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AN INTEGRAL UNIVALENT OPERATOR DEFINED BY GENERALIZED Al-OBOUDI DIFFERENTIAL OPERATOR ON THE CLASSES $\mathcal{T}_j$, $\mathcal{T}_{j,\mu}$ AND $\mathcal{S}_j(p)$

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Abstract. In [4], Breaz and Breaz gave the univalence conditions of the integral operator $F_{\alpha,n}$ of the analytic functions belonging to the classes $\mathcal{T}_2, \mathcal{T}_{2,\mu}$ and $\mathcal{S}(p)$.

The purpose of this paper is to generalize the integral operator $F_{\alpha,n}$ by means of the generalized Al-Oboudi differential operator and investigate univalence conditions of this generalized integral operator considering the classes $\mathcal{T}_j, \mathcal{T}_{j,\mu}$ and $\mathcal{S}_j(p)$ $(j = 2, 3, \ldots)$.

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1. Introduction

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

The following definition of fractional derivative given by Owa [7] (also by Srivastava and Owa [13]) will be required in our investigation.

The fractional derivative of order γ for a function $f$is defined by

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\gamma}} d\xi \quad (0 \leq \gamma < 1), \tag{1.2}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\gamma}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

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It readily follows from (1.2) that

$$D_z^{\gamma} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} z^{k-\gamma} \quad (0 \leq \gamma < 1, k \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using $D_z^{\gamma} f$, Owa and Srivastava [8] introduced the operator $\Omega^{\gamma} : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^{\gamma} f(z) &= \Gamma(2-\gamma) z^{\gamma} D_z^{\gamma} f(z), \quad \gamma \neq 2, 3, 4, \ldots \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k. \end{aligned} \tag{1.3}$$

Note that

$$\Omega^0 f(z) = f(z).$$

In [2], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator $D_{\lambda}^{n,\gamma}$ as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_{\lambda}^{1,\gamma} f(z) &= (1-\lambda) \Omega^{\gamma} f(z) + \lambda z (\Omega^{\gamma} f(z))' \\ &= D_{\lambda}^{\gamma} (f(z)), \quad \lambda \geq 0, \ 0 \leq \gamma < 1, \\ D_{\lambda}^{2,\gamma} f(z) &= D_{\lambda}^{\gamma} \left(D_{\lambda}^{1,\gamma} f(z) \right), \end{aligned} \tag{1.4}$$

$$\vdots$$

$$D_{\lambda}^{n,\gamma} f(z) = D_{\lambda}^{\gamma} \left(D_{\lambda}^{n-1,\gamma} f(z) \right), \quad n \in \mathbb{N}. \tag{1.5}$$

If f is given by (1.1), then by (1.3), (1.4) and (1.5), we see that

$$D_{\lambda}^{n,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\gamma, \lambda) a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{1.6}$$

where

$$\Psi_{k,n}(\gamma, \lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} (1 + (k-1)\lambda) \right]^n. \tag{1.7}$$

Remark 1. (**i**) When $\gamma = 0$, we get the Al-Oboudi differential operator [1].
(**ii**) When $\gamma = 0$ and $\lambda = 1$, we get the Sălăgean differential operator [10].
(**iii**) When $n = 1$ and $\lambda = 0$, we get the Owa-Srivastava fractional differential operator [8].

Let $\mathcal{A}_j$ be the subclass of $\mathcal{A}$ consisting of functions f given by

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (j \in \mathbb{N}_1^* := \mathbb{N} \setminus \{1\} = \{2, 3, \ldots\}). \tag{1.8}$$

Let $\mathcal{T}$ be the univalent subclass of $\mathcal{A}$ consisting of functions f which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1 \qquad (z \in \mathbb{U}). \tag{1.9}$$

Let $\mathcal{T}_j$ be the subclass of $\mathcal{T}$ for which $f^{(k)}(0) = 0$ $(k = 2, 3, \ldots, j)$. Let $\mathcal{T}_{j,\mu}$ be the subclass of $\mathcal{T}_j$ consisting of functions f of the form (1.8) which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < \mu \qquad (z \in \mathbb{U}) \tag{1.10}$$

for some μ $(0 < \mu \leq 1)$, and let us denote $\mathcal{T}_{j,1} \equiv \mathcal{T}_j$.

For some real p with $0 < p \leq 2$, we define the subclass $\mathcal{S}(p)$ of $\mathcal{A}$ consisting of all functions f which satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p \qquad (z \in \mathbb{U}). \tag{1.11}$$

In [12], Singh has shown that if $f \in \mathcal{S}(p)$, then f satisfies

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p\,|z|^2 \qquad (z \in \mathbb{U}). \tag{1.12}$$

Let $\mathcal{S}_j(p)$ be the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}_j$ which satisfy (1.11) and

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p\,|z|^j \qquad (z \in \mathbb{U}, j \in \mathbb{N}_1^*), \tag{1.13}$$

and let us denote $\mathcal{S}_2(p) \equiv \mathcal{S}(p)$.

The subclasses $\mathcal{T}_j, \mathcal{T}_{j,\mu}$ and $\mathcal{S}_j(p)$ are introduced by Seenivasagan [11].

The following results will be required in our investigation.

General Schwarz Lemma. [6] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}\,|z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if $f(z) = e^{i\theta}(M/R^m)\,z^m$, where θ is a constant.

Theorem A. [9] *Let $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ and $f \in \mathcal{A}$. If f satisfies*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \qquad (z \in \mathbb{U}),$$

then, for any complex number β *with* $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, *the integral operator*

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} f'(t) dt \right\}^{\frac{1}{\beta}} \tag{1.14}$$

is in the class $\mathcal{S}$.

In [4], Breaz and Breaz gave the following results.

Theorem B. [4] *Let* $g_i \in \mathcal{T}_2$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + \cdots$, $\forall i = \overline{1,n}$, $n \in \mathbb{N}$ *which satisfy the properties*

$$\left| \frac{z^2 g_i'(z)}{(g_i(z))^2} - 1 \right| < 1, \quad \forall z \in \mathbb{U}, \forall i = \overline{1,n}.$$

If $|g_i(z)| \leq 1$, $\forall z \in \mathbb{U}$, $\forall i = \overline{1,n}$, *then for every complex number* α, *satisfying the properties*

$$\operatorname{Re}\alpha > 0, \quad \operatorname{Re}(n(\alpha-1)+1) \geq \operatorname{Re}\alpha, \quad \text{and} \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{3n}$$

the function

$$F_{\alpha,n}(z) = \left\{ (n(\alpha-1)+1) \int_0^z (g_1(t))^{\alpha-1} \ldots (g_n(t))^{\alpha-1} dt \right\}^{\frac{1}{n(\alpha-1)+1}} \tag{1.15}$$

is univalent.

Theorem C. [4] *Let* $g_i \in \mathcal{T}_{2,\mu}$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + \cdots$, $\forall i = \overline{1,n}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ *so that*

$$\operatorname{Re}(n(\alpha-1)+1) \geq \operatorname{Re}\alpha, \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{n(\mu+2)}.$$

If $|g_i(z)| \leq 1$, $\forall z \in \mathbb{U}$, $i = \overline{1,n}$, *then we have* $F_{\alpha,n} \in \mathcal{S}$.

Theorem D. [4] *Let* $g_i \in \mathcal{S}(p)$, $0 < p < 2$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + \cdots$, $\forall i = \overline{1,n}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ *so that*

$$\operatorname{Re}(n(\alpha-1)+1) \geq \operatorname{Re}\alpha, \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{n(p+2)}.$$

If $|g_i(z)| \leq 1$, $\forall z \in \mathbb{U}$, $i = \overline{1,n}$, *then we have* $F_{\alpha,n} \in \mathcal{S}$.

In [3], Breaz gave the extensions of Theorems B, C, and D as follows.

Theorem B′. [3] *Let* $M \geq 1$, $g_i \in \mathcal{T}_2$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + \cdots$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$ *so that it satisfies the properties*

$$\left| \frac{z^2 g_i'(z)}{(g_i(z))^2} - 1 \right| < 1, \quad \forall z \in \mathbb{U}, \forall i \in \{1, \ldots, n\}.$$

If $|g_i(z)| \leq M$, $\forall z \in \mathbb{U}$, $\forall i \in \{1, \ldots, n\}$, *then for every complex number* α, *such that*

$$\operatorname{Re}\alpha \geq 1, \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{(2M+1)n}$$

the function $F_{\alpha,n}$ *is univalent.*

Theorem C′. [3] *Let* $M \geq 1$, $g_i \in \mathcal{T}_{2,\mu}$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + \cdots$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. *Let* $\alpha \in \mathbb{C}$, *be such that*

$$\operatorname{Re}\alpha \geq 1, \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{n(M\mu + M + 1)}.$$

If $|g_i(z)| \leq M$, $\forall z \in \mathbb{U}$, $i \in \{1, \ldots, n\}$, *then the function* $F_{\alpha,n} \in \mathcal{S}$.

Theorem D′. [3] *Let* $M \geq 1$, $g_i \in \mathcal{S}(p)$, $0 < p < 2$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + \cdots$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. *Let* $\alpha \in \mathbb{C}$ *be such that*

$$\operatorname{Re}\alpha \geq 1, \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{n(Mp + M + 1)}.$$

If $|g_i(z)| \leq M$, $\forall z \in \mathbb{U}$, $i \in \{1, \ldots, n\}$, *then the function* $F_{\alpha,n} \in \mathcal{S}$.

Now, we define a new general integral operator by means of the generalized Al-Oboudi differential operator as follows.

Definition 1. Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$, $\lambda \geq 0$, $0 \leq \gamma < 1$. We define the integral operator $G_{\lambda,\alpha}^{m,\gamma}$ by

(1.16)
$$G_{\lambda,\alpha}^{n,m,\gamma}(z) = \left\{ [n(\alpha - 1) + 1] \int_0^z \prod_{i=1}^{n} \left(D_\lambda^{m,\gamma} g_i(t)\right)^{\alpha - 1} dt \right\}^{\frac{1}{n(\alpha-1)+1}} \qquad (z \in \mathbb{U}),$$

where $g_1, \ldots, g_n \in \mathcal{A}$ and $D_\lambda^{m,\gamma}$ is the generalized Al-Oboudi differential operator.

Remark 2. In the special case $n = 1$ we obtain the integral operator

$$G_{\lambda,\alpha}^{m,\gamma}(z) = \left\{ \alpha \int_0^z \left(D_\lambda^{m,\gamma} g(t)\right)^{\alpha - 1} dt \right\}^{\frac{1}{\alpha}} \qquad (z \in \mathbb{U}). \tag{1.17}$$

Remark 3. If we set $\gamma = 0$ in (1.16) and (1.17), then we get the integral operators $G_{n,m,\alpha}$ and $G_{m,\alpha}$ respectively defined in [5].

In this paper we generalize the results of [3].

2. Main Results

Theorem 2.1. *Let g_i, defined by*

$$g_i(z) = z + \sum_{k=j+1}^{\infty} a_{k,i} z^k \tag{2.1}$$

be in the class $\mathcal{T}_j$ for $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$, and satisfy the properties

$$\left| \frac{z^2 \left(D_\lambda^{m,\gamma} g_i(z)\right)'}{(D_\lambda^{m,\gamma} g_i(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U},\ i \in \{1, \ldots, n\}).$$

If

$$|D_\lambda^{m,\gamma} g_i(z)| \leq M_i \ (M_i \geq 1\,;\ z \in \mathbb{U}\,;\ i \in \{1, \ldots, n\}),$$

and $\alpha \in \mathbb{C}$ be such that

$$\operatorname{Re}(n(\alpha - 1) + 1) \geq \operatorname{Re}\alpha > 0, \quad \text{and} \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{\sum_{i=1}^{n}(2M_i + 1)},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ defined by (1.16) is in the class $\mathcal{S}$.

Proof. Since $g_i \in \mathcal{T}_j$ $(i \in \{1, \ldots, n\}, n \in \mathbb{N}, j \in \mathbb{N}_1^*)$, by (1.6), we have

$$\frac{D_\lambda^{m,\gamma} g_i(z)}{z} = 1 + \sum_{k=j+1}^{\infty} \Psi_{k,n}(\gamma, \lambda)\, a_{k,i} z^{k-1} \quad (n \in \mathbb{N}_0)$$

and

$$\frac{D_\lambda^{m,\gamma} g_i(z)}{z} \neq 0$$

for all $z \in \mathbb{U}$.

From (1.16) we obtain that

$$G_{\lambda,\alpha}^{n,m,\gamma}(z) = \left\{ (n(\alpha - 1) + 1) \int_0^z t^{n(\alpha-1)} \prod_{i=1}^{n} \left(\frac{D_\lambda^{m,\gamma} g_i(t)}{t} \right)^{\alpha - 1} dt \right\}^{\frac{1}{n(\alpha-1)+1}}.$$

Define a function

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{D_\lambda^{m,\gamma} g_i(t)}{t} \right)^{\alpha-1} dt. \tag{2.2}$$

Then we obtain

$$h'(z) = \prod_{i=1}^{n} \left(\frac{D_\lambda^{m,\gamma} g_i(z)}{z} \right)^{\alpha-1}.$$

It is clear that $h(0) = h'(0) - 1 = 0$. Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1) \sum_{i=1}^{n} \left(\frac{z\left(D_\lambda^{m,\gamma} g_i(z)\right)'}{D_\lambda^{m,\gamma} g_i(z)} - 1 \right). \tag{2.3}$$

From (2.3), we get

$$
\begin{aligned}
&\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\
(2.4)\qquad &\le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\,|\alpha-1|\sum_{i=1}^{n}\left(\left|\frac{z^2\left(D_\lambda^{m,\gamma}g_i(z)\right)'}{(D_\lambda^{m,\gamma}g_i(z))^2}\right|\left|\frac{D_\lambda^{m,\gamma}g_i(z)}{z}\right|+1\right).
\end{aligned}
$$

From the hypothesis, we have $|D_\lambda^{m,\gamma}g_i(z)| \le M_i$ $(z \in \mathbb{U}\,;\, i \in \{1,\ldots,n\})$, then by the general Schwarz lemma we obtain that

$$|D_\lambda^{m,\gamma}g_i(z)| \le M_i\,|z| \quad (i \in \{1,\ldots,n\}\ ;\ z \in \mathbb{U}).$$

We apply this result in inequality (2.4), then we find

$$
\begin{aligned}
&\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\,|\alpha-1|\sum_{i=1}^{n}\left(\left|\frac{z^2\left(D_\lambda^{m,\gamma}g_i(z)\right)'}{(D_\lambda^{m,\gamma}g_i(z))^2}\right|M_i+1\right) \\
&\le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\,|\alpha-1|\sum_{i=1}^{n}\left(\left|\frac{z^2\left(D_\lambda^{m,\gamma}g_i(z)\right)'}{(D_\lambda^{m,\gamma}g_i(z))^2}-1\right|M_i+M_i+1\right) \\
&\le \frac{|\alpha-1|}{\operatorname{Re}\alpha}\sum_{i=1}^{n}(2M_i+1) \\
&\le 1
\end{aligned}
$$

since $|\alpha-1| \le \frac{\operatorname{Re}\alpha}{\sum_{i=1}^{n}(2M_i+1)}$. Applying Theorem A, we obtain that $G_{\lambda,\alpha}^{n,m,\gamma}$ is in the class $\mathcal{S}$. □

Corollary 2.2. *Let g_i, defined by* (2.1)*, be in the class $\mathcal{T}_j$ for $i \in \{1,\ldots,n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$, and satisfy the properties*

$$\left|\frac{z^2\left(D_\lambda^{m,\gamma}g_i(z)\right)'}{(D_\lambda^{m,\gamma}g_i(z))^2}-1\right| < 1 \quad (z \in \mathbb{U},\ i \in \{1,\ldots,n\})\,.$$

If

$$|D_\lambda^{m,\gamma}g_i(z)| \le M\ (M \ge 1\,;\, z \in \mathbb{U}\,;\, i \in \{1,\ldots,n\}),$$

and $\alpha \in \mathbb{C}$ be such that

$$\operatorname{Re}\left(n(\alpha-1)+1\right) \ge \operatorname{Re}\alpha > 0, \quad \text{and} \quad |\alpha-1| \le \frac{\operatorname{Re}\alpha}{(2M+1)n},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ defined by (1.16) *is in the class $\mathcal{S}$.*

Proof. In Theorem 2.1, we consider $M_1 = M_2 = \cdots = M_n = M$. □

Corollary 2.3. *In Corollary* 2.2*, if we set* $D_{\lambda}^{0,\gamma}g_i = D_0^{m,0}g_i = g_i$ $(i \in \{1,\dots,n\})$ *and*

(**i**) $j=2$*, then we have Theorem* B'*.*

(**ii**) $j=2$ *and* $M=1$*, then we have Theorem B.*

Theorem 2.4. *Let* g_i*, defined by* (2.1)*, be in the class* $\mathcal{T}_{j,\mu_i}$ *for* $i \in \{1,\dots,n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$*, and satisfy the properties*

$$\left| \frac{z^2 \left(D_{\lambda}^{m,\gamma} g_i(z)\right)'}{\left(D_{\lambda}^{m,\gamma} g_i(z)\right)^2} - 1 \right| < \mu_i \quad \left(0 < \mu_i \le 1,\ z \in \mathbb{U},\ i \in \{1,\dots,n\}\right).$$

If

$$|D_{\lambda}^{m,\gamma} g_i(z)| \le M_i \ (M_i \ge 1\,;\, z \in \mathbb{U}\,;\, i \in \{1,\dots,n\}),$$

and $\alpha \in \mathbb{C}$ *be such that*

$$\operatorname{Re}\left(n(\alpha-1)+1\right) \ge \operatorname{Re}\alpha > 0, \quad \textit{and} \quad |\alpha - 1| \le \frac{\operatorname{Re}\alpha}{\sum_{i=1}^{n}((\mu_i+1)M_i+1)},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ *defined by* (1.16) *is in the class* $\mathcal{S}$*.*

Proof. Considering the function h defined by (2.2), we take the same steps as in the proof of Theorem 2.1. Then, we obtain that

$$\begin{aligned} &\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \\ &\quad \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} |\alpha - 1| \sum_{i=1}^{n} \left(\left| \frac{z^2 \left(D_{\lambda}^{m,\gamma} g_i(z)\right)'}{\left(D_{\lambda}^{m,\gamma} g_i(z)\right)^2} - 1 \right| M_i + M_i + 1 \right) \\ &\quad \le \frac{|\alpha-1|}{\operatorname{Re}\alpha} \sum_{i=1}^{n} ((\mu_i+1)M_i+1) \le 1 \end{aligned}$$

for $g_i \in \mathcal{T}_{j,\mu_i}$ $(i \in \{1,\dots,n\})$. In view of Theorem A, we have $G_{\lambda,\alpha}^{n,m,\gamma} \in \mathcal{S}$. □

Corollary 2.5. *Let* g_i*, defined by* (2.1)*, be in the class* $\mathcal{T}_{j,\mu_i}$ *for* $i \in \{1,\dots,n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$*, and satisfy the properties*

$$\left| \frac{z^2 \left(D_{\lambda}^{m,\gamma} g_i(z)\right)'}{\left(D_{\lambda}^{m,\gamma} g_i(z)\right)^2} - 1 \right| < \mu_i \quad \left(0 < \mu_i \le 1,\ z \in \mathbb{U},\ i \in \{1,\dots,n\}\right).$$

If

$$|D_{\lambda}^{m,\gamma} g_i(z)| \le M \ (M \ge 1\,;\, z \in \mathbb{U}\,;\, i \in \{1,\dots,n\}),$$

and $\alpha \in \mathbb{C}$ *be such that*

$$\operatorname{Re}\left(n(\alpha-1)+1\right) \ge \operatorname{Re}\alpha > 0, \quad \textit{and} \quad |\alpha - 1| \le \frac{\operatorname{Re}\alpha}{\sum_{i=1}^{n}((\mu_i+1)M+1)},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ *defined by* (1.16) *is in the class* $\mathcal{S}$*.*

Proof. In Theorem 2.4, we consider $M_1 = M_2 = \cdots = M_n = M$. $\square$

Corollary 2.6. *Let g_i, defined by* (2.1)*, be in the class $\mathcal{T}_{j,\mu}$ for $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$, and satisfy the properties*

$$\left| \frac{z^2 \left(D_\lambda^{m,\gamma} g_i(z)\right)'}{(D_\lambda^{m,\gamma} g_i(z))^2} - 1 \right| < \mu \quad (0 < \mu \leq 1,\ z \in \mathbb{U},\ i \in \{1, \dots, n\}).$$

If

$$|D_\lambda^{m,\gamma} g_i(z)| \leq M \ (M \geq 1\,;\ z \in \mathbb{U}\,;\ i \in \{1, \dots, n\}),$$

and $\alpha \in \mathbb{C}$ be such that

$$\operatorname{Re}(n(\alpha - 1) + 1) \geq \operatorname{Re}\alpha > 0, \quad \textit{and} \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{((\mu + 1)M + 1)n},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ defined by (1.16) *is in the class $\mathcal{S}$.*

Proof. In Corollary 2.5, we consider $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$. $\square$

Corollary 2.7. *In Corollary* 2.6*, if we set $D_\lambda^{0,\gamma} g_i = D_0^{m,0} g_i = g_i$* $(i \in \{1, \dots, n\})$ *and*

(**i**) *$j = 2$, then we have Theorem C'.*

(**ii**) *$j = 2$ and $M = 1$, then we have Theorem C.*

Theorem 2.8. *Let g_i, defined by* (2.1)*, be in the class $\mathcal{S}_j(p_i)$ for $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$, and satisfy the properties*

$$\left| \frac{z^2 \left(D_\lambda^{m,\gamma} g_i(z)\right)'}{(D_\lambda^{m,\gamma} g_i(z))^2} - 1 \right| < p_i \quad (0 < p_i \leq 2,\ z \in \mathbb{U},\ i \in \{1, \dots, n\}).$$

If

$$|D_\lambda^{m,\gamma} g_i(z)| \leq M_i \ (M_i \geq 1\,;\ z \in \mathbb{U}\,;\ i \in \{1, \dots, n\}),$$

and $\alpha \in \mathbb{C}$ be such that

$$\operatorname{Re}(n(\alpha - 1) + 1) \geq \operatorname{Re}\alpha > 0, \quad \textit{and} \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{\sum_{i=1}^n ((p_i + 1)M_i + 1)},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ defined by (1.16) *is in the class $\mathcal{S}$.*

Proof. Considering the function h defined by (2.2) and following the same way as in the proof of Theorem 2.1, we see that

$$\begin{aligned}
&\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \\
&\leq \ \frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} |\alpha - 1| \sum_{i=1}^n \left(\left| \frac{z^2 \left(D_\lambda^{m,\gamma} g_i(z)\right)'}{(D_\lambda^{m,\gamma} g_i(z))^2} - 1 \right| M_i + M_i + 1 \right) \\
&\leq \ \frac{|\alpha - 1|}{\operatorname{Re}\alpha} \sum_{i=1}^n ((p_i + 1)M_i + 1) \leq 1
\end{aligned}$$

for $g_i \in \mathcal{S}_j(p_i)$ $(i \in \{1,\ldots,n\})$. Therefore, we get $G_{\lambda,\alpha}^{n,m,\gamma} \in \mathcal{S}$ by Theorem A. □

Corollary 2.9. *Let g_i, defined by* (2.1)*, be in the class $\mathcal{S}_j(p_i)$ for $i \in \{1,\ldots,n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$, and satisfy the properties*

$$\left|\frac{z^2\left(D_\lambda^{m,\gamma} g_i(z)\right)'}{(D_\lambda^{m,\gamma} g_i(z))^2} - 1\right| < p_i \quad (0 < p_i \leq 2,\ z \in \mathbb{U},\ i \in \{1,\ldots,n\}).$$

If

$$|D_\lambda^{m,\gamma} g_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}\,;\, i \in \{1,\ldots,n\}),$$

and $\alpha \in \mathbb{C}$ be such that

$$\operatorname{Re}(n(\alpha-1)+1) \geq \operatorname{Re}\alpha > 0, \quad \textit{and} \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{\sum_{i=1}^{n}((p_i+1)M+1)},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ defined by (1.16) *is in the class $\mathcal{S}$.*

Proof. In Theorem 2.8, we consider $M_1 = M_2 = \cdots = M_n = M$. □

Corollary 2.10. *Let g_i, defined by* (2.1)*, be in the class $\mathcal{S}_j(p)$ for $i \in \{1,\ldots,n\}$, $n \in \mathbb{N}$, $j \in \mathbb{N}_1^*$, and satisfy the properties*

$$\left|\frac{z^2\left(D_\lambda^{m,\gamma} g_i(z)\right)'}{(D_\lambda^{m,\gamma} g_i(z))^2} - 1\right| < p \quad (0 < p \leq 2,\ z \in \mathbb{U},\ i \in \{1,\ldots,n\}).$$

If

$$|D_\lambda^{m,\gamma} g_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}\,;\, i \in \{1,\ldots,n\}),$$

and $\alpha \in \mathbb{C}$ be such that

$$\operatorname{Re}(n(\alpha-1)+1) \geq \operatorname{Re}\alpha > 0, \quad \textit{and} \quad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{((p+1)M+1)n},$$

then the integral operator $G_{\lambda,\alpha}^{n,m,\gamma}$ defined by (1.16) *is in the class $\mathcal{S}$.*

Proof. In Corollary 2.9, we consider $p_1 = p_2 = \cdots = p_n = p$. □

Corollary 2.11. *In Corollary* 2.10*, if we set $D_\lambda^{0,\gamma} g_i = D_0^{m,0} g_i = g_i$ $(i \in \{1,\ldots,n\})$ and*

(i) *$j = 2$, then we have Theorem D$'$.*

(ii) *$j = 2$ and $M = 1$, then we have Theorem D.*

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