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EXISTENCE AND UNIQUENESS OF $\xi\eta$-MULTIPLE FIXED POINTS OF MIXED MONOTONE OPERATORS

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Abstract. In this paper, we firstly coin the concept of $\xi\eta$-multiple fixed point and then obtain necessary and sufficient conditions for a class of mixed monotone operators to have this point. We see that these conditions can be considerably loosened for monotone operators acting on pairs of points, which are multiples of the same point. Also when a cone is chosen to be a normal solid cone and monotone operator on interior of the cone, the necessary and sufficient conditions get reduced further. The introduction of adjoint sequence generalizes further the main result.

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1. Introduction

Monotone operators have been the center of attraction for mathematicians working on fixed point theory [1]-[8]. Monotonicity helps in convergence of schemes of iterates to fixed points in many situations [7]. Mixed monotone operators form further interesting class of mappings having combination of two reverse-directed properties. We devise some results which guarantee generalized fixed points for mixed monotone operators. We also consider concavity and convexity properties along with monotonicities as has been done earlier [8]. The uniqueness of the respective fixed points in each case is also a noteworthy property.

Before beginning our discussion of the basic definitions of the structures that we work with, we propose a new definition of λ-multiple fixed point.

Definition 1.1. *Suppose X and Y are two linear spaces over the same field F and $f : X \to Y$. For $0 \neq \lambda \in F$, a point $x \in X$ is said to be λ-multiple fixed point of f if, and only if, $f(\lambda x) = x$.*

This is a new concept introduced by us specially to incorporate a similar property in a wider sense. In fact, λ-multiple fixed point is generalization of the usual fixed point in which $\lambda = 1$.

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Now we recapitulate basic terminologies required for the development of the subject matter of this paper.

Definition 1.2. *A non-empty closed convex subset P of a real Banach space E is said to be a cone if*

$$x \in P \text{ and } \lambda \geq 0 \Rightarrow \lambda x \in P,$$
$$x, -x \in P \Rightarrow x = 0.$$

If a real Banach space contains a cone, then it can be provided with additional partial order structure.

Definition 1.3. *If P is a cone in a real Banach space E, then E is a partially ordered set with respect to the partial order relation induced by P given by $x \leq y$ if, and only if, $y - x \in P$.*

We introduce two types of cones which are of interest for us.

Definition 1.4. *A cone P is said to be solid cone if its interior $P^{\circ} = \{x \in P : x$ is interior point of $P\}$ is non-empty.*

Definition 1.5. *A cone P is said to be normal cone if there exists a constant $N > 0$ such that for $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$.*

The positive constant whose existence makes a cone normal is called normality constant.

Definition 1.6. *The set of all elements of the space, which are bounded by some positive multiples of h is denoted by P_h.*
i.e., $P_h = \{x \in E : \lambda(x) h \leq x \leq \mu(x) h$, for some $\lambda(x), \mu(x) > 0\}$.

By the very definition, it is clear that $P_h \subset P$ and P_h contains all positive multiples of its own elements. Now using the well-known properties of both monotonicities simultaneously, one gets the following.

Definition 1.7. *An operator $A : P_h \times P_h \to P_h$ is said to be a mixed monotone operator if $A(x, y)$ is non-decreasing in the first component and non-increasing in the second component, i.e., if $x_1 \leq x_2$ and $y_1 \geq y_2 \Rightarrow A(x_1, y_1) \leq A(x_2, y_2)$.*

The λ-multiple fixed point concept can be extended to monotone operators.

Definition 1.8. *A point $x \in E$ is called $\xi\eta$-multiple fixed point of a mixed monotone operator $A : P_h \times P_h \to P_h$, if and only if, $x = A(\xi x, \eta x) = A(\eta x, \xi x)$.*

The special case of $\xi = \eta = 1$ leads to usual fixed point.

2. Main Results

An obvious and straightforward lemma begins the journey of the main results.

Lemma 2.1. *Let E be a real Banach space, P a cone in $E, h > 0$ and $A : P_h \times P_h \to P_h$. Then the following two statements are equivalent:*
(a) *For all $0 < \alpha < 1$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \theta = \theta\left(\alpha^2\beta\right) < 1$ such that*

$$A(\alpha u, \beta v) \geq \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)} A(u, v) \quad (\textit{for all } u, v \in P_h, u \leq v) \tag{2.1}$$

(b) *For all $0 < \alpha < 1$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \psi = \psi\left(\alpha^2\beta\right) < 1$ such that*

$$A(\alpha u, \beta v) \geq \left(\alpha^2\beta\right)\left[1 + \psi\left(\alpha^2\beta\right)\right] A(u, v) \quad (\textit{for all } u, v \in P_h, u \leq v) \tag{2.2}$$

where $\left(\alpha^2\beta\right)\left[1 + \psi\left(\alpha^2\beta\right)\right] < 1$.

Proof. If (a) holds, let $\psi\left(\alpha^2\beta\right) = \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)-1} - 1$. Then (b) holds with $0 < \psi = \psi\left(\alpha^2\beta\right) < 1$ and $\left(\alpha^2\beta\right)\left[1 + \psi\left(\alpha^2\beta\right)\right] < 1$.
Conversely, if (b) holds, it is easy to see that (a) holds with the choice of $0 < \theta = \theta\left(\alpha^2\beta\right) < 1$ as

$$\theta\left(\alpha^2\beta\right) = \frac{\log\left[\left(\alpha^2\beta\right)\left(1 + \psi\left(\alpha^2\beta\right)\right)\right]}{\log\left(\alpha^2\beta\right)}.$$

This completes the proof.

The interchangeable usability of the two equivalent conditions in Lemma 2.1 is employed to prove the important $\xi\eta$-multiple fixed point results ahead.

Theorem 2.1. *(see [2],[3],[8]) Suppose that E is a real Banach space, P is a normal cone in E, $h > 0$, and $A : P_h \times P_h \to P_h$ is a mixed monotone operator such that for all $0 < \alpha < 1$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \theta = \theta\left(\alpha^2\beta\right) < 1$ such that*

$$A(\alpha u, \beta v) \geq \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)} A(u, v) \quad (\textit{for all } u, v \in P_h, u \leq v).$$

Then for any $\xi, \eta > 0, A$ has a unique $\xi\eta$-multiple fixed point x^* in P_h if, and only if, there exist $u_0, v_0 \in P_h$ with $u_0 \leq v_0, u_0 \leq A(\xi u_0, \eta v_0)$ and $A(\eta v_0, \xi u_0) \leq v_0$. Further, for every pair of sequences $\{x_n\}$ and $\{y_n\}$ constructed as

$$x_n = A(\xi x_{n-1}, \eta y_{n-1}), \tag{2.3}$$

$$y_n = A(\eta y_{n-1}, \xi x_{n-1}), \tag{2.4}$$

for each $n \geq 1$ and $x_0, y_0 \in [\xi x_0, \eta y_0]$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x^*$.

Proof. First we assume that for $\xi, \eta > 0$, the given condition is satisfied and there exist u_0 and v_0 with $u_0 \leq v_0, u_0 \leq A(\xi u_0, \eta v_0)$ and $A(\eta v_0, \xi u_0) \leq v_0$. Now starting with u_0 and v_0, we construct sequences $\{u_n\}$ and $\{v_n\}$ employing scheme of (2.3) and (2.4),

$$\begin{aligned} u_n &= A(\xi u_{n-1}, \eta v_{n-1}), \\ v_n &= A(\eta v_{n-1}, \xi u_{n-1}). \end{aligned}$$

By very definitions, $u_n, v_n \in P_h$ for all n. Also, by the given conditions and mixed monotonicity of A,

$$\begin{cases} u_0 \le A(\xi u_0, \eta v_0) = u_1 \\ v_1 = A(\eta v_0, \xi u_0) \le v_0 \end{cases}$$
$$\begin{cases} u_1 = A(\xi u_0, \eta v_0) \le A(\xi u_1, \eta v_1) = u_2 \\ v_2 = A(\eta v_1, \xi u_1) \le A(\eta v_0, \xi u_0) = v_1 \end{cases}$$
$$\vdots$$
$$\begin{cases} u_n = A(\xi u_{n-1}, \eta v_{n-1}) \le A(\xi u_n, \eta v_n) = u_{n+1} \\ v_{n+1} = A(\eta v_n, \xi u_n) \le A(\eta v_{n-1}, \xi u_{n-1}) = v_n \end{cases}$$

Thus, $u_0 \le u_1 \le u_2 \le \cdots \le u_n \le u_{n+1} \le \cdots \le v_{n+1} \le v_n \le \cdots \le v_2 \le v_1 \le v_0$. For each n, there exists $0 < \lambda \le 1$ such that $\lambda v_n \le u_n \le v_n$. Let $\mu_n = \sup\{0 < \lambda \le 1 | \lambda v_n \le u_n\}$. Clearly, $0 < \mu_n \le 1$, and as $\{u_n\}$ and $\{v_n\}$ are non-retreating, $\{\mu_n\}$ is non-decreasing sequence. Hence $\lim\limits_{n\to\infty} \mu_n = \mu$ with $0 < \mu \le 1$. If possible, suppose that $0 < \mu < 1$. For each n and corresponding μ_n, we choose $\beta_n > 0$ such that for $0 < \frac{\mu_n}{\sqrt{\mu\beta_n}} < 1$, the corresponding conditional value is $\frac{\beta_n}{\mu_n} \ge \frac{1}{\frac{\mu_n}{\sqrt{\mu\beta_n}}} = \frac{\sqrt{\mu\beta_n}}{\mu_n}$ with $A\left(\frac{\mu_n}{\sqrt{\mu\beta_n}}u, \frac{\beta_n}{\mu_n}v\right) \ge \left(\frac{\mu_n}{\mu}\right)^{\theta\left(\frac{\mu_n}{\mu}\right)} A(u,v)$ for $u, v \in P_h, u \le v$. Also, we choose $\beta > 0$, such that $\xi u_n \ge \frac{\mu_n}{\sqrt{\beta\beta_n}}\eta v_n, \eta v_n \le \frac{\beta\beta_n}{\mu_n}\xi u_n$, and for $0 < \sqrt{\frac{\mu}{\beta}} < 1$, the corresponding conditional value is $\beta \ge \frac{1}{\sqrt{\mu/\beta}} = \sqrt{\frac{\beta}{\mu}}$, with $A\left(\sqrt{\frac{\mu}{\beta}}u, \beta v\right) \ge \mu^{\theta(\mu)} A(u,v)$, for $u, v \in P_h, u \le v$. Now

$$\begin{aligned}
u_{n+1} &= A(\xi u_n, \eta v_n) \\
&\ge A\left(\frac{\mu_n}{\sqrt{\beta\beta_n}}\eta v_n, \frac{\beta\beta_n}{\mu_n}\xi u_n\right) \\
&= A\left(\frac{\mu_n}{\sqrt{\mu\beta_n}}\sqrt{\frac{\mu}{\beta}}\eta v_n, \frac{\beta_n}{\mu_n}(\beta\xi u_n)\right) \\
&\ge \left(\frac{\mu_n}{\sqrt{\mu\beta_n}}\right)^2 \frac{\beta_n}{\mu_n}\left(1 + \psi\left(\left(\frac{\mu_n}{\sqrt{\mu\beta_n}}\right)^2 \frac{\beta_n}{\mu_n}\right)\right) \\
&\quad \times A\left(\sqrt{\frac{\mu}{\beta}}\eta v_n, \beta\xi u_n\right) \\
&= \frac{\mu_n}{\mu}\left(1 + \psi\left(\frac{\mu_n}{\mu}\right)\right) A\left(\sqrt{\frac{\mu}{\beta}}\eta v_n, \beta\xi u_n\right) \\
&\ge \frac{\mu_n}{\mu} A\left(\sqrt{\frac{\mu}{\beta}}\eta v_n, \beta\xi u_n\right) \\
&\ge \frac{\mu_n}{\mu}\left(\left(\sqrt{\frac{\mu}{\beta}}\right)^2 \beta\right)\left(1 + \psi\left(\left(\sqrt{\frac{\mu}{\beta}}\right)^2 \beta\right)\right) A(\eta v_n, \xi u_n) \\
&= \frac{\mu_n}{\mu}\mu(1 + \psi(\mu)) A(\eta v_n, \xi u_n) \\
&= \mu_n(1 + \psi(\mu)) v_{n+1}
\end{aligned}$$

But by definition of μ_{n+1}, we have $\mu_{n+1} \geq \mu_n (1+\psi(\mu))$. As $n \to \infty, \mu \geq \mu(1+\psi(\mu))$, and this is a contradiction, since $\psi(\mu) > 0$. So the supposition that $0 < \mu < 1$ is wrong and $\mu = 1$.
For any positive integer p,

$$0 \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - \mu_n v_n = (1-\mu_n) v_n \leq (1-\mu_n) v_0.$$

Since P is a normal cone, $\|u_{n+p} - u_n\| \leq N(1-\mu_n)\|v_0\| \to N(1-\mu)\|v_0\| = 0$, as $n \to \infty$. Thus, $\{u_n\}$ is a Cauchy sequence. Similarly, it is seen that $\{v_n\}$ is also a Cauchy sequence. Completeness of E guarantees that there exist $u^*, v^* \in E$ such that $u_n \to u^*$ and $v_n \to v^*$. Again, since $\{u_n\}$ is non-decreasing and $\{v_n\}$ is non-increasing with $u_n \leq v_n$, $u_n \leq u^* \leq v^* \leq v_n$, in particular, $u^*, v^* \in E$. As earlier, $v^* - u^* \leq v_n - u_n \leq (1-\mu_n) v_n \leq (1-\mu_n) v_0$, and as $n \to \infty, \|v^* - u^*\| = 0$. This gives $u^* = v^* = x^*$, say. Now,
$u_{n+1} = A(\xi u_n, \eta v_n) \leq A(\xi u^*, \eta v^*) = A(\xi x^*, \eta x^*) \leq A(\eta v_n, \xi u_n) = v_{n+1}$.
As $n \to \infty, A(\xi x^*, \eta x^*) = x^*$, the $\xi\eta$-multiple fixed point of A.
We prove that this $\xi\eta$-multiple fixed point is unique.
If possible, suppose that there are two distinct $\xi\eta$-multiple fixed points, viz., x^* and y^* in P_h. So, $A(\xi x^*, \eta x^*) = x^*$ and $A(\xi y^*, \eta y^*) = y^*$.
Let $\lambda_0 = \sup\left\{\lambda > 0 : \lambda y^* \leq x^* \leq \left(\frac{1}{\lambda}\right) y^*\right\}, 0 < \lambda_0 \leq 1$.
But if $0 < \lambda_0 < 1$, by the given condition, there exists $\omega \geq \left(\frac{1}{\lambda_0}\right)$, such that $A(\lambda_0 u, \omega v) \geq \left(\lambda_0^2\omega\right)^{\theta\left(\lambda_0^2\omega\right)} A(u,v)$. Now,

$$\begin{aligned} x^* &= A(\xi x^*, \eta x^*) \\ &\geq A\left(\xi\lambda_0 y^*, \eta\left(\frac{1}{\lambda_0}\right) y^*\right) \\ &\geq A(\lambda_0 \xi y^*, \omega\eta y^*) \\ &\geq \left(\lambda_0^2\omega\right)^{\theta\left(\lambda_0^2\omega\right)} A(\xi y^*, \eta y^*) \\ &\geq \lambda_0^{\theta\left(\lambda_0^2\omega\right)} y^*, \end{aligned}$$

and this is not possible since it gives $\lambda_0^{\theta\left(\lambda_0^2\omega\right)} > \lambda_0$, contradicting the definition of λ_0. Therefore, $\lambda_0 = 1$. But this means that $x^* = y^*$. So, the supposition is wrong and the $\xi\eta$-multiple fixed point is unique.
Now, suppose conversely that A has a unique $\xi\eta$-multiple fixed point x^* in P_h. Taking $u_0 = v_0 = x^*$, it is straightforward that, $u_0 = x^* = A(\xi x^*, \eta x^*) \leq A(\xi u_0, \eta v_0)$ and $A(\xi u_0, \eta v_0) = A(\xi x^*, \eta x^*) \leq x^* = v_0$
Finally, for $x_0, y_0 \in [\xi u_0, \eta v_0]$ and the sequences $\{x_n\}$ and $\{y_n\}$ given by (2.3) and (2.4), respectively, since $\xi u_n \leq x_n \leq \eta v_n$ and $\xi u_n \leq y_n \leq \eta v_n$, and P is a normal cone, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x^*$. This completes the proof of the theorem.

3. Same Element Multiple Arguments Operators

At times, the initial condition for values of mixed monotone operator A may not be satisfied by all general members of P_h as the two argument components of A but only by multiples of the same member. For this situation, the following corollary comes into picture with the choice of special forms of the initial points in Theorem 2.1.

Corollary 3.1. *(see [8]) Suppose that E is a real Banach space, P is a normal cone in E, $h > 0$, and $A : P_h \times P_h \to P_h$ is a mixed monotone operator such that for all $0 < \alpha < 1$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \theta = \theta\left(\alpha^2\beta\right) < 1$ such that*

$$A(\alpha u, \beta u) \geq \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)} A(u, u), \quad (\text{for } u \in P_h).$$

Then A has a unique $\xi\eta$-multiple fixed point x^* in P_h if, and only if, there exists $x_0 \in P_h$ and $\alpha_0 \in (0, 1), \beta_0$ with $\alpha_0^2\beta_0 < 1$ such that $\alpha_0 x_0 \leq A(\xi\alpha_0 x_0, \eta\beta_0 x_0)$ and $A(\eta\beta_0 x_0, \xi\alpha_0 x_0) \leq \beta_0 x_0$.

Proof. Taking $u_0 = \alpha_0 x_0$ and $v_0 = \beta_0 x_0$, and proceeding as in proof of Theorem 2.1 above, the proof gets completed.

The condition for the two components in the class of mixed monotone operators considered by us when restricted to multiples of the same members of P°, offers more freedom to the θ function, and necessary and sufficient condition for the existence of $\xi\eta$-multiple fixed point is reduced.

Corollary 3.2. *(see [4], [8]) Suppose that E is a real Banach space, P is a normal solid cone in E, and $A : P^\circ \times P^\circ \to P^\circ$ is a mixed monotone operator such that for all $\alpha \in [\gamma, \delta] \subset (0, 1)$, there exists $\beta \geq \frac{1}{\alpha}$, and $0 < \theta = \theta(\gamma, \delta) < 1$ satisfying $A(\alpha x, \beta x) \geq \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)} A(x, x)$ for all $x \in P^\circ$. Then A has a unique $\xi\eta$-multiple fixed point x^* in P° and for all $x_0 \in P^\circ, A^n(x_0, x_0) = A(x_{n-1}, x_{n-1}) \to x^* = A(x^*, x^*)$.*

Proof. For every $x_0 \in P^\circ$, there exists a $0 < \delta < 1$, satisfying $\delta x_0 \leq A(\xi x_0, \eta x_0)$ and $A(\eta x_0, \xi x_0) \leq \frac{1}{\delta} x_0$. Using the given hypothesis, for every $\alpha_0 \in [\gamma, \delta] \subset (0, 1)$, there exist $\beta_0 \geq \frac{1}{\alpha_0}$ and $0 < \theta_0 = \theta_0(\gamma, \delta) < 1$, such that

$$A(\alpha_0 x_0, \beta_0 x_0) \geq \left(\alpha_0^2\beta_0\right)^{\theta_0(\gamma,\delta)} A(x_0, x_0).$$

Interestingly, using mixed monotonicity, this also gives

$$\begin{aligned} A(\alpha_0 x_0, \alpha_0 x_0) &\geq A(\alpha_0 x_0, \beta_0 x_0) \geq \left(\alpha_0^2\beta_0\right)^{\theta_0(\gamma,\delta)} A(x_0, x_0), \text{and} \\ A(\beta_0 x_0, \beta_0 x_0) &\leq \left(\alpha_0^2\beta_0\right)^{-\theta_0(\gamma,\delta)} A\left(\alpha_0\beta_0 x_0, \beta_0^2 x_0\right) \\ &\leq \left(\alpha_0^2\beta_0\right)^{-\theta_0(\gamma,\delta)} A(x_0, x_0) \end{aligned}$$

For all sequences $\{\gamma_n\}$ satisfying $0 < \gamma_n < 1$, and $\gamma_1 > \gamma_2 > \cdots > \gamma_n > \cdots > 0$, let $\forall n$, $\theta_n = \inf\left\{\theta \in (0, 1) \,|A(\alpha x, \alpha x) \geq \left(\alpha^2\beta\right)^\theta A(x, x) \,\forall \alpha \in [\gamma_n, \delta], x \in P^\circ\right\}$. Clearly, $\theta_1 < \theta_2 < \cdots < \theta_n < \cdots < 1, \{\theta_n\}$ is a monotonic increasing sequence bounded above by 1 and hence its limit exists. Let $\lim_{n\to\infty} \theta_n = \theta$ with $0 < \theta \leq 1$. Now there are two possible cases. Firstly, if $\delta^{\frac{1}{1-2\theta_1}} > \gamma_1$, we take such $\alpha_0 \in (\gamma_1, \delta_1) \subset [\gamma, \delta]$, where $\delta_1 = \min\left\{\delta^{\frac{1}{1-2\theta_1}}, \beta_0^{\frac{1}{-1-2\theta_1}}\right\}$, for the corresponding conditional $\beta_0, A(\eta\beta_0 x_0, \xi\alpha_0 x_0) \leq \beta_0^{2\theta_1} A(\eta x_0, \xi x_0)$. From the choice of δ_1, firstly, $\alpha_0 < \delta^{\frac{1}{1-2\theta_1}}$, i.e., $\alpha_0^{1-2\theta_1} < \delta$, i.e., $\alpha_0^{1-2\theta_1} < \beta_0^{\theta_1}\delta$ and $\alpha_0 < \left(\alpha_0^2\beta_0\right)^{\theta_1}\delta$; also, $\beta_0 \geq \frac{1}{\alpha_0} > \frac{1}{\delta^{\frac{1}{1-2\theta_1}}}$, i.e., $\beta_0^{1-2\theta_1} > \frac{1}{\delta}$ or $\beta_0 > \frac{\beta_0^{2\theta_1}}{\delta}$. Now the choice of $u_0 = \alpha_0 x_0$ and $v_0 = \beta_0 x_0$ gives $u_0 = \alpha_0 x_0 \leq \left(\alpha_0^2\beta_0\right)^{\theta_1}\delta x_0 \leq \left(\alpha_0^2\beta_0\right)^{\theta_1} A(\xi x_0, \eta x_0) \leq A(\xi\alpha_0 x_0, \eta\beta_0 x_0) = A(\xi u_0, \eta v_o)$, and $A(\eta v_0, \xi u_0) = A(\eta\beta_0 x_0, \xi\alpha_0 x_0) \leq \beta_0^{2\theta_1} A(\eta x_0, \xi x_0) \leq \beta_0^{2\theta_1}\frac{1}{\delta}x_0 \leq \beta_0 x_0 = v_0$. In this case, all conditions in Theorem 2.1 are satisfied and there exists the required unique $\xi\eta$-multiple fixed point.

In the other case, if $\delta^{\frac{1}{1-2\theta_1}} \leq \gamma_1$, for $n \geq 2$, we take $\gamma_n = \gamma_1\delta^{\frac{1}{1-2\theta_{n-1}}}$, which gives as required that $\gamma_n > 0$ and $\{\gamma_n\}$ is decreasing sequence. There exists some positive integer N_0 such that $\gamma_n = \gamma_1\delta^{\frac{1}{1-2\theta_{n-1}}} < \delta^{\frac{1}{1-2\theta_n}}$, for all $n \geq N_0$. We take such $\alpha_0 \in (\gamma_{N_0}, \delta_1) \subset [\gamma, \delta]$, where $\delta_1 = \min\left\{\delta^{\frac{1}{1-2\theta_{N_0}}}, \beta_0^{\frac{1}{-1-2\theta_{N_0}}}\right\}$, for the corresponding conditional β_0, $A(\eta\beta_0 x_0, \xi\alpha_0 x_0) \leq \beta_0^{2\theta_{N_0}} A(\eta x_0, \xi x_0)$. Just as in the previous case, now we also get the required conditions. This completes the proof of the theorem.

4. A Special Case

Both, Theorem 2.1 and Corollary 3.1 following it, have given the necessary and sufficient condition for the existence of a unique $\xi\eta$-multiple fixed point for the mixed monotone operator $A : P_h \times P_h \to P_h$. If instead of P_h, the interior P° for a solid cone is taken into account for the operator A, and the function $\theta = \theta\left(\alpha^2\beta\right)$ involved in the initial condition is restricted to a fixed fraction $0 < \theta < 1$, it is quite interesting to see that the necessary and sufficient condition is again reduced.

Corollary 4.1. *(see [2], [8]) Suppose that E is a real Banach space, P is a normal solid cone in E, and $A : P^\circ \times P^\circ \to P^\circ$ is a mixed monotone operator such that for all $0 < \alpha < 1$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \theta < 1$ such that*

$$A(\alpha u, \beta v) \geq \left(\alpha^2\beta\right)^{\theta} A(u, v) \ \ (\textit{for all } u, v \in P_h, u \leq v).$$

Then for every $\xi, \eta > 0$, A has a unique $\xi\eta$-multiple fixed point, and hence also the usual fixed point, x^* in P°. Also, for any $x_0, y_0 \in P^\circ$, the sequences $\{x_n\}$ and $\{y_n\}$ defined by,

$$x_n = A(\xi x_{n-1}, \eta y_{n-1}),$$
$$y_n = A(\eta y_{n-1}, \xi x_{n-1})$$

for each $n \geq 1$, are convergent with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$

Proof. For any $x_0, y_0 \in P^\circ$, and $\xi, \eta > 0$, we choose a sufficiently small μ_0 with the corresponding conditional value ω, satisfying $\mu_0 x_0 \leq x_0 \leq \omega x_0, \mu_0 x_0 \leq \left(\mu_0^2 \omega\right)^\theta A\left(\xi x_0, \eta x_0\right)$ and $A\left(\mu_0 \eta \omega x_0, \omega \xi \mu_0 x_0\right) \leq \left(\mu_0^2 \omega\right)^\theta \omega x_0$.
Then taking $u_0 = \mu_0 x_0$ and $v_0 = \omega x_0$, gives $u_0 \leq v_0$. Using the other two conditions on μ_0,

$$\begin{aligned}
u_0 = \mu_o x_0 &\leq \left(\mu_0^2 \omega\right)^\theta A\left(\xi x_0, \eta x_0\right) \\
&\leq A\left(\xi \mu_0 x_0, \eta \omega x_0\right) = A\left(\xi u_0, \eta v_0\right) = u_1, \\
v_1 = A\left(\eta v_0, \xi u_0\right) &= A\left(\eta \omega x_0, \xi \mu_0 x_0\right) \\
&\leq \left(\mu_0^2 \omega\right)^{-\theta} A\left(\mu_0 \eta \omega x_0, \omega \xi \mu_0 x_0\right) \leq \omega x_0 = v_0.
\end{aligned}$$

We get the sequences $\{u_n\}$ and $\{v_n\}$ like those in Theorem 2.1. Following the same steps as there, completes this proof.

The extra advantage of Corollary 4.1 is that it works for every ξ and η, and hence also guarantees the usual fixed point of A.

5. Convexity and Concavity

Like most of the authors working on this line, we now turn to convexity and concavity in the following well-known senses.

Definition 5.1. *An operator A on a real Banach space E is said to be $(-\gamma)$-convex if, and only if, for each x and for each $0 < \mu < 1$, $\mu^\gamma A(\mu x) \leq A(x)$.*

Definition 5.2. *An operator A on a real Banach space E is said to be concave if, and only if, for each x and for each $0 < \mu < 1$, $A(\mu x + (1-\mu) y) \geq \mu A(x) + (1-\mu) A(y)$.*

Theorem 2.1 can be again applied to obtain the fixed point for mixed monotone operator having concavity property at first component and some $(-\gamma)$-convexity property at the other.

Theorem 5.1. *(see [3],[8]) Suppose that E is a real Banach space, P is a normal solid cone in E, and $A : P^\circ \times P^\circ \to P^\circ$ is a mixed monotone operator such that*
(a) For fixed y, $A(\cdot, y) : P^\circ \to P^\circ$ is concave and for fixed x, $A(x, \cdot) : P^\circ \to P^\circ$ is $(-\gamma)$-convex.
(b) There exist $u_0, v_0 \in P, 0 < \epsilon < 1$, and $\xi, \eta > 0$ such that $0 \ll u_0 < v_0, u_0 \leq A(\xi u_0, \eta v_0), A(\eta v_0, \xi u_0) \leq v_0$, and $A(0, v_0) \geq \epsilon A(u_0, v_0)$.
Then A has a unique $\xi\eta$-multiple fixed point x^* in $[u_0, v_0]$. Further, for every pair of sequences $\{x_n\}$ and $\{y_n\}$ constructed as

$$\begin{aligned}
x_n &= A\left(\xi x_{n-1}, \eta y_{n-1}\right), \\
y_n &= A\left(\eta y_{n-1}, \xi x_{n-1}\right),
\end{aligned}$$

for each $n \geq 1$ and $x_0, y_0 \in [\xi u_0, \eta v_0]$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x^*$.

Proof. For all $h \in P^\circ, P_h = P^\circ$. So, $A : P_h \times P_h \to P_h$. For all $0 < \alpha < 1$, we choose $\beta \geq \frac{1}{\alpha}$ and $0 < \theta = \theta\left(\alpha^2\beta\right) < 1$ such that $\alpha^2\beta < 1$ and $\left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)}\beta^\gamma \leq \epsilon$. Now, first using the convexity of A in the second component and then concavity in the first,

$$\begin{aligned}
A(\alpha u, \beta v) &\geq \left(\frac{1}{\beta}\right)^\gamma A\left(\alpha u, \frac{1}{\beta}\beta v\right) = \left(\frac{1}{\beta}\right)^\gamma A(\alpha u, v) \\
&= \left(\frac{1}{\beta}\right)^\gamma A(\alpha u + (1-\alpha)\,0, v) \\
&\geq \left(\frac{1}{\beta}\right)^\gamma [\alpha A(u,v) + (1-\alpha) A(0,v)] \\
&\geq \left(\frac{1}{\beta}\right)^\gamma [\alpha A(u,v) + (1-\alpha)\,\varepsilon A(u,v)] \\
&= \left(\frac{1}{\beta}\right)^\gamma [\alpha + (1-\alpha)\,\varepsilon] A(u,v) \\
&= \left(\frac{1}{\beta}\right)^\gamma [\alpha + \varepsilon - \alpha\varepsilon] A(u,v) \\
&\geq \left(\frac{1}{\beta}\right)^\gamma [\alpha + \varepsilon - \alpha] A(u,v) = \left(\frac{1}{\beta}\right)^\gamma \varepsilon A(u,v) \\
&\geq \left(\frac{1}{\beta}\right)^\gamma \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)} \beta^\gamma A(u,v) \\
&= \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta\right)} A(u,v).
\end{aligned}$$

This ensures that all requirements are satisfied for the application of Theorem 2.1, and there exists a unique $\xi\eta$-multiple fixed point of A. The convergence of the given sequences to the fixed point is also an easy consequence. This completes the proof.

6. Further Generalizations

Now, while extending our own results, first of all, we give a generalization of our Lemma 2.1.

Lemma 6.1. *Let E be a real Banach space, P a cone in $E, h > 0$ and $A : P_h \times P_h \to P_h$. Then the following two statements are equivalent:*
(a) For all $0 < \alpha < 1$ and $u, v \in P_h$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \theta = \theta\left(\alpha^2\beta, u, v\right) < 1$ such that

$$A(\alpha u, \beta v) \geq \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta, u, v\right)} A(u,v)$$

(b) For all $0 < \alpha < 1$ and $u, v \in P_h$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \psi = \psi\left(\alpha^2\beta, u, v\right) < 1$ such that

$$A(\alpha u, \beta v) \geq \left(\alpha^2\beta\right)\left[1 + \psi\left(\alpha^2\beta, u, v\right)\right] A(u,v)$$

where $\left(\alpha^2\beta\right)\left[1+\psi\left(\alpha^2\beta,u,v\right)\right]<1$.

Proof. The proof is verbatim similar to that of Lemma 2.1, with replacement of $\theta\left(\alpha^2\beta\right)$ by $\theta\left(\alpha^2\beta,u,v\right)$ and $\psi\left(\alpha^2\beta\right)$ by $\psi\left(\alpha^2\beta,u,v\right)$.

In addition to $\alpha^2\beta$, the functions θ and ψ in Lemma 6.1 depend on the points u and v of P_h; whereas in Lemma 2.1, they are free of u and v.

Now we need the concept of adjoint sequence for extending few main results.

Definition 6.1. *Suppose E be a real Banach space, P a normal cone in $E, h>0$, $A: P_h\times P_h\to P_h$ be an operator, and $\xi,\eta>0$ and $u_0,v_0\in P_h$. If there exists $0<\lambda_0<\min\left\{\frac{\eta}{\xi},1\right\}$ such that $\lambda_0 v_0\le u_0\le v_0$, we define u_n and v_n for $n>0$ by*

$$\begin{aligned} u_n &= A\left(\xi u_{n-1},\eta v_{n-1}\right), \\ v_n &= A\left(\eta v_{n-1},\xi u_{n-1}\right). \end{aligned}$$

A sequence $\{\psi_n\}$ is called $\xi\eta$-adjoint sequence of A with respect to λ_0,u_0, and v_0 if, and only if, $0<\lambda_n=\lambda_0\left(1+\psi_n\right)^n<1$ for $\lambda_n v_n\le u_n\le v_n, n\ge 0$.

For mixed monotone operators, this hypothesis and any of the equivalent conditions (a) or (b) in Lemma 6.1 are sufficient to guarantee the existence of such $\xi\eta$-adjoint sequence.

Lemma 6.2. *Let E be a real Banach space, P a normal cone in $E, h>0$ and $A: P_h\times P_h\to P_h$ a mixed monotone operator such that for all $0<\alpha<1$ and $u,v\in P_h$, there exists $\beta\ge\frac{1}{\alpha}>1$ with $\alpha^2\beta<1$ and $0<\theta=\theta\left(\alpha^2\beta,u,v\right)<1$ such that $A\left(\alpha u,\beta v\right)\ge\left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta,u,v\right)}A\left(u,v\right)$. Then for any $0<\xi\le\eta, u_0,v_0\in P_h$ with $u_0\le v_0$, and $0<\lambda_0<1$ with $\lambda_0 v_0\le u_0\le v_0$, there exists a $\xi\eta$-adjoint sequence of A with respect to λ_0,u_0 and v_0.*

Proof. Given $\xi,\eta>0; u_0,v_0\in P_h$ with $\xi u_0\le\eta v_0$; and $0<\lambda_0<1$, we choose $\alpha, 0<\alpha<\frac{\xi\lambda_0}{\eta}$, with the corresponding conditional $\beta, \frac{\eta}{\xi\lambda_0}<\beta$, such that $\lambda_0\le\alpha^2\beta<1$. Clearly, $v_1=A\left(\eta v_0,\xi u_0\right)\ge A\left(\xi u_0,\eta v_0\right)=u_1$.
Now, by Lemma 6.1 there exists $\psi_1'=\psi_1'\left(\alpha^2\beta,u_0,v_0\right)$ such that

$$\begin{aligned} u_1 &= A\left(\xi u_0,\eta v_0\right) \\ &\ge A\left(\xi\lambda_0 v_0,\eta\frac{1}{\lambda_0}u_0\right) \\ &\ge A\left(\alpha\eta v_0,\beta\xi u_0\right) \\ &\ge \left(\alpha^2\beta\right)\left[1+\psi_1'\left(\alpha^2\beta,u_0,v_0\right)\right]A\left(\eta v_0,\xi u_0\right) \\ &\ge \lambda_0\left[1+\psi_1'\left(\alpha^2\beta,u_0,v_0\right)\right]v_1 \\ &\ge \lambda_1 v_1, \end{aligned}$$

taking $0<\psi_1\le\psi_1'$ and $0<\lambda_1=\lambda_0\left(1+\psi_1\right)<1$.
Thus, $\lambda_1 v_1\le u_1\le v_1$.
Again, clearly, $v_2=A\left(\eta v_1,\xi u_1\right)\ge A\left(\xi u_1,\eta v_1\right)=u_2$.

We choose $\alpha_1, 0 < \alpha_1 < \frac{\xi\lambda_1}{\eta}$, with the corresponding $\beta_1, \frac{\eta}{\xi\lambda_1} < \beta_1$, such that $\lambda_1 \leq \alpha_1^2\beta_1 < 1$.
By Lemma 6.1, there exists $\psi_2' = \psi_2'\left(\alpha^2\beta, u_1, v_1\right)$ such that

$$\begin{aligned} u_2 &= A\left(\xi u_1, \eta v_1\right) \\ &\geq A\left(\xi\lambda_1 v_1, \eta\frac{1}{\lambda_1}u_1\right) \\ &\geq A\left(\alpha_1\eta v_1, \beta_1\xi u_1\right) \\ &\geq \left(\alpha_1^2\beta_1\right)\left[1+\psi_2'\left(\alpha_1^2\beta_1, u_1, v_1\right)\right] A\left(\eta v_1, \xi u_1\right) \\ &\geq \lambda_1\left[1+\psi_2'\left(\alpha_1^2\beta_1, u_1, v_1\right)\right] v_2, \end{aligned}$$

where $0 < \lambda_1\left(1+\psi_2'\right) < 1$. Taking $\psi_2 = \min\{\psi_1, \psi_2'\}$,

$$u_2 \geq \lambda_1\left(1+\psi_2\right) v_2 \geq \lambda_0\left(1+\psi_2\right)^2 v_2 = \lambda_2 v_2$$

with $0 < \lambda_2 = \lambda_0\left(1+\psi_2\right)^2 < 1$.
Thus, this time also, $\lambda_2 v_2 \leq u_2 \leq v_2$.
Continuing by induction, $\forall n$, we get λ_n such that $0 < \lambda_n = \lambda_0\left(1+\psi_n\right)^n < 1$ and $\lambda_n v_n \leq u_n \leq v_n$. The sequence $\{\psi_n\}$ obtained in the process is the required $\xi\eta$-adjoint sequence of A with respect to λ_0, u_0, and v_0. This completes the proof.

We are all set to give the most general result.

Theorem 6.1. *(see [8])Suppose that E is a real Banach space, P is a normal cone in E, $h > 0$, and $A : P_h \times P_h \to P_h$ is a mixed monotone operator such that for all $0 < \alpha < 1$ and $u, v \in P_h$, there exists $\beta \geq \frac{1}{\alpha} > 1$ with $\alpha^2\beta < 1$ and $0 < \theta = \theta\left(\alpha^2\beta, u, v\right) < 1$ such that $A\left(\alpha u, \beta v\right) \geq \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta, u, v\right)} A\left(u, v\right)$. Then for any $\xi, \eta > 0, A$ has a unique $\xi\eta$-multiple fixed point x^* in P_h if, and only if, there exist $u_0, v_0 \in P_h$, satisfying*
(a) $u_0 \leq v_0, u_0 \leq A\left(\xi u_0, \eta v_0\right)$ and $A\left(\eta v_0, \xi u_0\right) \leq v_0$,
(b) If there exists $\lambda_0 > 0$ such that $\lambda_0 v_0 \leq u_0$, then there exists a $\xi\eta$-adjoint sequence $\{\psi_n\}$ of A with respect to λ_0, u_0, and v_0 such that $\lim\limits_{n\to\infty} n\psi_n = \ln\frac{1}{\lambda_0}$

Proof. First we assume that for any $\xi, \eta > 0$, the given conditions are satisfied and there exist u_0 and v_0 in P_h with $u_0 \leq v_0, u_0 \leq A\left(\xi u_0, \eta v_0\right)$ and $A\left(\eta v_0, \xi u_0\right) \leq v_0$. Now starting with u_0 and v_0, we construct the sequences $\{u_n\}$ and $\{v_n\}$ employing the same recursive scheme,

$$\begin{aligned} u_n &= A\left(\xi u_{n-1}, \eta v_{n-1}\right), \\ v_n &= A\left(\eta v_{n-1}, \xi u_{n-1}\right). \end{aligned}$$

By very definitions, $u_n, v_n \in P_h$ for all n. Also, by the given conditions and mixed monotonicity of A,

$$\begin{cases} u_0 \leq A(\xi u_0, \eta v_0) = u_1 \\ v_1 = A(\eta v_0, \xi u_0) \leq v_0 \end{cases}$$
$$\begin{cases} u_1 = A(\xi u_0, \eta v_0) \leq A(\xi u_1, \eta v_1) = u_2 \\ v_2 = A(\eta v_1, \xi u_1) \leq A(\eta v_0, \xi u_0) = v_1 \end{cases}$$
$$\vdots$$
$$\begin{cases} u_n = A(\xi u_{n-1}, \eta v_{n-1}) \leq A(\xi u_n, \eta v_n) = u_{n+1} \\ v_{n+1} = A(\eta v_n, \xi u_n) \leq A(\eta v_{n-1}, \xi u_{n-1}) = v_n \end{cases}$$

Thus, $u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \leq v_{n+1} \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0$. This guarantees the existence of $0 < \lambda_0 < 1$, for which $\lambda_0 v_0 \leq u_0 \leq v_0$. So, by the second condition, we get the $\xi\eta$-adjoint sequence $\{\psi_n\}$ of A with respect to λ_0, u_0, and v_0 such that $u_n \geq \lambda_0 (1+\psi_n)^n v_n$ and $\lim\limits_{n\to\infty} n\psi_n = \ln \frac{1}{\lambda_0}$. Now

$$v_n - u_n \leq v_n - \lambda_0 (1+\psi_n)^n v_n = [1 - \lambda_0 (1+\psi_n)^n] v_n \leq [1 - \lambda_0 (1+\psi_n)^n] v_0.$$

If N is the normality constant of the normal cone P,

$$\|v_n - u_n\| \leq N [1 - \lambda_0 (1+\psi_n)^n] \|v_0\|$$

The adjoint sequence $\{\psi_n\}$ is such that $\psi_n \to 0$ as $n \to \infty$. So,

$$\lambda_0 (1+\psi_n)^n = \lambda_0 \left[(1+\psi_n)^{\frac{1}{\psi_n}}\right]^{n\psi_n} \to \lambda_0 e^{^{n\psi_n}} \to \lambda_0 \frac{1}{\lambda_0} = 1$$

This shows that $\|v_n - u_n\| \to 0$ as $n \to \infty$. $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. As E is complete, u_n is non-decreasing, v_n is non-increasing, P_h is closed and $u_n \leq v_n$, there exist $u^*, v^* \in P_h$, such that $u_n \to u^*$ and $v_n \to v^*$ as $n \to \infty$. Since $u_n \leq u^* \leq v^* \leq v_n$, we must have $u^* = v^* = x^*$, say. Now,
$u_{n+1} = A(\xi u_n, \eta v_n) \leq A(\xi u^*, \eta v^*) = A(\xi x^*, \eta x^*) \leq A(\eta v_n, \xi u_n) = v_{n+1}$. As $n \to \infty$, $x^* \leq A(\xi x^*, \eta x^*) \leq x^*$, implies that $A(\xi x^*, \eta x^*) = x^*$, i.e., x^* is the $\xi\eta$-multiple fixed point of A.
We prove that this $\xi\eta$-multiple fixed point is unique.
If possible, suppose that there are two distinct $\xi\eta$-multiple fixed points, viz., x^* and y^*, in P_h. So, $A(\xi x^*, \eta x^*) = x^*$ and $A(\xi y^*, \eta y^*) = y^*$.
Let $\lambda_0 = \sup\left\{\lambda > 0 | \lambda y^* \leq x^* \leq \left(\frac{1}{\lambda}\right) y^*\right\}, 0 < \lambda_0 \leq 1$.
But if $0 < \lambda_0 < 1$, then by the given condition, there exists $\omega \geq \frac{1}{\lambda_0}$, such that $A(\lambda_0 u, \omega v) \geq \left(\lambda_0^2 \omega\right)^{\theta\left(\lambda_0^2 \omega\right)} A(u, v)$. Now,

$$\begin{aligned} x^* &= A(\xi x^*, \eta x^*) \\ &\geq A\left(\xi \lambda_0 y^*, \eta \left(\frac{1}{\lambda_0}\right) y^*\right) \\ &\geq A(\lambda_0 \xi y^*, \omega \eta y^*) \\ &\geq \left(\lambda_0^2 \omega\right)^{\theta\left(\lambda_0^2 \omega\right)} A(\xi y^*, \eta y^*) \\ &\geq \lambda_0^{\theta\left(\lambda_0^2 \omega\right)} y^*, \end{aligned}$$

and this is not possible since it gives $\lambda_0^{\theta\left(\lambda_0^2\omega\right)} > \lambda_0$, contradicting the definition of λ_0. Therefore, $\lambda_0 = 1$. But this means that $x^* = y^*$. So, the supposition is wrong and the $\xi\eta$-multiple fixed point is unique.
Now, suppose conversely that x^* is a $\xi\eta$-multiple fixed point of A. We choose that $0 < \alpha < 1$ in our initial condition for which the corresponding β is such that $\left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta,x^*\right)} x^* \geq \alpha$ and $\alpha\xi \leq \beta\eta$. Now, taking $u_0 = \alpha x^*$ and $v_0 = \beta x^*$, we get $u_0 \leq v_0$. Also,

$$\begin{aligned}
A\left(\xi u_0, \eta v_0\right) &= A\left(\alpha\xi x^*, \beta\eta x^*\right) \\
&\geq \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta,x^*\right)} A\left(\xi x^*, \eta x^*\right) \\
&= \left(\alpha^2\beta\right)^{\theta\left(\alpha^2\beta,x^*\right)} x^* \\
&\geq \alpha x^* = u_0, \\
A\left(\eta v_0, \xi u_0\right) &= A\left(\beta\eta x^*, \alpha\xi x^*\right) \\
&\leq \left(\alpha^2\beta\right)^{-\theta\left(\alpha^2\beta,x^*\right)} A\left(\eta x^*, \xi x^*\right) \\
&= \left(\alpha^2\beta\right)^{-\theta\left(\alpha^2\beta,x^*\right)} x^* \\
&\leq \beta x^* = v_0.
\end{aligned}$$

Thus the first required condition is satisfied. Now again we define $\{u_n\}$ and $\{v_n\}$ by

$$\begin{aligned}
u_n &= A\left(\xi u_{n-1}, \eta v_{n-1}\right), \\
v_n &= A\left(\eta v_{n-1}, \xi u_{n-1}\right),
\end{aligned}$$

so that as in proof of Thoerem 2.1,

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \leq v_{n+1} \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0;$$

$\{u_n\}$ and $\{v_n\}$ happen to be Cauchy sequences and hence convergent and both converge to x^*. By monotonocities of $\{u_n\}$ and $\{v_n\}$, we can find $\{\tau_n\}$ such that $\tau_n v_n \leq u_n$, so that $0 \leq \tau_n \leq 1$ and $\tau_n \to 1$. Now taking $\psi_n = \left(\frac{\tau_n}{\lambda_0}\right)^{\frac{1}{n}} - 1$, $\tau_n = \lambda_0\left(1+\psi_n\right)^n$ and $\lambda_0\left(1+\psi_n\right)^n v_n \leq u_n$, By definition, $\{\psi_n\}$ is adjoint sequence of A with respect to λ_0, u_0, and v_0. And finally,

$$\begin{aligned}
0 = \lim_{n\to\infty} \ln \tau_n &= \lim_{n\to\infty} \left(\ln\lambda_0 + \ln\left[\left(1+\psi_n\right)^{\frac{1}{\psi_n}}\right]^{n\psi_n}\right) \\
&= \lim_{n\to\infty} \left(\ln\lambda_0 + n\psi_n \ln\left[\left(1+\psi_n\right)^{\frac{1}{\psi_n}}\right]\right)
\end{aligned}$$

giving $\lim\limits_{n\to\infty} n\psi_n = \ln\frac{1}{\lambda_0}$ as required. This completes the proof of the theorem.

Applications. This paper generalizes many results of earlier works [2], [3], [4], [5], [8]. In most of our hypotheses, the entity θ is not constant, but a function; in fact, at times a very general function. The conditions used are both sufficient as well as necessary to guarantee the existence and uniqueness of the required fixed point. We use simple metric everywhere instead of more demanding Thompson or Hilbert metric. In view of all this, the previously established theorems become simple consequences of the applications of results of this paper. Further, all the results proved here are for $\xi\eta$-multiple fixed point and naturally, as the application, they also guarantee the usual fixed point under the choice of $\xi = \eta = 1$.

References

[1] Chen, Y. Z., Existence theorem of coupled fixed point. J. Math. Anal. Appl., 154 (1991), 142-150.

[2] Chen, Y. Z., Thompson's metric and mixed monotone operators. J. Math. Anal. Appl., 177 (1993), 31-37.

[3] Guo, D. J., Fixed points of mixed monotone operators with applications. Appl. Anal., 31 (1988), 215-224.

[4] Guo, D. J., Existence and uniqueness of positive fixed points for mixed monotone operators and applications, Appl. Anl., 46 (1992), 91 - 100.

[5] Guo, D. J., Lakshmikantham, V., Coupled fixed points of nonlinear operators with applications. Nonlin. Anal., TMA 11 (1987), 623 - 632.

[6] Sun, Y., A fixed point theorem for mixed monotone operators with applications. J. Math. Anal. Appl., 156 (1991), 240 - 252.

[7] Sun J. X., Liu L. S., Iterative method for coupled quasi-solution of mixed monotone operator equations. Appl. Math. Comput., 52 (1992), 301 - 308.

[8] Zandong, Liang et al., Fixed point theorems for a class of mixed monotone operators. J. Anal. Appl., Vol. 22 (3) (2003), 529 - 542.

[9] Zang, S. S., Ma Y. H., Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solution for a class of functional equations arising in dynamic programming. J. Math. Anal. Appl., 160 (1991), 468 - 479.

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