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ZERMELLO'S CONDITIONS AND ENERGIES OF HIGHER ORDER IN GENERALIZED LAGRANGE-HAMILTON SPACES

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Abstract. Many significant geometers contributed to the generalization of Riemann spaces in different directions. An almost complete list of them can be found in Miron's books. Here are mention [1], [2], [17] and [18] in which Hamilton and Finsler spaces are examined, further [3–9], where generalized Hamilton spaces are studied; [10] is most connected with the subject, and in [11–16] this problem also appears. Zermello's condition in Miron's $Osc^k M$ was examined in [6]. Here, the Zermello's conditions are given in Lagrange-Hamilton spaces, introduced in [9] and presented at the Workshop on Finsler Geometry 2009, Debrecen. It is proved that for a fundamental function for which the Zermello's conditions are satisfied all energies of higher order are equal to zero.

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1. Some invariants in generalized Lagrange-Hamilton spaces

Group of coordinate transformations. Generalized Lagrange-Hamilton spaces are introduced in [9]. We shall recall only those parts of which are necessary for the understanding of Zermello's condition in these spaces.

Let us denote by $(LH)^{(nk)}$ the $(2k+1)n$ dimensional C^∞ manifold in which a point $(y,p) = (x = y^{(0)}, y^{(1)}, y^{(2)}, \dots, y^{(k)}, p_{(1)}, p_{(2)}, \dots, p_{(k)})$ has the coordinates

$$(x^a = y^{0a}, y^{1a}, y^{2a}, \dots, y^{ka}, p_{1a}, p_{2a}, \dots, p_{ka}), \quad a = \overline{1,n}.$$

Some curve $c \in (LH)^{(nk)}$ is given by $c : t \in [a,b] \to c(t) \in (LH)^{(nk)}$. A point $(y,p) \in c(t)$ has the coordinates $(x^a(t) = y^{0a}(t), y^{1a}(t), \dots, y^{ka}(t), p_{1a}(t), \dots, p_{ka}(t))$, where

$$y^{Aa}(t) = d_t^A y^{0a}(t) \quad A = \overline{1,k}, \quad d_t^A = \frac{d^A}{dt^A}, \tag{1.1}$$

$$p_{\alpha a}(t) = d_t^{\alpha-1} p_{1a}(t), \quad \alpha = \overline{1,k}, \quad d_t^{\alpha-1} = \frac{d^{\alpha-1}}{dt^{\alpha-1}}.$$

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The allowable coordinate transformations are given by

$$
\begin{aligned}
(1.2)\quad x^{a'} =& x^{a'}(x^a) \Leftrightarrow x^a = x^a(x^{a'})\\
y^{1a'} =& B^{a'}_a y^{1a}, \quad B^{a'}_a = \partial_{0a} x^{a'} = \partial_a x^{a'}, \partial_{Aa} = \frac{\partial}{\partial y^{Aa}} \quad A = \overline{0,k},\\
y^{2a'} =& \binom{1}{0}(d^1_t B^{a'}_a) y^{1a} + \binom{1}{1} B^{a'}_a y^{2a} = d^1_t(B^{a'}_a y^{1a}),\\
y^{3a'} =& \binom{2}{0}(d^2_t B^{a'}_a) y^{1a} + \binom{2}{1}(d^1_t B^{a'}_a) y^{2a} + \binom{2}{2} B^{a'}_a y^{3a} = d^2_t(B^{a'}_a y^{1a}), \dots,\\
y^{Aa'} =& \binom{A-1}{0}(d^{A-1}_t B^{a'}_a) y^{1a} + \binom{A-1}{1}(d^{A-2}_t B^{a'}_a) y^{2a} + \cdots\\
& \cdots + \binom{A-1}{A-1} B^{a'}_a y^{Aa} = d^{A-1}_t(B^{a'}_a y^{1a}), \dots,\\
y^{ka'} =& \binom{k-1}{0}(d^{k-1}_t B^{a'}_a) y^{1a} + \binom{k-1}{1}(d^{k-2}_t B^{a'}_a) y^{2a} + \cdots\\
& \cdots + \binom{k-1}{k-1} B^{a'}_a y^{ka} = d^{k-1}_t(B^{a'}_a y^{1a}),\\
p_{1a'} =& B^a_{a'} p_{1a} \quad B^a_{a'} = \partial_{0a'} x^a = \frac{\partial x^a}{\partial x^{a'}} = B^a_{a'}(t),\\
p_{2a'} =& \binom{1}{0}(d^1_t B^a_{a'}) p_{1a} + \binom{1}{1} B^a_{a'} p_{2a} = d^1_t(B^a_{a'} p_{1a}),\\
p_{3a'} =& \binom{2}{0}(d^2_t B^a_{a'}) p_{1a} + \binom{2}{1}(d^1_t B^a_{a'}) p_{2a} + \binom{2}{2} B^a_{a'} p_{3a} = d^2_t(B^a_{a'} p_{1a}),\\
p_{4a'} =& \binom{3}{0}(d^3_t B^c_{a'}) p_{1c} + \binom{3}{1}(d^2_t B^c_{a'}) p_{2c} + \binom{3}{2}(d^1_t B^c_{a'}) p_{3c} + \binom{3}{3} B^c_{a'} p_{4c}, \dots,\\
p_{\alpha a'} =& \binom{\alpha-1}{0}(d^{\alpha-1}_t B^a_{a'}) p_{1a} + \binom{\alpha-1}{1}(d^{\alpha-2}_t B^a_{a'}) p_{2a} + \cdots\\
& \cdots + \binom{\alpha-1}{\alpha-1} B^a_{a'} p_{\alpha a}, \dots,\\
p_{ka'} =& \binom{k-1}{0}(d^{k-1}_t B^a_{a'}) p_{1a} + \binom{k-1}{1}(d^{k-2}_t B^a_{a'}) p_{2a} + \cdots
\end{aligned}
$$

$$\cdots + \binom{k-1}{k-1} B^a_{a'} p_{ka}.$$

Theorem 1.1. *The transformations of type (1.3) on the common domain form a group.*

Definition 1.1. *The generalized Lagrange-Hamilton space of order k, $(GLH)^{(nk)}$, is an $(LH)^{(nk)}$ space, where the group of allowable transformations is given by (1.2) and in which a fundamental function $F(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)}, p_{(1)}, p_{(2)}, \ldots, p_{(k)})$ is given, where $F : U \to R$ is differentiable on $\tilde{U}$ (where rank $[y^{1a}] = 1$, rank $[p_{1a}] = 1$) and continuous in those points of U, where y^{1a} and p_{1a} are equal to zero, U is a domain in $(GLH)^{(nk)}$.*

The natural and special adapted bases in $T(GLH)^{(nk)}$ and $T^*(GLH)^{(nk)}$. The natural basis, $\bar{B}_{LH}$ of $T(GLH)^{(nk)}$ as usual consists of partial derivatives of variables, i.e.

$$\text{(1.3)} \quad \bar{B}_{LH} = \{\partial_{0a}, \partial_{1a}, \ldots, \partial_{ka}, \partial^{1a}, \partial^{2a}, \ldots, \partial^{ka}\}, \quad \partial_{0a} = \partial_a = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}},$$

$$\partial_{Aa} = \frac{\partial}{\partial y^{Aa}} \; A = \overline{1,k}, \quad \partial^{\alpha a} = \frac{\partial}{\partial p_{\alpha a}}, \; \alpha = \overline{1,k}.$$

Theorem 1.2. *The elements of $\bar{B}_{LH}$ transform in the following way:*

(1.4)

$$\begin{aligned}
\partial_{0a} =& (\partial_{0a} y^{0a'})\partial_{0a'} + (\partial_{0a} y^{1a'})\partial_{1a'} + (\partial_{0a} y^{2a'})\partial_{2a'} + (\partial_{0a} y^{3a'})\partial_{3a'} + \cdots + (\partial_{0a} y^{ka'})\partial_{ka'} + \\
& (\partial_{0a} p_{1a'})\partial^{1a'} + (\partial_{0a} p_{2a'})\partial^{2a'} + (\partial_{0a} p_{3a'})\partial^{3a'} + \cdots + (\partial_{0a} p_{ka'})\partial^{ka'}, \\
\partial_{1a} =& (\partial_{1a} y^{1a'})\partial_{1a'} + (\partial_{1a} y^{2a'})\partial_{2a'} + (\partial_{1a} y^{3a'})\partial_{3a'} + \cdots + (\partial_{1a} y^{ka'})\partial_{ka'} + \\
& (\partial_{1a} p_{2a'})\partial^{2a'} + (\partial_{1a} p_{3a'})\partial^{3a'} + \cdots + (\partial_{1a} p_{ka'})\partial^{ka'}, \\
\partial_{2a} =& (\partial_{2a} y^{2a'})\partial_{2a'} + (\partial_{2a} y^{3a'})\partial_{3a'} + \cdots + (\partial_{2a} y^{ka'})\partial_{ka'} + \\
& (\partial_{2a} p_{3a'})\partial^{3a'} + \cdots + (\partial_{2a} p_{ka'})\partial^{ka'}, \\
&\vdots \\
\partial_{ka} =& (\partial_{ka} y^{ka'})\partial_{ka'} \\
\partial^{1a} =& (\partial^{1a} p_{1a'})\partial^{1a'} + (\partial^{1a} p_{2a'})\partial^{2a'} + (\partial^{1a} p_{3a'})\partial^{3a'} + \cdots + (\partial^{1a} p_{ka'})\partial^{ka'}, \\
\partial^{2a} =& (\partial^{2a} p_{2a'})\partial^{2a'} + (\partial^{2a} p_{3a'})\partial^{3a'} + \cdots + (\partial^{2a} p_{ka'})\partial^{ka'}, \\
\partial^{3a} =& (\partial^{3a} p_{3a'})\partial^{3a'} + \cdots + (\partial^{3a} p_{ka'})\partial^{ka'}, \\
&\vdots \\
\partial^{ka} =& (\partial^{ka} p_{ka'})\partial^{ka'}.
\end{aligned}$$

The natural basis of $T^*(GLH)^{(nk)}$ is

$$\bar{B}^*_{LH} = \{dy^{0a}, dy^{1a}, \ldots, dy^{ka}, dp_{1a}, dp_{2a}, \ldots, dp_{ka}\}.$$

Theorem 1.3. *The elements of $\bar{B}^*_{LH}$ transform in the following way:*

(1.5)

$$\begin{aligned}
dy^{0a'} &= (\partial_{0a} y^{0a'}) dy^{0a} \\
dy^{1a'} &= (\partial_{0a} y^{1a'}) dy^{0a} + (\partial_{1a} y^{1a'}) dy^{1a}, \ldots, \\
dy^{ka'} &= (\partial_{0a} y^{ka'}) dy^{0a} + (\partial_{1a} y^{ka'}) dy^{1a} + \cdots + (\partial_{ka} y^{ka'}) dy^{ka}, \\
dp_{1a'} &= (\partial_{0a} p_{1a'}) dy^{0a} + (\partial^{1a} p_{1a'}) dp_{1a}, \\
dp_{2a'} &= (\partial_{0a} p_{2a'}) dy^{0a} + (\partial_{1a} p_{2a'}) dy^{1a} + \\
&\quad (\partial^{1a} p_{2a'}) dp_{1a} + (\partial^{2a} p_{2a'}) dp_{2a}, \ldots, \\
dp_{ka'} &= (\partial_{0a} p_{ka'}) dy^{0a} + (\partial_{1a} p_{ka'}) dy^{1a} + \cdots + (\partial_{(k-1)a} p_{ka'}) dy^{(k-1)a} + \\
&\quad (\partial^{1a} p_{ka'}) dp_{1a} + \cdots + (\partial^{ka} p_{ka'}) dp_{ka}.
\end{aligned}$$

Definition 1.2. *The special adapted basis B_{LH} of $T(GLH)^{(nk)}$*

$$B_{LH} = \{\delta_{0a}, \delta_{1a}, \ldots, \delta_{ka}, \delta^{1a}, \delta^{2a}, \ldots, \delta^{ka}\} \tag{1.6}$$

is defined by

(1.7)

$$\begin{aligned}
\delta_{0a} =& \tbinom{0}{0}\partial_{0a} - \tbinom{1}{0}N^{1b}_{0a}\partial_{1b} - \tbinom{2}{0}N^{2b}_{0a}\partial_{2b} - \tbinom{3}{0}N^{3b}_{0a}\partial_{3b} - \cdots - \tbinom{k}{0}N^{kb}_{0a}\partial_{kb}\\
&- \tbinom{0}{0}N_{0a1b}\partial^{1b} - \tbinom{1}{0}N_{0a2b}\partial^{2b} - \tbinom{2}{0}N_{0a3b}\partial^{3b} - \cdots - \tbinom{k-1}{0}N_{0akb}\partial^{kb}\\
\delta_{1a} =& \tbinom{1}{1}\partial_{1a} - \tbinom{2}{1}N^{1b}_{0a}\partial_{2b} - \tbinom{3}{1}N^{2b}_{0a}\partial_{3b} - \cdots - \tbinom{k}{1}N^{(k-1)b}_{0a}\partial_{kb}\\
&- \tbinom{1}{1}N_{0a1b}\partial^{2b} - \tbinom{2}{1}N_{0a2b}\partial^{3b} - \cdots - \tbinom{k-1}{1}N_{0a(k-1)b}\partial^{kb}\\
\delta_{2a} =& \tbinom{2}{2}\partial_{2a} - \tbinom{3}{2}N^{1b}_{0a}\partial_{3b} - \cdots - \tbinom{k}{2}N^{(k-2)b}_{0a}\partial_{kb}\\
&- \tbinom{2}{2}N_{0a1b}\partial^{3b} - \cdots - \tbinom{k-1}{2}N_{0a(k-2)b}\partial^{kb}\\
\delta_{3a} =& \tbinom{3}{3}\partial_{3a} - \cdots - \tbinom{k}{3}N^{(k-3)b}_{0a}\partial_{kb}\\
&- \cdots - \tbinom{k-1}{3}N_{0a(k-3)b}\partial^{kb}, \ldots,\\
\delta_{ka} =& \tbinom{k}{k}\partial_{ka}\\
\delta^{1a} =& \tbinom{0}{0}\partial^{1a} - \tbinom{1}{0}N^{0a}_{2b}\partial^{2b} - \tbinom{2}{0}N^{0a}_{3b}\partial^{3b} - \cdots - \tbinom{k}{0}N^{0a}_{kb}\partial^{kb}\\
\delta^{2a} =& \tbinom{1}{1}\partial^{2a} - \tbinom{2}{1}N^{0a}_{2b}\partial^{3b} - \cdots - \tbinom{k-1}{1}N^{0a}_{(k-1)b}\partial^{kb}\\
\delta^{3b} =& \tbinom{2}{2}\partial^{3b} - \cdots - \tbinom{k-2}{2}N^{0a}_{(k-2)b}\partial^{kb}, \ldots,\\
\delta^{kb} =& \tbinom{k-1}{k-1}\partial^{kb}.
\end{aligned}$$

Definition 1.3. *The special adapted basis B^*_{LH} of $T^*(GLH)^{(nk)}$ is*

$$B^*_{LH} = \{\delta y^{0a}, \delta y^{1a}, \ldots, \delta y^{ka}, \delta p_{1a}, \delta p_{2a}, \ldots, \delta p_{ka}\}, \tag{1.8}$$

where

$$\begin{aligned}
\delta y^{0a} =& dy^{0a} = dx^a \\
\delta y^{1a} =& \binom{1}{1} dy^{1a} + \binom{1}{0} M^{1a}_{0b} dy^{0b}\\
\delta y^{2a} =& \binom{2}{2} dy^{2a} + \binom{2}{1} M^{1a}_{0b} dy^{1b} + \binom{2}{0} M^{2a}_{0b} dy^{0b},\\
\delta y^{3a} =& \binom{3}{3} dy^{3a} + \binom{3}{2} M^{1a}_{0b} dy^{2b} + \binom{3}{1} M^{2a}_{0b} dy^{1b} + \binom{3}{0} M^{3a}_{0b} dy^{0b}, \ldots,\\
\delta y^{ka} =& \binom{k}{k} dy^{ka} + \binom{k}{k-1} M^{1a}_{0b} dy^{(k-1)b} + \binom{k}{k-2} M^{2a}_{0b} dy^{(k-2)b} + \cdots\\
&\cdots + \binom{k}{0} M^{ka}_{0b} dy^{0b},
\end{aligned} \tag{1.9}$$

$$\delta p_{1a}=\binom{0}{0}M_{0a1b}dy^{0b}+\binom{0}{0}dp_{1a},$$

$$\delta p_{2a}=\binom{1}{0}M_{0a2b}dy^{0b}+\binom{1}{1}M_{0a1b}dy^{1b}+\binom{1}{0}M^{1b}_{0a}dp_{1b}+\binom{1}{1}dp_{2a},$$

$$\delta p_{3a}=\binom{2}{0}M_{0a3b}dy^{0b}+\binom{2}{1}M_{0a2b}dy^{1b}+\binom{2}{2}M_{0a1b}dy^{2b}+$$

$$\binom{2}{0}M^{2b}_{0a}dp_{1b}+\binom{2}{1}M^{1b}_{0a}dp_{2b}+\binom{2}{2}dp_{3a},\dots,$$

$$\delta p_{ka}=\binom{k-1}{0}M_{0akb}dy^{0b}+\binom{k-1}{1}M_{0a(k-1)b}dy^{1b}+\cdots$$

$$\cdots+\binom{k-1}{k-1}M_{0a1b}dy^{(k-1)b}+\binom{k-1}{0}M^{(k-1)b}_{0a}dp_{1b}$$

$$+\binom{k-1}{1}M^{(k-2)b}_{0a}dp_{2b}+\cdots+\binom{k-1}{k-1}dp_{ka}.$$

In [9], there are given the conditions for M's and N's such that the elements of B_{LH} and B^*_{LH} are tensors and when these bases are dual to each other.

The $\bar{J}$ structure in $(GLH)^{(nk)}$

Definition 1.4. *The k-tangent structure $\bar{J}$ is a $\mathcal{F}$ linear mapping*

$$\bar{J}: T^*(GLH)^{(nk)} \to T^*(GLH)^{(nk)}$$

defined by

$$(1.10)\quad \bar{J}dy^{0a}=0, \bar{J}dy^{1a}=dy^{0a}, \bar{J}dy^{2a}=2dy^{1a},\dots,\bar{J}dy^{ka}=kdy^{(k-1)a}$$

$$\bar{J}dp_{1a}=0, \bar{J}dp_{2a}=dp_{1a}, \bar{J}dp_{3a}=2dp_{2a},\dots,\bar{J}dp_{ka}=(k-1)dp_{(k-1)a},$$

from which it follows

$$(1.11)\quad \bar{J} = dy^{0a}\otimes\partial_{1a}+2dy^{1a}\otimes\partial_{2a}+\cdots+kdy^{(k-1)a}\otimes\partial_{ka}+$$

$$dp_{1a}\otimes\partial^{2a}+2dp_{2a}\otimes\partial^{3a}+\cdots+(k-1)dp_{(k-1)a}\otimes\partial^{ka}.$$

In [9], it is proved that the $\bar{J}$ structure in the special adapted bases B_{LH} and B^*_{LH} is given by

$$(1.12)\bar{J} = \delta y^{0a}\otimes\delta_{1a}+2\delta y^{1a}\otimes\delta_{2a}+3\delta y^{2a}\otimes\delta_{3a}+\cdots+k\delta y^{(k-1)a}\otimes\delta_{ka}+$$

$$\delta p_{1a}\otimes\delta^{2a}+2\delta p_{2a}\otimes\delta^{3a}+3\delta p_{3a}\otimes\delta^{4a}+\cdots+(k-1)\delta p_{(k-1)a}\otimes\delta^{ka}.$$

Remark 1.1. From (1.11) and (1.12) it follows that the k-tangent structure $\bar{J}$ in the natural and special adapted bases has the same coordinates.

It is also proved that the following relations are valid

$$\bar{J}\delta y^{0a}=0,\bar{J}\delta y^{1a}=\delta y^{0a},\bar{J}\delta y^{2a}=2\delta y^{1a},\dots,\bar{J}\delta y^{ka}=k\delta y^{(k-1)a} \tag{1.13}$$

$$\bar{J}\delta p_{1a}=0,\bar{J}\delta p_{2a}=\delta p_{1a},\bar{J}\delta p_{3a}=2\delta p_{2a},\dots,\bar{J}\delta p_{ka}=(k-1)\delta p_{(k-1)a},$$

The Liouville vector field. If $M(y^{0a},y^{1a},\dots,y^{ka},p_{1a},p_{2a},\dots,p_{ka})$ and $M'(y^{0a}+dy^{0a},y^{1a}+dy^{1a},\dots,y^{ka}+dy^{ka},p_{1a}+dp_{1a},p_{2a}+dp_{2a},\dots,p_{ka}+dp_{ka})$ are two points in $(GLH)^{(nk)}$, then the vector MM' expressed in the natural basis $T(GLH)^{(nk)}$ has the form [9]

$$\begin{aligned} MM'=dr &= dy^{0a}\partial_{0a}+dy^{1a}\partial_{1a}+\cdots+dy^{ka}\partial_{ka}+ \\ & dp_{1a}\partial^{1a}+dp_{2a}\partial^{2a}+\cdots+dp_{ka}\partial^{ka}. \end{aligned} \tag{1.14}$$

It is proved ([9]) that dr is coordinate invariant, i.e.

$$\begin{aligned} dr &= \delta y^{0a}\delta_{0a}+\delta y^{1a}\delta_{1a}+\cdots+\delta y^{ka}\delta_{ka}+ \\ & \delta p_{1a}\delta^{1a}+\delta p_{2a}\delta^{2a}+\cdots+\delta p_{ka}\delta^{ka}. \end{aligned} \tag{1.15}$$

Definition 1.5. *The Liouville vector fields $\bar{\Gamma}_0,\bar{\Gamma}_1,\bar{\Gamma}_2,\dots,\bar{\Gamma}_k$ are defined by*

(1.16)

$$\bar{\Gamma}_k=dr,\quad \bar{J}\bar{\Gamma}_A=\bar{\Gamma}_A\bar{J}=(k-(A-1))\bar{\Gamma}_{A-1},\quad \bar{A}=\overline{1,k},\quad \bar{J}\bar{\Gamma}_0=\bar{\Gamma}_0\bar{J}=0.$$

Remark 1.2. From (1.14)-(1.15) it is obvious that dr has the same components in the natural and special adapted bases. The same property has the structure $\bar{J}$ (see Remark 1.1). This fact allows that the action $\bar{J}$ on dr can be written by the equations of the same form in both coordinate systems.

In $(GLH)^{(nk)}$ it is difficult to construct vector fields, but using dr, the structure $\bar{J}$, one family of the Liouville vector field can, be constructed.

From (1.16) it follows

$$\begin{aligned} &\bar{J}\bar{\Gamma}_0=\bar{\Gamma}_0\bar{J}=0,\quad \bar{J}\bar{\Gamma}_1=\bar{\Gamma}_1\bar{J}=k\bar{\Gamma}_0,\quad \bar{J}\bar{\Gamma}_2=\bar{\Gamma}_2\bar{J}=(k-1)\bar{\Gamma}_1, \\ &\dots,\bar{J}\bar{\Gamma}_{k-1}=\bar{\Gamma}_{k-1}\bar{J}=2\bar{\Gamma}_{k-2},\quad \bar{J}\bar{\Gamma}_k=\bar{\Gamma}_k\bar{J}=\bar{\Gamma}_{k-1}. \end{aligned} \tag{1.17}$$

Theorem 1.4. *The Liouville vector fields $\bar{\Gamma}_0,\bar{\Gamma}_1,\dots,\bar{\Gamma}_k$ from $(GLH)^{(nk)}$ expressed in the special adapted basis B of $T(GLH)^{(nk)}$, have the form [9]*

(1.18)

$$\bar{\Gamma}_0 = \binom{k}{0}\delta y^{0a}\delta_{ka},$$

$$\begin{aligned}
\bar\Gamma_1 &= \binom{k}{1}\delta y^{1a}\delta_{ka} + \binom{k-1}{0}\delta y^{0a}\delta_{(k-1)a} + \\
&\binom{k-1}{0}\delta p_{1a}\delta^{ka}, \\
\bar\Gamma_2 &= \binom{k}{2}\delta y^{2a}\delta_{ka} + \binom{k-1}{1}\delta y^{1a}\delta_{(k-1)a} + \binom{k-2}{0}\delta y^{0a}\delta_{(k-2)a} + \\
&\binom{k-1}{1}\delta p_{2a}\delta^{ka} + \binom{k-2}{0}\delta p_{1a}\delta^{(k-1)a}, \\
\bar\Gamma_3 &= \binom{k}{3}\delta y^{3a}\delta_{ka} + \binom{k-1}{2}\delta y^{2a}\delta_{(k-1)a} + \binom{k-2}{1}\delta y^{1a}\delta_{(k-2)a} + \\
&\binom{k-3}{0}\delta y^{0a}\delta_{(k-3)a} + \binom{k-1}{2}\delta p_{3a}\delta^{ka} + \binom{k-2}{1}\delta p_{2a}\delta^{(k-1)a} + \\
&\binom{k-3}{0}\delta p_{1a}\delta^{(k-2)a}, \dots, \\
\bar\Gamma_{k-1} &= \binom{k}{k-1}\delta y^{(k-1)a}\delta_{ka} + \binom{k-1}{k-2}\delta y^{(k-2)a}\delta_{(k-1)a} + \dots + \binom{2}{1}\delta y^{1a}\delta_{2a} + \\
&\binom{1}{0}\delta y^{0a}\delta_{1a} + \binom{k-1}{k-2}\delta p_{(k-1)a}\delta^{ka} + \\
&\binom{k-2}{k-3}\delta p_{(k-2)a}\delta^{(k-1)a} + \dots + \binom{1}{0}\delta p_{1a}\delta^{2a}, \\
\bar\Gamma_k &= \binom{k}{k}\delta y^{ka}\delta_{ka} + \binom{k-1}{k-1}\delta y^{(k-1)a}\delta_{(k-1)A} + \dots + \binom{1}{1}\delta y^{1a}\delta_{1a} + \\
&\binom{0}{0}\delta y^{0a}\delta_{0a} + \binom{k-1}{k-1}\delta p_{ka}\delta^{ka} + \\
&\binom{k-2}{k-2}\delta p_{(k-1)a}\delta^{(k-1)a} + \dots + \binom{0}{0}\delta p_{1a}\delta^{1a}.
\end{aligned}$$

Theorem 1.5. *The Liouville vector fields $\bar\Gamma_0, \bar\Gamma_1, \dots, \bar\Gamma_k$ in $(GLH)^{(nk)}$ in the natural basis $\bar B$ of $T(GLH)^{(nk)}$ have the form obtained from (1.18) if $\delta y^{Aa}, \delta p_{\alpha a}, \delta_{Aa}, \delta^{\alpha a}$, are substituted by $dy^{Aa}, dp_{\alpha a}, \partial_{Aa}, \partial^{\alpha a}$ respectively for every $A = \overline{0,k}$, $\alpha = \overline{1,k}$.*

Proof. The proof follows from Definition 1.5 and Remark 1.2.

2. The Zermello's conditions

Let c^* be a curve in $(GLH)^{(nk)}$ such that

$$c^* : t \in [0,1] \to x^a(t)\partial_a + d_t^1 x^a(t)\partial_{1a} + \cdots + d_t^k x^a(t)\partial_{ka} + p_{1a}(t)\partial^{1a} + \cdots + d_t^{k-1} p_{1a}\partial^{ka} = = y^{0a}(t)\partial_{0a} + y^{1a}\partial_{1a} + \cdots + y^{ka}\partial_{ka} + p_{1a}\partial^{1a} + \cdots + p_{ka}\partial^{ka} \tag{2.1}$$

and $Imc^* \subset U$.

The integral of action I_{c^*} is

$$I_{c^*} = \int_0^1 F(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)}, p_{(1)}, p_{(2)}, \ldots, p_{(k)})dt. \tag{2.2}$$

I_{c^*} does not depend on the parametrization of the curve c^*:

$$x^a = y^{0a}(t) = y^{0a}, \quad y^{Aa} = d_t^A x^a = \frac{d^A x^a}{dt^A}, \quad A = \overline{1,k},$$

$$p_{1a} = p_{1a}(t), \quad p_{\alpha a} = d_t^{\alpha-1} p_{1a} = \frac{d^{\alpha-1} p_{1a}}{dt^{\alpha-1}} \quad \alpha = \overline{1,k}$$

if

$$\int_0^1 F(x, y^1, y^2, \ldots, y^k, p_1, p_2, \ldots, p_k)dt = = \int_0^1 F(x, y^{1'}, y^{2'}, \ldots, y^{k'}, p_{1'}, p_{2'}, \ldots, p_{k'})ds, \tag{2.3}$$

where $s = s(t)$ is at least C^k function, $s'(t) > 0$ for $t \in [0,1]$, $s(0) = 0$, $s(1) = 1$ and

$$y^{Aa'} = ds^A x^a = \frac{d^A x^a}{ds^A}, \quad A = \overline{1,k}, \tag{2.4}$$

$$p_{\alpha a'} = ds^\alpha p_{1a} = \frac{d^{\alpha-1} p_{1a}}{ds^{\alpha-1}}, \quad \alpha = \overline{1,k}.$$

The equations, which give the conditions when (2.3) is satisfied are called Zermello's conditions. The equality (2.3) will be satisfied if along the curve c^* we have

$$F(x, y^1, y^2, \ldots, y^k, p_1, p_2, \ldots, p_k) = F(x, y^{1'}, y^{2'}, \ldots, y^{k'}, p_{1'}, p_{2'}, \ldots, p_{k'})s', \tag{2.5}$$

where $s' = \frac{ds}{dt}$. To express (2.5) in invariant form we need some relations. We shall use the notation

$$s^{(\alpha)} = \frac{d^\alpha s}{dt^\alpha}, \quad \alpha = \overline{1,k}.$$

As $x^a = x^a(s)$, $s = s(t)$, we have

$$\begin{aligned}
(2.6) \qquad y^{1a} &= \frac{dx^a}{ds}s' = y^{1a}(s,s'),\\
y^{2a} &= \frac{\partial y^{1a}}{\partial s}s' + \frac{\partial y^{1a}}{\partial s'}s'' = y^{2a}(s,s's''),\\
y^{3a} &= \frac{\partial y^{2a}}{\partial s}s' + \frac{\partial y^{2a}}{\partial s'}s'' + \frac{\partial y^{2a}}{\partial s''}s''' = y^{3a}(s,s',s'',s'''),\dots,\\
y^{ka} &= \frac{\partial y^{(k-1)a}}{\partial s}s' + \frac{\partial y^{(k-1)a}}{\partial s'}s'' + \cdots + \frac{\partial y^{(k-1)a}}{\partial s^{(k-1)}}s^{(k)},\\
p_{1a} &= p_{1a}(s), \quad s = s(t),\\
p_{2a} &= \frac{\partial p_{1a}}{\partial s}s' = p_{2a}(s,s'),\\
p_{3a} &= \frac{\partial p_{2a}}{\partial s}s' + \frac{\partial p_{2a}}{\partial s'}s'' = p_{3a}(s,s',s''),\dots,\\
p_{ka} &= \frac{\partial p_{(k-1)a}}{\partial s}s' + \frac{\partial p_{(k-1)a}}{\partial s'}s'' + \cdots + \frac{\partial p_{(k-1)a}}{\partial s^{(k-2)}}s^{(k-1)}\\
&= p_{ka}(s,s'\dots,s^{(k-1)}).
\end{aligned}$$

Using the notations

$$(2.7) \qquad A_0^a = \frac{dy^{0a}}{ds} = \frac{dx^a}{ds} = y^{1a'}, \quad A_A^a = \frac{d^A}{dt^A}A_0^a = d_t^A A_0^a \quad A = \overline{1,k-1},$$

$$(2.8) \qquad B_a^1 = \frac{dp_{1a}}{ds} = p_{2a'}, \quad B_a^\alpha = \frac{d^{\alpha-1}B_a^1}{dt^{\alpha-1}} = d_t^{\alpha-1}B_a^1, \quad \alpha = \overline{1,k-1}$$

and the Leibniz rule for differentiation we can prove the following theorem.

Theorem 2.1. *y^{Aa}, $p_{\alpha a}$ and $s^{(\alpha)}$, A, $\alpha = \overline{1,k}$ are connected by formulae:*

(2.9) $y^{1a} = A_0^a s'$,

$$y^{2a} = \binom{1}{0}A_1^a s' + \binom{1}{1}A_0^a s'',$$

$$y^{3a} = \binom{2}{0}A_2^a s' + \binom{2}{1}A_1^a s'' + \binom{2}{2}A_0^a s''',\dots,$$

$$y^{Aa} = \binom{A-1}{0} A^a_{A-1} s' + \binom{A-1}{1} A^a_{A-2} s'' + \cdots + \binom{A-1}{A-1} A^a_0 s^{(A)}, \ldots,$$

$$y^{ka} = \binom{k-1}{0} A^a_{k-1} s' + \binom{k-1}{1} A^a_{k-2} s'' + \cdots + \binom{k-1}{k-1} A^a_0 s^{(k)},$$

$$p_{1a} = p_{1a}(s), \quad s = s(t)$$

$$p_{2a} = B^1_a s'$$

$$p_{3a} = \binom{1}{0} B^2_a s' + \binom{1}{1} B^1_a s'', \ldots,$$

$$p_{\alpha a} = \binom{\alpha-2}{0} B^{\alpha-1}_a s' + \binom{\alpha-2}{1} B^{\alpha-2}_a s'' + \cdots + \binom{\alpha-2}{\alpha-2} B^1_a s^{(\alpha-1)}, \ldots,$$

$$p_{ka} = \binom{k-2}{0} B^{k-1}_a s' + \binom{k-2}{1} B^{k-2}_a s'' + \cdots + \binom{k-2}{k-2} B^1_a s^{(k-1)}.$$

Theorem 2.2. *The following relations are valid:*

$$(2.10)\quad A^a_A = \binom{A-1}{0} \frac{\partial A^a_{A-1}}{\partial s} s' + \binom{A-1}{1} \frac{\partial A_{A-2}}{\partial s} s'' + \cdots + \binom{A-1}{A-1} \frac{\partial A^a_0}{\partial s} s^{(A)},$$

$$(2.11)\qquad A^a_A = \frac{dy^{Aa}}{ds}, \quad A = \overline{1, k-1},$$

$$(2.12)\quad B^\alpha_a = \binom{\alpha-2}{0} \frac{\partial B^{\alpha-1}_a}{\partial s} s' + \binom{\alpha-2}{1} \frac{\partial B^{\alpha-2}_a}{\partial s} s'' + \cdots + \binom{\alpha-2}{\alpha-2} B^1_a s^{(\alpha-1)},$$

$$(2.13)\qquad B^\alpha_a = \frac{dp_{\alpha a}}{ds}, \quad \alpha = \overline{2, k-1}.$$

Proof.

$$A^a_A = d^A_t A^a_0 = d^{A-1}_t (d^1_t A^a_0) = d^{A-1}_t \left(\frac{\partial A^a_0}{\partial s} s' \right) =$$

$$d^{A-1}_t \frac{\partial}{\partial s} (A^a_0 s') = \frac{\partial}{\partial s} d^{A-1}_t (A^a_0 s') =$$

$$\frac{\partial}{\partial s} \left[\binom{A-1}{0} A^a_{A-1} s' + \binom{A-1}{1} A^a_{A-2} s'' + \cdots + \binom{A-1}{A-1} A^a_0 s^{(A)} \right].$$

If we take $\partial/\partial s$ from the sum in the middle bracket we obtain (2.10), and if we substitute y^{Aa} from (2.9) we obtain (2.11). In the similar way we have

$$B^\alpha_a = d^{\alpha-1}_t B^1_a = d^{\alpha-2}_t (d^1_t B^1_a) = d^{\alpha-2}_t \left(d^1_t \frac{dp_{1a}}{ds} \right) =$$

$$d_t^{\alpha-2}\left[\frac{\partial}{\partial s}\left(\frac{\partial p_{1a}}{\partial s}s'\right)\right] = \frac{\partial}{\partial s}d_t^{\alpha-2}(B_a^1 s') =$$

$$\frac{\partial}{\partial s}\left[\binom{\alpha-2}{0}B_a^{\alpha-1}s' + \binom{\alpha-2}{1}B_a^{\alpha-2}s'' + \cdots + \binom{\alpha-2}{\alpha-2}B_a^1 s^{(\alpha-1)}\right].$$

If we take $\frac{\partial}{\partial s}$ from the sum in the middle bracket, we obtain (2.12) and if we substitute $p_{\alpha a}$ from (2.9) we obtain (2.13).

The explicit form of (2.6) as follows

$$(2.14)\quad y^{1a} = y^{1a'}s', \quad y^{Aa'} = d_s^A y^{0a}(s), \quad A = \overline{1,k},$$

$$y^{2a} = y^{2a'}(s')^2 + y^{1a'}s'',$$

$$y^{3a} = y^{3a'}(s')^3 + y^{2a'}3s's'' + y^{1a'}s''',$$

$$y^{4a} = y^{4a'}(s')^4 + y^{3a'}6(s')^2 s'' + y^{2a'}(3(s'')^2 + 4s's''') + y^{1a'}s'^v, \ldots,$$

$$p_{1a} = p_{1a}(s), \quad p_{\alpha a'} = d_s^{\alpha-1}p_{1a}, \quad \alpha = \overline{2,k},$$

$$p_{2a} = p_{2a'}s',$$

$$p_{3a} = p_{3a'}s'^2 + p_{2a'}s'',$$

$$p_{4a} = p_{4a'}(s')^3 + 3p_{3a'}s's'' + p_{2a'}s''',$$

$$p_{5a} = p_{5a'}(s')^4 + p_{4a'}6(s'^2)s'' + p_{3a'}(3(s'')^2 + 4s's''') + p_{2a'}s'^v, \ldots.$$

From (2.14) it follows

Theorem 2.3. *The following relations are valid:*

$$(2.15)\quad \frac{\partial y^{1a}}{\partial s'} = \frac{\partial y^{2a}}{\partial s''} = \cdots = \frac{\partial y^{ka}}{\partial s^{(k)}} = y^{1a'} = \frac{dy^{0a}}{ds}$$

$$(2.16)\quad \frac{\partial y^{Aa}}{\partial s^{(B)}} = \frac{A}{B}\frac{\partial y^{(A-1)a}}{\partial s^{(B-1)}} = \cdots = \binom{A}{B}\frac{\partial y^{(A-B)a}}{\partial s}$$

$$(2.17)\quad \frac{\partial p_{2a}}{\partial s'} = \frac{\partial p_{3a}}{\partial s''} = \cdots = \frac{\partial p_{ka}}{\partial s^{(k-1)}} = \frac{dp_{1a}}{ds} = p_{2a'}$$

$$(2.18)\quad \frac{\partial p_{\alpha a}}{\partial s^{(\beta)}} = \frac{(\alpha-1)}{\beta}\frac{\partial p_{(\alpha-1)a}}{\partial s^{(\beta-1)}} = \cdots = \binom{\alpha-1}{\beta}\frac{\partial p_{(\alpha-\beta)a}}{\partial s},$$

$$k \geq A \geq B \geq 0, \quad k \geq \alpha \geq \beta + 1 \geq 1.$$

Now we return to the purpose of this examination.

If we take the partial derivatives of F given by (2.5) with respect to $s', s'', \dots, s^{(k)}$, taking into account (2.6), we get

(2.19)

$$(\partial_{1a}F)\frac{\partial y^{1a}}{\partial s'} + (\partial_{2a}F)\frac{\partial y^{2a}}{\partial s'} + (\partial_{3a}F)\frac{\partial y^{3a}}{\partial s'} + \cdots + (\partial_{ka}F)\frac{\partial y^{ka}}{\partial s'} +$$

$$(\partial^{2a}F)\frac{\partial p_{2a}}{\partial s'} + (\partial^{3a}F)\frac{\partial p_{3a}}{\partial s'} + \cdots + (\partial^{ka}F)\frac{\partial p_{ka}}{\partial s'} = F(x, y^{1'}, \dots, y^{k'}, p_1, \dots, p_{k'})$$

$$(\partial_{2a}F)\frac{\partial y^{2a}}{\partial s''} + (\partial_{3a}F)\frac{\partial y^{3a}}{\partial s''} + \cdots + (\partial_{ka}F)\frac{\partial y^{ka}}{\partial s''} +$$

$$(\partial^{3a}F)\frac{\partial p_{3a}}{\partial s''} + (\partial^{4a}F)\frac{\partial p_{4a}}{\partial s''} + \cdots + (\partial^{ka}F)\frac{\partial p_{ka}}{\partial s''} = 0, \dots,$$

$$(\partial_{(k-1)a}F)\frac{\partial y^{(k-1)a}}{\partial s^{(k-1)}} + (\partial_{ka}F)\frac{\partial y^{ka}}{\partial s^{(k-1)}} +$$

$$(\partial^{ka}F)\frac{\partial p_{ka}}{\partial s^{(k-1)}} = 0,$$

$$(\partial_{ka}F)\frac{\partial y^{ka}}{\partial s^{(k)}} = 0.$$

On the left-hand side of (2.19) in all equations $F = F(x, y^1, \dots, y^k, p_1, \dots, p_k)$. If in (2.19) we substitute (2.16) and (2.18) in the form

$$\frac{\partial y^{Aa}}{\partial s^{(B)}} = \binom{A}{B}\frac{\partial y^{(A-B)a}}{\partial s} = \binom{A}{B} y^{(A-B+1)a}\frac{dt}{ds}$$

$$\frac{\partial p_{\alpha a}}{\partial s^{(\beta)}} = \binom{\alpha-1}{\beta}\frac{\partial p_{(\alpha-\beta)a}}{\partial s} = \binom{\alpha-1}{\beta} p_{(\alpha-\beta+1)a}\frac{dt}{ds}$$

we obtain

(2.20)
$$[\binom{1}{1} y^{1a}\partial_{1a} + \binom{2}{1} y^{2a}\partial_{2a} + \cdots + \binom{k}{1} y^{ka}\partial_{ka}](F) +$$

$$[\binom{1}{1} p_{2a}\partial^{2a} + \binom{2}{1} p_{3a}\partial^{3a} + \cdots + \binom{k-1}{1} p_{ka}\partial^{ka}](F) =$$

$$F(x, y^{1'}, \dots, y^{k'}, p_1', \dots, p_k')\frac{ds}{dt} = F(x, y^1, \dots, y^k, p_1, \dots, p_k) = F$$

$$\{[\binom{2}{2} y^{1a}\partial_{2a} + \binom{3}{2} y^{2a}\partial_{3a} + \cdots + \binom{k}{2} y^{(k-1)a}\partial_{ka}] +$$

$$[\binom{2}{2}p_{2a}\partial^{3a}+\binom{3}{2}p_{3a}\partial^{4a}+\cdots+\binom{k-1}{2}p_{(k-1)a}\partial^{ka}]\}(F)\frac{dt}{ds}=0,\ldots,$$

$$\{[\binom{k-1}{k-1}y^{1a}\partial_{(k-1)a}+\binom{k}{k-1}y^{2a}\partial_{ka}]+$$

$$\binom{k-1}{k-1}p_{2a}\partial^{ka}\}(F)\frac{dt}{ds}=0,$$

$$[\binom{k}{k}y^{1a}\partial_{ka}](F)\frac{dt}{ds}.$$

We shall introduce the notations

(2.21)

$$I'_1=\binom{k}{k}y^{1a}\partial_{ka}$$

$$I'_2=\binom{k-1}{k-1}y^{1a}\partial_{(k-1)a}+\binom{k}{k-1}y^{2a}\partial_{ka}$$

$$I'_3=\binom{k-2}{k-2}y^{1a}\partial_{(k-2)a}+\binom{k-1}{k-2}y^{2a}\partial_{(k-1)a}+\binom{k}{k-2}y^{3a}\partial_{ka},\ldots,$$

$$I'_{k-i+1}=\binom{i}{i}y^{1a}\partial_{ia}+\binom{i+1}{i}y^{2a}\partial_{(i+1)a}+\cdots+\binom{k}{i}y^{(k-i+1)a}\partial_{ka},\ldots,$$

$$I'_{k-2}=\binom{3}{3}y^{1a}\partial_{3a}+\binom{4}{3}y^{2a}\partial_{4a}+\cdots+\binom{k}{3}y^{(k-2)a}\partial_{ka}$$

$$I'_{k-1}=\binom{2}{2}y^{1a}\partial_{2a}+\binom{3}{2}y^{2a}\partial_{3a}+\cdots+\binom{k}{2}y^{(k-1)a}\partial_{ka}$$

$$I'_k=\binom{1}{1}y^{1a}\partial_{1a}+\binom{2}{1}y^{2a}\partial_{2a}+\cdots+\binom{k}{1}y^{ka}\partial_{ka}$$

$$I''_2=\binom{k-1}{k-1}p_{2a}\partial^{ka},\cdots,$$

$$I''_{k-i+1}=\binom{i}{i}p_{2a}\partial^{(i+1)a}+\binom{i+1}{i}p_{3a}\partial^{(i+2)a}+\cdots+\binom{k-1}{i}p_{k-i+1}\partial^{ka},\ldots,$$

$$I''_{k-2}=\binom{3}{3}p_{2a}\partial^{4a}+\binom{4}{3}p_{3a}\partial^{5a}+\cdots+\binom{k-1}{3}p_{(k-2)a}\partial^{ka},$$

$$I''_{k-1} = \binom{2}{2} p_{2a}\partial^{3a} + \binom{3}{2} p_{3a}\partial^{4a} + \cdots + \binom{k-1}{2} p_{(k-1)a}\partial^{ka},$$

$$I''_{k} = \binom{1}{1} p_{2a}\partial^{2a} + \binom{2}{1} p_{3a}\partial^{3a} + \cdots + \binom{k-1}{1} p_{ka}\partial^{ka}.$$

Using the above notations, from (2.20) and (2.5), we obtain the following theorem.

Theorem 2.4. *The necessary conditions that the integral of action (see (2.2)) I_{c^*} does not depend on the parametrization of the curve are:*

$$
\begin{aligned}
&(I'_k + I''_k)F = F,\\
&(I'_{k-1} + I''_{k-1})F = 0,\\
&(I'_{k-2} + I''_{k-2})F = 0, \dots,\\
&(I'_{k-i+1} + I''_{k-i+1})F = 0,\\
&(I'_2 + I''_2)F = 0,\\
&I'_1 F = 0.
\end{aligned}
\tag{2.22}
$$

The above equations are called by A. Kawaguchi and K. Kondo [10] Zermello's conditions.

The vector fields $I'_k, \dots, I'_1$, $I''_k, \dots, I''_2$ are connected with $\bar{\Gamma}_0, \bar{\Gamma}_1, \dots, \bar{\Gamma}_k$ given by (1.18). If we write these vector fields in the natural bases substituting δy^{Aa}, δ^{Aa}, $\delta p_{\alpha a}$, $\delta^{\alpha a}$ with dy^{Aa}, ∂^{Aa}, $dp_{\alpha a}$, $\partial^{\alpha a}$ respectively (see Theorem 1.5) using the symmetry of binomial coefficients and the relations

$$y^{Aa} = \frac{dy^{(A-1)a}}{dt}, \quad p_{\alpha a} = \frac{dp_{(\alpha-1)a}}{dt}, \quad A = \overline{1,k}, \quad \alpha = \overline{2,k},$$

we get

Theorem 2.5. *The following relations are valid:*

$$
\begin{aligned}
&\bar{\Gamma}_0 = I'_1 dt,\\
&\bar{\Gamma}_1 = (I'_2 + I''_2)dt,\\
&\bar{\Gamma}_2 = (I'_3 + I''_3)dt, \dots,\\
&\bar{\Gamma}_{k-1} = (I'_k + I''_k)dt.
\end{aligned}
\tag{2.23}
$$

From (1.27) and (1.28) it follows that the first equation of Zermello's condition has the following explicit form:

$$\frac{1}{dt}\left\{\left[\binom{1}{0} dy^{0a}\partial_{1a} + \binom{2}{1} dy^{1a}\partial_{2a} + \cdots + \binom{k}{k-1} dy^{(k-1)a}\partial_{ka}\right] + \right.$$

$$\left[\binom{1}{0}dp_{1a}\partial^{2a}+\binom{2}{1}dp_{2a}\partial^{3a}+\cdots+\binom{k-1}{k-2}dp_{(k-1)a}\partial^{ka}\right]\Bigg\}F=F.$$

Using the notations

$$I_1=\frac{1}{dt}\bar{\Gamma}_0, I_2=\frac{1}{dt}\bar{\Gamma}_1,\dots,I_k=\frac{1}{dt}\bar{\Gamma}_{k-1} \tag{2.24}$$

(2.22) can be written in the form

$$I_kF=F, I_{k-1}F=0,\dots,I_1F=0.$$

Theorem 2.6. *If we suppose*

$$F(x,y^{(1)},y^{(2)},\dots,y^{(k)},p_{(1)},p_{(2)},\dots,p_{(k)})= \tag{2.25}$$

$$L(x,y^{(1)},y^{(2)},\dots,y^{(k)})+H(x,p_{(1)},p_{(2)},\dots,p_{(k)}),$$

then the Zermello's conditions reduce to

$$\begin{aligned} I'_kL&=L, & I''_kH&=H,\\ I'_{k-1}L&=0, & I''_{k-1}H&=0,\\ I'_{k-2}L&=0, & I''_{k-2}H&=0,\dots,\\ I'_{k-i+1}L&=0, & I''_{k-i+1}H&=0,\dots,\\ I'_2L&=0, & I''_2H&=0,\\ I'_1L&=0 & &. \end{aligned} \tag{2.26}$$

Proof. From

$$\begin{aligned} &F(x,y^{(1)},y^{(2)},\dots,y^{(k)},p_{(1)},p_{(2)},\dots,p_{(k)})\\ =\ &L(x,y^{(1)},y^{(2)},\dots,y^{(k)})+H(x,p_{(1)},p_{(2)},\dots,p_{(k)}) \end{aligned}$$

and (2.20) it is obvious that

$$\begin{aligned} I'_1H=0,\qquad & I'_2H=0,\dots,I'_kH=0,\\ & I''_2L=0,\dots,I''_kL=0, \end{aligned}$$

from which it follows (2.26).

The obtained results are in accordance with the results appearing in the references. The main difference is that there $y^{Ai}=\frac{1}{A!}\frac{d^Ax^i}{dt^A}$, and here the factor $\frac{1}{A!}$ is missing.

In the generalized Lagrange space, the Zermello's conditions are given by the first column of (2.26), where usual notation

$$\Gamma_A=I'_A\quad A=\overline{1,k}$$

is used.

In [13], where the generalized Lagrange space is studied ($F = L$, $H = 0$), the Zermello's conditions are given by

$$I^1(L) = \cdots = I^{k-1}(L) = 0, \quad I^k(L) = L.$$

I^1, I^k are called by R. Miron main invariants. Their relation to $I'_1, I'_2, \ldots, I'_k$ (see (2.21)) is given by

$$I^1 = k!I'_1, \quad I^2 = (k-1)!I'_2, \ldots, I^k = 1!I'_k.$$

3. Energies of higher order

Definition 3.1. *We call $\varepsilon_A(F)$ energies of order A, $A = \overline{1,k}$ of the fundamental function*
$F(y^0, y^1, \ldots, y^k, p_1, \ldots, p_k)$*. They are defined along the curve c^* (see (2.1)) by the invariants I_A in the following way*
(3.1)

$$\begin{aligned}
\varepsilon_k(F) &= [I_k - d_t^1 I_{k-1} + d_t^2 I_{k-2} - \cdots + (-1)^{k-1} d_t^{(k-1)} I_1](F) - F,\\
\varepsilon_{k-1}(F) &= [-I_{k-1} + d_t^1 I_{k-2} - \cdots + (-1)^{k-1} d_t^{k-2} I_1](F),\\
\varepsilon_{k-2}(F) &= [I_{k-2} - \cdots + (-1)^{k-1} dt^{k-3} I_1](F),\\
&\vdots\\
\varepsilon_2(F) &= [(-1)^{k-2} I_2 + (-1)^{k-1} d_t^1 I_1](F),\\
\varepsilon_1(F) &= [(-1)^{k-1} I_1](F).
\end{aligned}$$

Proposition 3.1. *The following identities hold:*

$$(\varepsilon_k - d_t^1 \varepsilon_{k-1})(F) = I_k(F) - F, \tag{3.2}$$

$$(\varepsilon_{k-1} - d_t^1 \varepsilon_{k-2})(F) = -I_{k-1}(F), \ldots,$$

$$(\varepsilon_2 - d_t^1 \varepsilon_1)(F) = (-1)^{k-2} I_2(F).$$

Theorem 3.1. *For the fundamental function $F(y^0, y^1, \ldots, y^k, p_1, p_2, \ldots, p_k)$, for which the Zermello's conditions are satisfied, the higher order energies are equal to zero, i.e.*

$$\varepsilon_1(F) = 0, \varepsilon_2(F) = 0, \ldots, \varepsilon_k(F) = 0. \tag{3.3}$$

Proof. From (2.25) it follows $I_1(F) = 0$, $\Rightarrow \varepsilon_1(F) = 0 \Rightarrow (d_t^1 \varepsilon_1)(F) = 0$. From this equation, $I_2(F) = 0$ and the last equation of (3.2) it follows $\varepsilon_2(F) = 0$. The other equations of (3.3) can be proved in the same way.

Definition 3.2. *If the fundamental function F satisfies relation (2.26), i.e. F is the sum of Lagrangian L and Hamiltonian H, then in the $(GLH)^{(nk)}$ space we can define energies of order A, $\varepsilon'_A(L)$, $A=\overline{1,k}$ of the Lagrangian $L(y^0,y^1,\dots,y^k)$ and energies of order α, $\varepsilon''_\alpha(H)$ $\alpha=\overline{2,k}$ of the Hamiltonian $H(x,p_1,\dots,p_k)$. They are defined along the curve c^* (see (2.1)) by the invariants I'_A and I''_α in the following way:*

$$\begin{aligned}
\varepsilon'_k(L) &= [I'_k - d^1_t I'_{k-1} + d^2_t I'_{k-2} - \dots + (-1)^{k-1} d^{k-1}_t I'_1(L) - L,\\
\varepsilon'_{k-1}(L) &= [-I'_{k-1} + d^1_t I'_{k-2} - \dots + (-1)^{k-1} d^{k-2}_t I'_1](L),\\
&\vdots\\
\varepsilon'_2(L) &= [(-1)^{k-2} I'_2 + (-1)^{k-1} d^1_t I'_1](L),\\
\varepsilon'_1(L) &= [(-1)^{k-1} I'_1](L)\\
\varepsilon''_k(H) &= [I''_k - d^1_t I''_{k-2} + d^2_t I''_{k-2} - \dots + (-1)^{k-2} d^{k-2}_t I''_2](H) - H\\
\varepsilon''_{k-1}(H) &= [-I''_{k-2} + d^1_t I''_{k-2} - \dots + (-1)^{k-2} d^{k-3}_t I''_2]H\\
&\vdots\\
\varepsilon''_2(H) &= [(-1)^{k-2} I''_2](H).
\end{aligned}$$

Proposition 3.2. *If the fundamental function satisfies (2.25) $(F = L + H)$, then the following relations are valid:*

$$\begin{aligned}
\varepsilon_k(F) &= \varepsilon'_k(L) + \varepsilon''_k(H),\\
\varepsilon_{k-1}(F) &= \varepsilon'_{k-1}(L) + \varepsilon''_k(H),\\
&\vdots\\
\varepsilon_2(F) &= \varepsilon'_2(L) + \varepsilon''_2(H)\\
\varepsilon_1(F) &= \varepsilon'_1(L).
\end{aligned}\tag{3.4}$$

Theorem 3.2. *If for the fundamental function F determined by (2.25) the Zermello's conditions are satisfied, then all types of energies of higher order are equal to zero, namely*

$$\begin{aligned}
&\varepsilon'_1(L) = 0, \varepsilon'_2(L) = 0, \dots, \varepsilon'_k(L) = 0,\\
&\varepsilon''_2(H) = 0, \dots, \varepsilon''_k(H) = 0.
\end{aligned}\tag{3.5}$$

This theorem follows from Theorem 3.1.

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