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A COMPARISON THEOREM OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper it is proved that for comparing the solutions of two differential equations it is enough that one of them is unique; i.e. no Lipschitz condition is needed.

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1. Introduction

Differential inequalities are the basic tools in the qualitative theory of differential equations. In many relevant textbooks one can find the following.

Proposition. *Suppose that the functions f and g are continuous in the domain*

$$D = \{(x, y) : |x - x_0| < a,\ |y - y_0| < b\},$$

and denote by $y_0(x)$, $z_0(x)$ any solution of the initial value problems

(1) $y'(x) = f(x, y)$, $y_0(x_0) = y_0$

(2) $z'(x) = g(x, z)$, $z_0(x_0) = y_0$

respectively.

If $(f(x, y) > g(x, y)$ in D then $y_0(x) > z_0(x)$ for $x > x_0$ and $y_0(x) < z_0(x)$ for $x < x_0$.

However, $f(x, y) \geq g(x, y)$ in D does not imply $y_0(x) \geq z_0(x)$ for $x > x_0$, without some additional conditions on f or g.

2. Results

In the existing literature it is usually required that one of these functions, e.g. f, belongs to the Lipschitz class in D, implying the uniqueness of solutions of equation (1).

The purpose of this paper is to show that the uniqueness of solutions of one of the equation (1), (2) is a relevant sufficient condition.

We prove

Theorem. *Suppose that the equation (1) has unique solutions in D. Then if $f(x, y) \geq g(x, y)$ there follows $y_0(x) \geq z_0(x)$ for $x \geq x_0$, where $y_0(x)$ is the*

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unique solution of initial problem (1), while $z_0(x)$ is any solution of the initial problem (2).

Proof. Put
$$S = \{(x, y) \in D \ : \ f(x, y) > g(x, y)\}.$$
Notice that, due to the continuity of f and g, S is an open set. Suppose on the contrary that there exists some solution of the initial problem (2), denote it by $z_0^*(x)$, and some $x_1 > x_0$ such that $z_0^*(x_1) > y_0(x_1)$. Evidently, $z_0^*(x)$ does not belong to $D\backslash S$ for all $x \geq x_0$ because it would imply $y_0(x) \equiv z_0^*(x)$. Let $x \geq x_0$ and denote
$$I = \{x \ : \ (x, z_0^*(x)) \in S\}.$$
Due to the continuity of $z_0^*(x)$ I is an open set and so, an at most countable union of open intervals ([1, Prop. 8, p.39]) i.e.
$$I = \bigcup_{i=1}^{n} (x_i, x_i^0) : \ x_0 \leq x_i < x_i^0, \ x_i^0 \leq x_0 + a, \ n \leq \infty.$$
Let (x_k, x_k^0) be one of those intervals such that $z_0^*(x) > y_0(x)$ for $x \in (x_k, x_k^0)$. Consider now the initial problem
$$y' = f(x, y) \ , \ y(\overline{x}) = z_0^*(\overline{x}) \ , \ \overline{x} \in (x_k, x_k^0).$$

Its solution is, according to our Proposition, less than $z_0^*(x)$ in some left neighborhood of $\overline{x}$ but greater than $y_0(x)$ for all $x_0 \leq x < \overline{x}$. So, it intersects the solution $z_0^*(x)$ at some point x_i or x_i^0, $x_i^0 \leq x_k$. It is a contradiction because we get a one-to-one correspondence between a countable and an uncountable set. □

Corollary. Solutions ones separated remain separated.

Proof. Suppose that the distinct solutions $y_0(x)$ and $z_0(x)$ connect at the point $x_c, \ x_c > x_0$. Putting $t = x_c - x$ in equations (1) and (2) we get the contradiction according to our Theorem. □

Remark 1. If D is a closed domain we endowed it with the topology induced by the usual one and the statement of our Theorem remains the same.

Remark 2. Without any difficulties we can generalize our Theorem to a system of differential equations if for vector functions f and g the relation $f \geq g$ on D means that $f_i(t, x_1, \ldots, x_n) \geq g_i(t, x_1, \ldots, x_n)$ for all $i = 1, 2, \ldots, n$.

References

[1] Royden, H. L., Real Analysis. New York: Macmillan, 1968.

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