \mathcal{P}_m$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. If X is (m)normal for all $m \in N$, then it is said to be $(\aleph_0)$*normal.*

From the definition, it follows that X is (m)normal iff for any $(\mathcal{P}_m)$closed set A and any $(\mathcal{P}_n)$open set W containing A with $n \neq m$, there exists a $U \in \bigcup_{n\neq m} \mathcal{P}_n$ such that $A \subset U \subset (\mathcal{P}_m)clU \subset W$.

Definition 2.11. X is said to be *completely* (m)*normal* if for each pair A, B of subsets of X satisfying $(A \cap (\mathcal{P}_n)clB) \cup ((\mathcal{P}_m)clA \cap B) = \emptyset, n \neq m$, there exist $U \in \bigcup_{n\neq m} \mathcal{P}_n$ and $V \in \mathcal{P}_m$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Complete $(\aleph_0)$*normality* is defined in the obvious way.

The $(\aleph_0)$topological space $(R, \{\mathcal{Q}_n\})$ considered in Example 2.1, is completely $(\aleph_0)$normal but for no n, the topological space $(R, \mathcal{Q}_n)$ is normal. Also, for no n, $(R, \{\mathcal{Q}_n\})$ is (n)regular.

Example 2.3. Let X be an infinite set and $a, b, c \in X$. Suppose $\mathcal{G}$ (resp. $\mathcal{H}$) is a collection of subsets of X, which contains, in addition to $\emptyset$ and X, all those subsets E of X for which $a \notin E$ and $c \in E$ (resp. $a, b \notin E$ and $c \in E$). Then $\mathcal{G}$ and $\mathcal{H}$ form topologies on X. Let $\mathcal{P}_n = \mathcal{G}$ for odd n, and $= \mathcal{H}$ for even n.

Then the $(\aleph_0)$topological space $(X, \{\mathcal{P}_n\})$ is $(\aleph_0)$normal but not completely $(\aleph_0)$normal. In fact, it is completely (m)normal for no m.

Definition 2.12. A collection $\mathcal{C}$ of subsets of X is said to be $(\aleph_0)$*locally finite* if each $x \in X$ has a $(\mathcal{P}_n)$open nbd, for at least two values of n, intersecting at most finitely many $U \in \mathcal{C}$. For any $m \in N$, $\mathcal{C}$ is said to be $(\bigcup_{n\neq m} \mathcal{P}_n)$*locally finite* if for each $x \in X$, there exists a set $\in \bigcup_{n\neq m} \mathcal{P}_n$, containing x and intersecting at most finitely many $U \in \mathcal{C}$.

Definition 2.13. A refinement $\mathcal{V}$ of an $(\aleph_0)$open cover $\mathcal{U}$ is said to be a *parallel refinement* if $V \in \mathcal{V}$ is $(\mathcal{P}_n)$open whenever $V \subset U \in \mathcal{U}$ and U is $(\mathcal{P}_n)$open.

Definition 2.14. X is said to be $(\aleph_0)$*paracompact* if every $(\aleph_0)$open cover $\mathcal{U}$ of X with $\mathcal{U} \cap \mathcal{P}_n \neq \emptyset$ for at least two values of n has an $(\aleph_0)$locally finite parallel refinement.

It follows from the definitions that an $(\aleph_0)$compact space is $(\aleph_0)$paracompact. The converse is not true:

Example 2.4. Let $\mathcal{T}$ be the indiscrete topology on R, and $\mathcal{U}_n$ be the subspace topology $\mathcal{U} \mid I_n$ on $I_n = (-n, n)$, of the usual topology $\mathcal{U}$ on R. If $\mathcal{P}_n = \mathcal{T} \cup \mathcal{U}_n$, then $(R, \{\mathcal{P}_n\})$ is an $(\aleph_0)$topological space, which is $(\aleph_0)$paracompact but not $(\aleph_0)$compact.

Let $\mathcal{P}$ denote the smallest topology containing all the topologies $\mathcal{P}_n, n \in N$. We call a cover $\mathcal{C}$ of X, a $(\bigcup_{n\neq m} \mathcal{P}_n)$*cover* if $\mathcal{C} \subset \bigcup_{n\neq m} \mathcal{P}_n$.

Definition 2.15. X is said to be $(\beta\text{-}\aleph_0)$*paracompact* if for all $m \in N$, every $(\bigcup_{n\neq m} \mathcal{P}_n)$cover of X has a $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite $(\mathcal{P})$open refinement.

The following example shows that a (β-$\aleph_0$)paracompact space may not be ($\aleph_0$)paracompact.

Example 2.5. Suppose $X = [0, \infty)$ and $\mathcal{U}$ is the usual topology on X. Let $\mathcal{P}_1$ = discrete topology on X, $\mathcal{P}_2 = \mathcal{U}$ and for $n > 2$, $\mathcal{P}_n = \{\emptyset\} \cup \{G \cup (n, \infty) \mid G \in \mathcal{U}\}$. Then the ($\aleph_0$)topological space $(X, \{\mathcal{P}_n\})$ is (β-$\aleph_0$)paracompact but not ($\aleph_0$)paracompact.

Open question. Does ($\aleph_0$)paracompactness of a space imply (β-$\aleph_0$)paracompactness of the space?

3. Results

Theorem 3.1. *If X is ($\aleph_0$)compact, and K is ($\mathcal{P}_m$)closed for some m, then K is ($\aleph_0$)compact. If K is a proper subset of X, then K is ($\mathcal{P}_n$)compact for all $n \neq m$.*

Proof. We prove only the second part. Choose $n_0 \neq m$. Let $\mathcal{U}$ be a ($\mathcal{P}_{n_0}$)open cover of K. Then $\mathcal{U}_1 = \mathcal{U} \cup \{X - K\}$ is an ($\aleph_0$)open cover of X with $\mathcal{U}_1 \cap \mathcal{P}_n \neq \emptyset$ for two values of n. Therefore $\mathcal{U}_1$ has a finite subcover. Hence $\mathcal{U}$ has a finite subcover. □

Theorem 3.2. *If $(X, \mathcal{P}_n)$ is a Hausdorff topological space for each n, and $(X, \{\mathcal{P}_n\})$ is ($\aleph_0$)compact, then all $\mathcal{P}_n$ are identical.*

The proof is straightforward.

Theorem 3.3. *Let $M = \{n \in N \mid n \neq m\}$. If X is (m)Hausdorff and (M)compact, then $\mathcal{P}_n \subset \mathcal{P}_m$ for all n.*

Proof. Let $n \neq m$. If $\mathcal{P}_n \not\subset \mathcal{P}_m$, then there exists a set $U \in \mathcal{P}_n$ such that $U \notin \mathcal{P}_m$. Then $X - U$ is ($\mathcal{P}_n$)closed, and hence (M)compact. Since $X - U$ is not ($\mathcal{P}_m$)closed, there is a point $p \in U$ which is a ($\mathcal{P}_m$)limit point of $X - U$. Again, since X is (m)Hausdorff, for each $x \in X - U$, there exist $U_x \in \underset{n \neq m}{U} \mathcal{P}_n$ and $V_x \in \mathcal{P}_m$ such that $x \in U_x, p \in V_x$ and $U_x \cap V_x = \emptyset$. Then $\{U_x \mid x \in X - U\}$ is a cover of $X - U$ by sets $\in \bigcup_{n \in M} \mathcal{P}_n$. Therefore it has a finite subcover $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$. If $V = \bigcap_{i=1}^{n} V_{x_i}$, then $V \in \mathcal{P}_m$, $p \in V$ and $V \cap (X - U) = \emptyset$, which is a contradiction, since p is a ($\mathcal{P}_m$)limit point of $X - U$. □

Theorem 3.4. *X is (m)Hausdorff iff for all x,*

$$\{x\} = \bigcap \{(\mathcal{P}_n)clU \mid n \neq m, U \in \mathcal{P}_m, x \in U\}.$$

The proof is straightforward.

Theorem 3.5. *If X is (m)Hausdorff, then for each filterbase $\mathcal{F}$ on X such that $\mathcal{F}$ is ($\mathcal{P}_m$)convergent to x and ($\mathcal{P}_n$)convergent to y for some $n \neq m$, $x = y$. The converse is also true if $\{\mathcal{P}_n\}$ satisfies the condition (a).*

Proof. Let X be (m)Hausdorff, and $\mathcal{F}$ be a filterbase on X, which is $(\mathcal{P}_m)$convergent to x. Let $y \neq x$. Then there exist $U \in \mathcal{P}_m$ and $V \in \bigcup_{n\neq m} \mathcal{P}_n$ such that $x \in U, y \in V$, $U \cap V = \emptyset$. There is an $A \in \mathcal{F}$ with $A \subset U$. If $V \in \mathcal{P}_n, n \neq m$, then $\mathcal{F}$ cannot be $(\mathcal{P}_n)$convergent to y. For if it is so, there must be a $B \in \mathcal{F}$ with $B \subset V$, which is not possible, since any two elements of $\mathcal{F}$ have a non-empty intersection.

To prove the converse, suppose X is not (m)Hausdorff. Then there exists a pair x, y $(x \neq y)$ such that for any $U \in \mathcal{P}_m$ and $V \in \bigcup_{n\neq m} \mathcal{P}_n$ with $x \in U, y \in V$, we have $U \cap V \neq \emptyset$. So the class $\mathcal{C} = \{U \cap V \mid U \in \mathcal{P}_m, V \in \bigcup_{n\neq m} \mathcal{P}_n, x \in U, y \in V\}$ is a filterbase on X, which is $(\mathcal{P}_m)$convergent to x and $(\mathcal{P}_n)$convergent to y for all $n \neq m$. □

If X is $(\aleph_0)$Hausdorff, then Theorem 3.5 is true for any pair of elements $m, n \in N$.

Likewise, each of the following theorems with (m) has a corresponding result with $(\aleph_0)$.

Theorem 3.6. *If X is (m)Hausdorff and $M = \{n \mid n \neq m\}$, then (M)compact subsets are $(\mathcal{P}_m)$closed.*

Proof. Let K be an (M)compact subset of X and $x \in X - K$. If $y \in K$, then there exist sets $U_{xy} \in \mathcal{P}_m$ and $V_{xy} \in \bigcup_{n\neq m} \mathcal{P}_n$ such that $x \in U_{xy}, y \in V_{xy}$ and $U_{xy} \cap V_{xy} = \emptyset$. Then $\mathcal{V} = \{V_{xy} \mid y \in K\}$ forms a cover of K by sets $\in \bigcup_{n\in M} \mathcal{P}_n$. Therefore $\mathcal{V}$ has a finite subcover $V_{xy_1}, V_{xy_2}, \ldots, V_{xy_k}$. If $V_x = \bigcup_{i=1}^{k} V_{xy_i}$ and $U_x = \bigcap_{i=1}^{k} U_{xy_i}$, then $x \in U_x \in \mathcal{P}_m$, $K \subset V_x \in \bigcup_{n\neq m} \mathcal{P}_n$ and $U_x \cap V_x = \emptyset$. Now

$$\begin{aligned} X - K &\subset \bigcup\{U_x \mid x \in X - K\} \\ &\subset \bigcup\{X - V_x \mid x \in X - K\} \\ &\subset X - K. \\ \Rightarrow X - K &= \bigcup\{U_x \mid x \in X - K\}. \end{aligned}$$

Thus K is $(\mathcal{P}_m)$closed. □

Theorem 3.7. *X is (m)regular iff for $n_0 \neq m$, any $(\mathcal{P}_{n_0})$closed set A is the intersection of all $(\mathcal{P}_n)$closed $(\mathcal{P}_m)$nbds of A, $n \neq m$.*

Proof. Let X be (m)regular. Suppose x belongs to the complement $X - A$ of the $(\mathcal{P}_{n_0})$closed set A with $n_0 \neq m$. Since $X - A$ is $(\mathcal{P}_{n_0})$open, $n_0 \neq m$, there exists a set $G \in \bigcup_{n\neq m} \mathcal{P}_n$ such that $x \in G \subset (\mathcal{P}_m)clG \subset X - A$. Then $A \subset X - (\mathcal{P}_m)clG \subset X - G$. Thus $X - G$ is a $(\mathcal{P}_n)$closed $(\mathcal{P}_m)$nbd of A with

$n \neq m$ and $x \notin X - G$. Hence the intersection of all $(\mathcal{P}_n)$closed $(\mathcal{P}_m)$nbds of A with $n \neq m$, is the set A.

To prove the converse, suppose for $n_0 \neq m$, A is a $(\mathcal{P}_{n_0})$closed set not containing the point x. Then there exists a $(\mathcal{P}_n)$closed $(\mathcal{P}_m)$nbd F of A with $n \neq m$ and $x \notin F$. If $U = X - F$ and $V = (\mathcal{P}_m)intF$, then $U \in \bigcup_{n\neq m} \mathcal{P}_n, V \in \mathcal{P}_m, x \in U, A \subset V$ and $U \cap V = \emptyset$. □

Theorem 3.8. *If X is $(\aleph_0)$compact and (m)Hausdorff, and if $\{\mathcal{P}_n\}$ satisfies the condition (a), then X is (m)regular.*

Proof. Let $x \in X$ and A be a $(\mathcal{P}_n)$closed set with $n \neq m$ and $x \notin A$. If $y \in A$, then by (m)Hausdorffness of X, there exist $U_y \in \bigcup_{n\neq m} \mathcal{P}_n$ and $V_y \in \mathcal{P}_m$ such that $x \in U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$. Then $\mathcal{V} = \{V_y \mid y \in A\}$ forms a $(\mathcal{P}_m)$open cover of A. Since A is $(\mathcal{P}_n)$closed, by Theorem 3.1, it is $(\mathcal{P}_m)$compact, and hence $\mathcal{V}$ has a finite subcover $V_{y_1}, \ldots, V_{y_k}$. If $U = \bigcap_{i=1}^{k} U_{y_i}$ and $V = \bigcup_{i=1}^{k} V_{y_i}$, then $U \in \bigcup_{n\neq m} \mathcal{P}_n, V \in \mathcal{P}_m, x \in U, A \subset V$ and $U \cap V = \emptyset$. □

Theorem 3.9. *Every completely (m)regular space is (m)regular.*

The proof is straightforward.

Theorem 3.10. *If X is $(\aleph_0)$compact and (m)regular, then X is (m)normal.*

Proof. Suppose A is a $(\mathcal{P}_m)$closed set and B is a $(\mathcal{P}_n)$closed set with $n \neq m$ and $A \cap B = \emptyset$. For $x \in A$, there is a $U_x \in \bigcup_{n\neq m} \mathcal{P}_n$ and $V_x \in \mathcal{P}_m$ such that $x \in U_x, B \subset V_x$, $U_x \cap V_x = \emptyset$. Since $\mathcal{U} = \{U_x \mid x \in A\}$ is a cover of A with sets $\in \bigcup_{n\neq m} \mathcal{P}_n$, and X is $(\aleph_0)$compact, it follows that there is a finite subcollection $U_{x_1}, U_{x_2}, \ldots, U_{x_k}$ of $\mathcal{U}$ covering A. If $U = \bigcup_{i=1}^{k} U_{x_i}$ and $V = \bigcap_{i=1}^{k} V_{x_i}$, then $U \in \bigcup_{n\neq m} \mathcal{P}_n$, $V \in \mathcal{P}_m, U \cap V = \emptyset, A \subset U, B \subset V$. □

Corollary 3.1. *If X is $(\aleph_0)$compact and (m)Hausdorff, and if $\{\mathcal{P}_n\}$ satisfies the condition (a), then X is (m)normal.*

The following theorem is a generalization of Urysohn's Lemma for $(\aleph_0)$topological spaces.

Theorem 3.11. *Suppose X is (m)normal. Let A be a $(\mathcal{P}_m)$closed set and B be a $(\mathcal{P}_n)$closed set with $n \neq m$ and $A \cap B = \emptyset$. Then there exists a function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A, f(x) = 1$ for all $x \in B, f$ is $(\bigcup_{n\neq m} \mathcal{P}_n)$upper semi-continuous and $(\mathcal{P}_m)$lower semi-continuous.*

The proof is omitted. It is similar to the proof of Theorem 2.7 (Kelly [5]). It is easy to see the converse of the above theorem is also true.

From the definitions, it is clear that a completely (m)normal space is (m)normal. Also, it can be shown that complete (m)normality is a hereditary property but (m)normality is not hereditary. The (m)Hausdorff property, (m)regularity and complete (m)regularity are hereditary.

Theorem 3.12. *X is completely (m)normal iff every subspace of X is (m)normal.*

The proof is omitted.

Theorem 3.13. *If X is $(\aleph_0)$normal, then every finite $(\aleph_0)$open cover $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$ of X with $U_i \in \mathcal{P}_{n_i}$ and $n_i \neq n_{i'}$ if $i \neq i'$, is shrinkable.*

Proof. Suppose X is $(\aleph_0)$normal. We choose a finite $(\aleph_0)$open cover $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$ of X, satisfying the above conditions. Then $(X - U_1) \cap (X - U_2) \cap \dots \cap (X - U_k) = \emptyset$, and so $(X - U_1)$ and $(X - U_2) \cap \dots \cap (X - U_k)$ are disjoint. Again, $X - U_1$ is $(\mathcal{P}_{n_1})$closed and $(X - U_2) \cap \dots \cap (X - U_k)$ is $(\mathcal{P}_{n_0})$closed for some $n_0 \neq n_1$. So there exist $V_1 \in \mathcal{P}_n$ for some $n \neq n_1$ and $W_1 \in \mathcal{P}_{n_1}$ such that $X - U_1 \subset V_1,\ (X - U_2) \cap \dots \cap (X - U_k) \subset W_1$ and $V_1 \cap W_1 = \emptyset$. Then, $(\mathcal{P}_n)clW_1 \subset X - V_1 \subset U_1,\ n \neq n_1$. Therefore it follows that $\{W_1, U_2, \dots, U_k\}$ forms an $(\aleph_0)$open cover of X with $(\mathcal{P}_n)clW_1 \subset U_1,\ n \neq n_1$. If we apply the process k times, we obtain an $(\aleph_0)$open cover $\{W_1, W_2, \dots, W_k\}$ such that $W_i \in \mathcal{P}_{n_i}$ and $(\mathcal{P}_n)clW_i \subset U_i,\ n \neq n_i$. □

Theorem 3.14. *If X is $(\aleph_0)$paracompact, and if K is a $(\mathcal{P}_m)$closed subset of X for some m, then K is $(\aleph_0)$paracompact.*

The proof is straightforward.

Theorem 3.15. *If X is (m)Hausdorff and $(\aleph_0)$paracompact, and if $\{\mathcal{P}_n\}$ satisfies the condition (a), then X is (m)regular.*

The proof is omitted.

Theorem 3.16. *Let X be $(\aleph_0)$regular, and let $\{\mathcal{P}_n\}$ satisfy the condition (a). Then X is $(\beta$-$\aleph_0)$paracompact iff for all $m \in N$, every $(\bigcup\limits_{n \neq m} \mathcal{P}_n)$cover of X has a $(\mathcal{P})$open refinement $\mathcal{V} = \bigcup\limits_{k=1}^{\infty} \mathcal{V}_k$, where each $\mathcal{V}_k$ is $(\bigcup\limits_{n \neq m} \mathcal{P}_n)$locally finite.*

Proof. Since the 'only if ' part of the theorem is obviously true, we prove the 'if' part. It is done in three steps.

Step I. For $m \in N$, let $\mathcal{G}$ be a $(\bigcup\limits_{n \neq m} \mathcal{P}_n)$cover of X. Then it has a $(\mathcal{P})$open refinement $\mathcal{V} = \bigcup\limits_{k=1}^{\infty} \mathcal{V}_k$, where each $\mathcal{V}_k$ is $(\bigcup\limits_{n \neq m} \mathcal{P}_n)$locally finite. Suppose $\mathcal{V}_k = \{V_{k\alpha} \mid \alpha \in A\}$. Let $V_{k\alpha} \subset G_{k\alpha} \in \mathcal{G}$. We write $W_k = \bigcup\limits_{\alpha} G_{k\alpha},\ W^i = \bigcup\limits_{k=1}^{i} W_k,\ A_1 = W_1$, and for $i > 1,\ A_i = W_i - W^{i-1}$. Then $\{W_k \mid k \in N\}$ is a cover of X, and so $\{A_i \mid i \in N\}$ is a cover of X. This cover is $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite. In fact, if $k(x)$ is the first k for which $x \in W_{k(x)}$, then $x \in G_{k(x)\alpha}$ for some α, $G_{k(x)\alpha} \in \bigcup_{n\neq m} \mathcal{P}_n$, and $G_{k(x)\alpha}$ does not intersect any A_i for $i > k(x)$. Thus $\mathcal{E}(\mathcal{G}) = \{E_{k\alpha} \mid k \in N, \alpha \in A\}$, where $E_{k\alpha} = A_k \cap V_{k\alpha}$, is a $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite refinement of $\mathcal{G}$.

Step II. For $m \in N$, let $\mathcal{H}$ be a $(\bigcup_{n\neq m} \mathcal{P}_n)$cover of X. For $x \in X$, we choose an $H_x \in \mathcal{H}$ such that $x \in H_x$. Since X is $(\aleph_0)$regular, and since H_x is $(\mathcal{P}_n)$open with $n \neq m$, there exists a set $H_x^1 \in \bigcup_{n\neq m} \mathcal{P}_n$ such that $x \in H_x^1 \subset (\mathcal{P}_m)clH_x^1 \subset H_x$. Since $\mathcal{P}_m \subset \mathcal{P}$, we have

$$(1) \qquad (\mathcal{P})clH_x^1 \subset (\mathcal{P}_m)clH_x^1 \subset H_x.$$

Now $\mathcal{H}^1 = \{H_x^1 \mid x \in X\}$ is also a $(\bigcup_{n\neq m} \mathcal{P}_n)$cover of X. Hence by Step I, we get a $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite refinement $\mathcal{E}(\mathcal{H}^1)$ of $\mathcal{H}^1$. For $E \in \mathcal{E}(\mathcal{H}^1)$, we get an $H_x^1 \in \mathcal{H}^1$ with $E \subset H_x^1$ and hence by (1), $(\mathcal{P})clE \subset H_x$. Thus $\mathcal{F}(\mathcal{H}) = \{(\mathcal{P})clE \mid E \in \mathcal{E}(\mathcal{H}^1)\}$ is a $(\mathcal{P})$closed refinement of $\mathcal{H}$. Also $\mathcal{F}(\mathcal{H})$ is $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite: Since $\mathcal{E}(\mathcal{H}^1)$ is $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite, there exists, for $x \in X$, a $G \in \mathcal{P}_n$, $n \neq m$ such that $x \in G$, and for all but finitely many $E \in \mathcal{E}(\mathcal{H}^1)$, $G \cap E = \emptyset \Rightarrow G \cap (\mathcal{P}_n)clE = \emptyset \Rightarrow G \cap (\mathcal{P})clE = \emptyset$.

Step III. We choose an $m \in N$. Let $\mathcal{U}$ be a $(\bigcup_{n\neq m} \mathcal{P}_n)$cover of X. By Step I, $\mathcal{U}$ has a $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite refinement $\mathcal{E}(\mathcal{U})$. For $x \in X$, suppose $W_x \in \bigcup_{n\neq m} \mathcal{P}_n$ is a set containing x and intersecting a finite number of members of $\mathcal{E}(\mathcal{U})$. Then $\mathcal{W} = \{W_x \mid x \in X\}$ forms a $(\bigcup_{n\neq m} \mathcal{P}_n)$cover of X. Therefore by Step II, $\mathcal{W}$ has a $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite $(\mathcal{P})$closed refinement $\mathcal{F}(\mathcal{W})$. For $E \in \mathcal{E}(\mathcal{U})$, let

$$S_E = X - \bigcup\{F \in \mathcal{F}(\mathcal{W}) \mid F \cap E = \emptyset\}.$$

Since the collection $\mathcal{F}(\mathcal{W})$ of $(\mathcal{P})$closed sets is $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite, and since $\bigcup_{n\neq m} \mathcal{P}_n \subset \mathcal{P}$, by 9.2 (Dugundji [3], p. 82), it follows that the set $\bigcup\{F \in \mathcal{F}(\mathcal{W}) \mid F \cap E = \emptyset\}$ is $(\mathcal{P})$closed, and hence S_E is $(\mathcal{P})$open. Since $\mathcal{E}(\mathcal{U})$ is a cover of X, $\mathcal{S} = \{S_E \mid E \in \mathcal{E}(\mathcal{U})\}$ is a $(\mathcal{P})$open cover of X. $\mathcal{S}$ is also $(\bigcup_{n\neq m} \mathcal{P}_n)$locally finite. For, consider a set $D_x \in \bigcup_{n\neq m} \mathcal{P}_n$ containing x and intersecting $F_1, F_2, \ldots, F_k \in$

$\mathcal{F}(\mathcal{W})$. Now

$$\begin{aligned} & D_x \cap S_E \neq \emptyset, \\ \Rightarrow\ & F_i \cap S_E \neq \emptyset \text{ for some } i = 1, 2, \ldots, k, \\ \Rightarrow\ & F_i \cap E \neq \emptyset \text{ for some } i = 1, 2, \ldots, k. \end{aligned}$$

Since each F_i is contained in some W_x, it can intersect at most finitely many $E \in \mathcal{E}(\mathcal{U})$. Therefore D_x can intersect at most finitely many $S_E \in \mathcal{S}$.

For every $E \in \mathcal{E}(\mathcal{U})$, we take $U_E \in \mathcal{U}$ such that $E \subset U_E$. Then the collection $\{U_E \cap S_E \mid E \in \mathcal{E}(\mathcal{U})\}$ is a $(\bigcup_{n \neq m} \mathcal{P}_n)$locally finite $(\mathcal{P})$open refinement of $\mathcal{U}$. Therefore X is $(\beta\text{-}\aleph_0)$paracompact. □

Acknowledgement. We are grateful to the referee for careful reading the paper and for certain suggestions towards the improvement of the paper.

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Received by the editors April 12, 2008