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A NOTE ON FRACTIONAL DERIVATIVES OF COLOMBEAU GENERALIZED STOCHASTIC PROCESSES

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Abstract. In this paper we consider a Caputo fractional derivative of a Colombeau generalized stochastic process G. In general, with some restrictions given on G, it is a Colombeau generalized stochastic process itself. Here we explore some other possible approaches in defining algebras of generalized fractional derivatives.

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1. Introduction

The past few decades have witnessed an increasing interest in fractional derivatives, mainly due to many applications. Fractional processes defined by using fractional calculus are convenient for describing a number of problems appearing very often in applications, especially in physics, meteorology, climatology, hydrology, geophysics, economy. For more about fractional processes we refer, for instance, to [1], [3] and [8].

Fractional derivatives of Colombeau generalized stochastic processes are introduced in [13], where it is proved that a Caputo fractional derivative of a Colombeau generalized stochastic process G is a Colombeau generalized stochastic process itself only if G satisfies certain conditions. One of the possible approaches to get rid of these restrictions is to make a regularization of the fractional derivative, as done in [13]. Here we explore some other approaches in studying fractional derivatives.

The paper is organized as follows. After the introductory part, in the second section we give some basic preliminaries such as notation and definitions of the objects we shall work with. We also introduce different spaces of Colombeau generalized stochastic processes.

In the third section we define the Caputo αth fractional derivative and the regularized Caputo αth fractional derivative of a Colombeau generalized stochastic process, for $\alpha > 0$. In this section we repeat some basic results from [13] in order to make the motivation for exploring some other possible approaches more deeply. The fourth section is devoted to a certain modification

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of the Colombeau space of fractional derivatives. Finally, in the fifth section we introduce a Colombeau fractional derivative stochastic process as one of the interesting possible approaches in studying fractional derivatives in Colombeau algebras.

2. Preliminaries

Let (Ω, Σ, μ) be a probability space. Generalized stochastic process on $\mathbb{R}$ is a weakly measurable mapping $X : \Omega \to \mathcal{D}'(\mathbb{R})$. We denote by $\mathcal{D}'_\Omega(\mathbb{R})$ the space of generalized stochastic processes. For each fixed function $\varphi \in \mathcal{D}(\mathbb{R})$, the mapping $\Omega \to \mathbb{R}$ defined by $\omega \mapsto \langle X(\omega), \varphi \rangle$ is a random variable.

White noise $\dot{W} : \Omega \to \mathcal{D}'(\mathbb{R})$ is the identity mapping $\dot{W}(\omega) = \omega$, i.e.,

$$\langle \dot{W}(\omega), \varphi \rangle = \langle \omega, \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

It is a generalized Gaussian process with mean zero and variance

$$V(\dot{W}(\varphi)) = E(\dot{W}(\varphi)^2) = \|\varphi\|^2_{L^2(\mathbb{R})},$$

where E denotes expectation. Its covariance is the bilinear functional

$$E\left(\dot{W}(\varphi)\dot{W}(\psi)\right) = \int_{\mathbb{R}} \varphi(y)\psi(y)\, dy$$

represented by Dirac's measure on the diagonal $\mathbb{R} \times \mathbb{R}$, showing the singular nature of white noise.

It we denote by W the (generalized) Brownian motion, it is well known that

$$\dot{W}(t) = \frac{d}{dt}\, W(t), \quad \text{almost surely in } \mathcal{D}'_\Omega(\mathbb{R}),$$

i.e., the white noise in $\mathbb{R}$ can be viewed as the derivative of the (generalized) Brownian motion.

A net φ_ε of mollifiers given by

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right), \quad \varphi \in \mathcal{D}(\mathbb{R}), \quad \int \varphi(t)dt = 1,$$

is called a model delta net.

Smoothed white noise process on $\mathbb{R}$ is defined as

$$\dot{W}_\varepsilon(t) = \langle \dot{W}(t), \varphi_\varepsilon(s-t) \rangle, \tag{2.1}$$

where $\dot{W}$ is the white noise process on $\mathbb{R}$ and φ_ε is a model delta net.

In the sequel we introduce Colombeau generalized stochastic processes as done in [10] and [11]. (For some other possible approaches in working with generalized stochastic processes see, e.g. [12] and [5]). We confine ourselves to the one-dimensional case.

Denote by $\mathcal{E}^{\Omega}([0,\infty))$ the space of nets $(X_\varepsilon)_{\varepsilon\in(0,1)} = (X_\varepsilon)_\varepsilon$, of stochastic processes X_ε with almost surely continuous paths, i.e., the space of nets of processes

$$X_\varepsilon : (0,1)\times[0,\infty)\times\Omega\to\mathbb{R}$$

such that

$$(t,\omega)\mapsto X_\varepsilon(t,\omega) \text{ is jointly measurable, for all } \varepsilon\in(0,1);$$
$$t\mapsto X_\varepsilon(t,\omega) \text{ belongs to } \mathcal{C}^\infty([0,\infty)), \text{ for all } \varepsilon\in(0,1) \text{ and almost all } \omega\in\Omega.$$

By $\mathcal{E}^{\Omega}_M([0,\infty))$ we denote the space of nets of processes $(X_\varepsilon)_\varepsilon\in\mathcal{E}^{\Omega}([0,\infty))$, with the property that for almost all $\omega\in\Omega$, for all $T>0$ and $\alpha\in\mathbb{N}_0$, there exist constants N, $C>0$ and $\varepsilon_0\in(0,1)$ such that $\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)|$ has a moderate bound, i.e.,

$$\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)|\le C\,\varepsilon^{-N},\quad \varepsilon\le\varepsilon_0.$$

$\mathcal{N}^{\Omega}([0,\infty))$ is the space of nets of processes $(X_\varepsilon)_\varepsilon\in\mathcal{E}^{\Omega}_M([0,\infty))$, with the property that for almost all $\omega\in\Omega$, for all $T>0$ and $\alpha\in\mathbb{N}_0$ and all $b\in\mathbb{R}$, there exist constants $C>0$ and $\varepsilon_0\in(0,1)$ such that

$$\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)|\le C\,\varepsilon^{b},\quad \varepsilon\le\varepsilon_0.$$

Then we say that $\sup_{t\in[0,T]}|\partial^\alpha X_\varepsilon(t,\omega)|$ is negligible.

Then

$$\mathcal{G}^{\Omega}([0,\infty))=\mathcal{E}^{\Omega}_M([0,\infty))/\mathcal{N}^{\Omega}([0,\infty))$$

is a differential algebra (differentiation with respect to t and pointwise multiplication) called *algebra of Colombeau generalized stochastic processes*. The elements of $\mathcal{G}^{\Omega}([0,\infty))$ will be denoted by $X=[X_\varepsilon]$, where $(X_\varepsilon)_\varepsilon$ is a representative of the class.

Both Brownian motion and white noise process can be viewed as Colombeau generalized stochastic processes. It follows from the usual imbedding arguments of Colombeau theory (see [9]). For instance, the Colombeau generalized white noise process has the representative given by (2.1).

Finally, we introduce $\mathcal{C}^k$-Colombeau generalized stochastic processes in the following way.

Denote by $\mathcal{E}^{\Omega}_{M,\mathcal{C}^k}([0,\infty))$ the space of nets of continuous processes $(X_\varepsilon)_\varepsilon$ on $[0,\infty)$, with the property that for almost all $\omega\in\Omega$ and for all $T>0$, there exist constants N, $C>0$ and $\varepsilon_0\in(0,1)$ such that $\sup_{t\in[0,T]}|\partial^m X_\varepsilon(t,\omega)|$ has a moderate bound for $m\in\{0,\dots,k\}$, $k\in\mathbb{N}$, i.e.,

$$\sup_{t\in[0,T]}|\partial^m X_\varepsilon(t,\omega)|\le C\,\varepsilon^{-N},\ \ m\in\{0,\dots,k\},\ \ k\in\mathbb{N},\ \ \varepsilon\le\varepsilon_0.$$

By $\mathcal{N}^{\Omega}_{\mathcal{C}^k}([0,\infty))$ we denote the space of nets of processes $(X_\varepsilon)_\varepsilon\in\mathcal{E}^{\Omega}_{M,\mathcal{C}^k}([0,\infty))$, with the property that for almost all $\omega\in\Omega$ and for all $T>0$ and all $b\in\mathbb{R}$, there exist constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T]} |\partial^m X_\varepsilon(t,\omega)| \le C\,\varepsilon^b, \quad m \in \{0,\ldots,k\}, \quad k \in \mathbb{N}, \quad \varepsilon \le \varepsilon_0.$$

Then we say that $\sup_{t\in[0,T]} |\partial^m X_\varepsilon(t,\omega)|$ is negligible for $m \in \{0,\ldots,k\}$, $k \in \mathbb{N}$.

Then

$$\mathcal{G}^\Omega_{\mathcal{C}^k}([0,\infty)) = \mathcal{E}^\Omega_{M,\mathcal{C}^k}([0,\infty))/\mathcal{N}^\Omega_{\mathcal{C}^k}([0,\infty))$$

is an algebra, and it is called *algebra of* $\mathcal{C}^k$*-Colombeau generalized stochastic processes.*

3. Fractional derivatives of Colombeau generalized stochastic processes

In [13] the Caputo αth fractional derivative and the regularized Caputo αth fractional derivative of a Colombeau generalized stochastic process are introduced. Here we recall definitions and some basic properties.

Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$. The Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha > 0$, is defined by

$$(3.1) \qquad {}^c_0D^\alpha_t G_\varepsilon(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{G^{(m)}_\varepsilon(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, & m-1 < \alpha < m \\ G^{(m)}_\varepsilon(t) = \dfrac{d^m}{dt^m}\, G_\varepsilon(t), & \alpha = m \end{cases},$$

for $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$.

For $m-1 < \alpha < m$, $m \in \mathbb{N}$, by using a simple change of variables one obtains

$${}^c_0D^\alpha_t G_\varepsilon(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{G^{(m)}_\varepsilon(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{G^{(m)}_\varepsilon(t-s)}{s^{\alpha+1-m}}\, ds.$$

The following proposition holds.

Proposition 3.1. *Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$ and let the Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha > 0$, be given by (3.1). Then, for every $\alpha > 0$, $\sup_{t\in[0,T]} |{}^c_0D^\alpha_t G_\varepsilon(t)|$ has a moderate bound.*

Let $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ be two different representatives of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$. Then, for every $\alpha > 0$, $\sup_{t\in[0,T]} |{}^c_0D^\alpha_t G_{1\varepsilon}(t) - {}^c_0D^\alpha_t G_{2\varepsilon}(t)|$ is negligible.

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. First, note that for $\alpha \in \mathbb{N}$, ${}^c_0D^\alpha_t G_\varepsilon(t)$ is the usual derivative of order α of $G_\varepsilon(t)$ and since $(G_\varepsilon(t))_\varepsilon \in \mathcal{E}^\Omega_M([0,\infty))$ the assertion immediately follows.

Now, consider the case when $m-1<\alpha<m$, $m\in\mathbb{N}$. Then, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} |\,{}_0^cD_t^\alpha G_\varepsilon(t)| \\
&\qquad\le \frac{1}{\Gamma(m-\alpha)} \sup_{t\in[0,T]} \int_0^t \left| \frac{G_\varepsilon^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau \right| \\
&\qquad\le \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,T]} |G_\varepsilon^{(m)}(\tau)| \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-\tau)^{\alpha+1-m}}\, d\tau \\
&\qquad= \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,T]} |G_\varepsilon^{(m)}(\tau)| \sup_{t\in[0,T]} \frac{t^{m-\alpha}}{m-\alpha}, \quad \text{since } m-1<\alpha<m \\
&\qquad\le \frac{1}{\Gamma(m-\alpha)} \frac{T^{m-\alpha}}{m-\alpha} \sup_{\tau\in[0,T]} |G_\varepsilon^{(m)}(\tau)|.
\end{aligned}$$

Since $G(t)\in\mathcal{G}^\Omega([0,\infty))$, it follows that $\sup_{\tau\in[0,T]}|G_\varepsilon^{(m)}(\tau)|$ has a moderate bound. Therefore, $\sup_{t\in[0,T]}|D^\alpha G_\varepsilon(t)|$ has a moderate bound, for every $\alpha>0$, as claimed.

In order to prove the second assertion, first, note that for $\alpha\in\mathbb{N}$, ${}_0^cD_t^\alpha G_{1\varepsilon}(t)$ and ${}_0^cD_t^\alpha G_{2\varepsilon}(t)$ are the usual derivatives of order α and since they represent the same Colombeau generalized stochastic process $G(t)\in\mathcal{G}^\Omega([0,\infty))$ it follows that $(\,{}_0^cD_t^\alpha G_{1\varepsilon}(t))_\varepsilon-(\,{}_0^cD_t^\alpha G_{2\varepsilon}(t))_\varepsilon\in\mathcal{N}^\Omega([0,\infty))$ and the assertion immediately follows.

Now, consider the case when $m-1<\alpha<m$, $m\in\mathbb{N}$. Then, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} |\,{}_0^cD_t^\alpha G_{1\varepsilon}(t) - {}_0^cD_t^\alpha G_{2\varepsilon}(t)| \\
&\qquad\le \frac{1}{\Gamma(m-\alpha)} \sup_{t\in[0,T]} \int_0^t \left| \frac{G_{1\varepsilon}^{(m)}(\tau)-G_{2\varepsilon}^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau \right| \\
&\qquad\le \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,T]} |G_{1\varepsilon}^{(m)}(\tau)-G_{2\varepsilon}^{(m)}(\tau)| \sup_{t\in[0,T]} \int_0^t \frac{d\tau}{(t-\tau)^{\alpha+1-m}} \\
&\qquad\le \frac{1}{\Gamma(m-\alpha)} \frac{T^{m-\alpha}}{m-\alpha} \sup_{\tau\in[0,T]} |G_{1\varepsilon}^{(m)}(\tau)-G_{2\varepsilon}^{(m)}(\tau)|.
\end{aligned}$$

Since $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ are both the representatives of $G(t)\in\mathcal{G}^\Omega([0,\infty))$ then $(G_{1\varepsilon}^{(m)}(t))_\varepsilon-(G_{2\varepsilon}^{(m)}(t))_\varepsilon\in\mathcal{N}^\Omega([0,\infty))$, i.e., $\sup_{\tau\in[0,T]}|G_{1\varepsilon}^{(m)}(\tau)-G_{2\varepsilon}^{(m)}(\tau)|$ is negligible. Therefore, $\sup_{t\in[0,T]}|\,{}_0^cD_t^\alpha G_{1\varepsilon}(t)-\,{}_0^cD_t^\alpha G_{2\varepsilon}(t)|$ is negligible. □

According to Proposition 3.1 the Caputo αth fractional derivative of a Colombeau generalized stochastic process on $[0,\infty)$ can be defined as an element of $\mathcal{G}^\Omega_{\mathcal{C}^0}([0,\infty))$.

Definition 3.1. *Let $G(t)\in\mathcal{G}^\Omega([0,\infty))$ be a Colombeau generalized stochastic process on $[0,\infty)$. The Caputo αth fractional derivative of $G(t)$, in notation ${}_0^cD_t^\alpha G(t)=[(\,{}_0^cD_t^\alpha G_\varepsilon(t))_\varepsilon]$, $\alpha>0$, is an element of $\mathcal{G}^\Omega_{\mathcal{C}^0}([0,\infty))$ satisfying (3.1).*

Note that, in general, the first order derivative of ${}_0^cD_t^\alpha G_\varepsilon(t)$, $\dfrac{d}{dt}\,{}_0^cD_t^\alpha G_\varepsilon(t)$, for $m-1<\alpha<m$, $m\in\mathbb{N}$, is not defined at the point $t=0$. Indeed,

$$\begin{aligned}\frac{d}{dt}\,{}_0^cD_t^\alpha G_\varepsilon(t) &= \frac{d}{dt}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{G_\varepsilon^{(m)}(t-s)}{s^{\alpha+1-m}}\,ds\right]\\ &= \frac{1}{\Gamma(m-\alpha)}\left[\int_0^t \frac{G_\varepsilon^{(m+1)}(t-s)}{s^{\alpha+1-m}}\,ds+\frac{G_\varepsilon^{(m)}(0)}{t^{\alpha+1-m}}\right]\end{aligned}$$

which is not defined in zero, unless $G_\varepsilon^{(m)}(0)=0$. In order to have the second order derivative $\dfrac{d^2}{dt^2}\,{}_0^cD_t^\alpha G_\varepsilon(t)$, $m-1<\alpha<m$, $m\in\mathbb{N}$, defined on the whole interval $[0,\infty)$, one additionally needs the condition $G_\varepsilon^{(m+1)}(0)=0$. In general, the kth order derivative $\dfrac{d^k}{dt^k}\,{}_0^cD_t^\alpha G_\varepsilon(t)$, $m-1<\alpha<m$, $m,k\in\mathbb{N}$, is defined on the whole interval $[0,\infty)$, if $G_\varepsilon^{(m+l)}(0)=0$, for all $l=0,\dots,k-1$.

The following assertion holds (for the details of the proof, see [13]).

Theorem 3.1. *Let $G(t)\in\mathcal{G}^\Omega([0,\infty))$ be a Colombeau generalized stochastic process on $[0,\infty)$. The Caputo αth fractional derivative ${}_0^cD_t^\alpha G(t)$ is a Colombeau generalized stochastic process (an element of $\mathcal{G}^\Omega([0,\infty))$) for $m-1<\alpha<m$, $m\in\mathbb{N}$, if $G_\varepsilon^{(m)}(0)=G_\varepsilon^{(m+1)}(0)=G_\varepsilon^{(m+2)}(0)=\cdots=0$.*

Moreover, if $G_\varepsilon^{(m)}(0)=0$, for every $m=1,2,\dots$, then, for every $\alpha>0$, the Caputo αth fractional derivative ${}_0^cD_t^\alpha G(t)$ is a Colombeau generalized stochastic process, i.e., an element of $\mathcal{G}^\Omega([0,\infty))$.

The previous theorem illustrates that a Caputo fractional derivative of a Colombeau generalized stochastic process $G(t)\in\mathcal{G}^\Omega([0,\infty))$ is a Colombeau generalized stochastic process itself only if G satisfies certain conditions. If one wants this to be satisfied for an arbitrary $G(t)\in\mathcal{G}^\Omega([0,\infty))$, one of the possible approaches is to make a regularization of the fractional derivative, as done in [13].

Definition 3.2. *Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t)\in\mathcal{G}^\Omega([0,\infty))$. The regularized Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha>0$, is defined by*

$$(3.2)\qquad {}_0^c\tilde{D}_t^\alpha G_\varepsilon(t)=\begin{cases}({}_0^cD_t^\alpha G_\varepsilon*\varphi_\varepsilon)(t), & m-1<\alpha<m\\ G_\varepsilon^{(m)}(t)=\dfrac{d^m}{dt^m}\,G_\varepsilon(t), & \alpha=m\end{cases},$$

for $m\in\mathbb{N}$ and $\varepsilon\in(0,1)$, where ${}_0^cD_t^\alpha G_\varepsilon(t)$ is given by (3.1) and φ_ε is a model delta net. The convolution in (3.2) is

$$({}_0^cD_t^\alpha G_\varepsilon*\varphi_\varepsilon)(t)=\int_0^\infty {}_0^cD_t^\alpha G_\varepsilon(s)\,\varphi_\varepsilon(t-s)\,ds.$$

Proposition 3.2. *([13]) Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$ and let the regularized Caputo αth fractional derivative of $(G_\varepsilon(t))_\varepsilon$, $\alpha > 0$, be given by (3.2). Then, for every $\alpha > 0$ and every $k \in \{0,1,2,\dots\}$, $\sup_{t\in[0,T]} \left| \frac{d^k}{dt^k}\, {}_0^c\tilde{D}_t^\alpha G_\varepsilon(t) \right|$ has a moderate bound.*

Let $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ be two different representatives of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$. Then, for every $\alpha > 0$ and every $k \in \{0,1,2,\dots\}$,

$$\sup_{t\in[0,T]} \left| \frac{d^k}{dt^k} \Big({}_0^c\tilde{D}_t^\alpha G_{1\varepsilon}(t) - {}_0^c\tilde{D}_t^\alpha G_{2\varepsilon}(t) \Big) \right| \quad \textit{is negligible.}$$

Definition 3.3. *Let $G(t) \in \mathcal{G}^\Omega([0,\infty))$ be a Colombeau generalized stochastic process. The regularized Caputo αth fractional derivative of $G(t)$, in the notation ${}_0^c\tilde{D}_t^\alpha G(t) = \left[\Big({}_0^c\tilde{D}_t^\alpha G_\varepsilon(t) \Big)_\varepsilon \right]$, $\alpha > 0$, is an element of $\mathcal{G}^\Omega([0,\infty))$ satisfying (3.2).*

Unlike the nonreglarized case, the regularized Caputo αth fractional derivative of a Colombeau generalized stochastic process is a Colombeau generalized stochastic process itself.

4. Modification of Colombeau space of fractional derivatives

According to Theorem 3.1, a Caputo αth fractional derivative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$ is, for $\alpha > 0$, a Colombeau generalized stochastic process itself, if all (usual) derivatives of $G_\varepsilon(t)$ are equal to zero at the point $t = 0$. As we have seen, one of the possible ways to get rid of this restriction is to make the regularization of the fractional derivative. The other possible way is to make a modification of the Colombeau space of fractional derivatives and this will be done here.

First, note that the following assertion holds.

Lemma 4.1. *Let $(G_\varepsilon(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$. Then, for every fixed $\alpha > 0$,*

$$\sup_{t\in[0,T]} \left| {}_0^cD_t^{\alpha+\alpha_0} G_\varepsilon(t) \right| \quad \textit{has a moderate bound, for every } \alpha_0 \in \mathbb{N}. \tag{4.1}$$

If $(G_{1\varepsilon}(t))_\varepsilon$ and $(G_{2\varepsilon}(t))_\varepsilon$ are two different representatives of a Colombeau generalized stochastic process $G(t) \in \mathcal{G}^\Omega([0,\infty))$, then

$$\sup_{t\in[0,T]} \left| {}_0^cD_t^{\alpha+\alpha_0} G_{1\varepsilon}(t) - {}_0^cD_t^{\alpha+\alpha_0} G_{2\varepsilon}(t) \right| \quad \textit{is negligible, for every } \alpha_0 \in \mathbb{N}. \tag{4.2}$$

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. If $\alpha \in \mathbb{N}$ then $\alpha + \alpha_0 \in \mathbb{N}$ and the assertion immediately follows.

For $m-1<\alpha<m$, $m\in\mathbb{N}$, and arbitrary $\alpha_0\in\mathbb{N}$, one has that $m-1+\alpha_0<\alpha+\alpha_0<m+\alpha_0$, $m\in\mathbb{N}$, and

$$
\begin{aligned}
&\sup_{t\in[0,T]} |\,{}_0^cD_t^{\alpha+\alpha_0}G_\varepsilon(t)| \\
&\quad\le \frac{1}{\Gamma(m+\alpha_0-\alpha-\alpha_0)} \sup_{t\in[0,T]} \int_0^t \left| \frac{G_\varepsilon^{(m)}(\tau)}{(t-\tau)^{\alpha+\alpha_0+1-m-\alpha_0}}\, d\tau \right| \\
&\quad\le \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,T]} |G_\varepsilon^{(m)}(\tau)| \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-\tau)^{\alpha+1-m}}\, d\tau \\
&\quad= \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,T]} |G_\varepsilon^{(m)}(\tau)| \sup_{t\in[0,T]} \frac{t^{m-\alpha}}{m-\alpha}, \ \text{ since } m-1<\alpha<m \\
&\quad\le \frac{1}{\Gamma(m-\alpha)} \frac{T^{m-\alpha}}{m-\alpha} \sup_{\tau\in[0,T]} |G_\varepsilon^{(m)}(\tau)|.
\end{aligned}
$$

Thus, (4.1) is satisfied. The assertion (4.2) follows from

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left|\,{}_0^cD_t^{\alpha+\alpha_0}G_{1\varepsilon}(t) - {}_0^cD_t^{\alpha+\alpha_0}G_{2\varepsilon}(t)\right| \\
&\qquad\qquad\le \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,T]} |G_{1\varepsilon}^{(m)}(\tau) - G_{2\varepsilon}^{(m)}(\tau)| \cdot \frac{T^{m-\alpha}}{m-\alpha},
\end{aligned}
$$

since, according to the assertion in Proposition 3.1, $\sup_{\tau\in[0,T]} |G_\varepsilon^{(m)}(\tau)|$ has a moderate bound and $\sup_{\tau\in[0,T]} |G_{1\varepsilon}^{(m)}(\tau) - G_{2\varepsilon}^{(m)}(\tau)|$ is negligible. □

Definition 4.1. *Let $(G(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t)\in\mathcal{G}^\Omega([0,\infty))$. The Caputo αth fractional derivative of $G(t)$, denoted by ${}_0^cD_t^\alpha G(t)$, is of Colombeau type if it satisfies (4.1).*

The assertion from Lemma 4.1 now can be written in the following way:

Theorem 4.1. *Let $G(t)\in\mathcal{G}^\Omega([0,\infty))$ be a Colombeau generalized stochastic process. Then, for every $\alpha>0$, the Caputo αth fractional derivative of $G(t)$, ${}_0^cD_t^\alpha G(t)$, is of Colombeau type.*

5. Colombeau fractional derivative stochastic processes

One of the possible approaches in studying fractional derivatives in Colombeau algebras is to define Colomebau fractional derivatives stochastic processes. We start with an appropriate definition for representatives. Namely, first we define a fractional derivative stochastic process.

Definition 5.1. *Let $(G(t))_\varepsilon$ be a representative of a Colombeau generalized stochastic process $G(t)\in\mathcal{G}^\Omega([0,\infty))$. The Caputo αth fractional derivative*

${}_0^cD_t^\alpha G_\varepsilon(t)$ *is a fractional derivative stochastic process if, for almost all* $\omega \in \Omega$, *for all* $T > 0$ *and for every* $\beta \geq 0$, *there exist constants* $N,\ C > 0$ *and* $\varepsilon_0 \in (0,1)$ *such that* $\sup_{t\in[0,T]} \left| {}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_\varepsilon(t)\right)\right|$ *has a moderate bound, i.e.,*

$$\sup_{t\in[0,T]} \left| {}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_\varepsilon(t)\right)\right| \leq C\,\varepsilon^{-N}, \quad \varepsilon \leq \varepsilon_0. \tag{5.1}$$

Theorem 5.1. *Let* $(G(t))_\varepsilon$ *be a representative of a Colombeau generalized stochastic process* $G(t) \in \mathcal{G}^\Omega([0,\infty))$ *and let* $\alpha > 0$. *If* $\alpha \notin \mathbb{N}$ *then* ${}_0^cD_t^\alpha G_\varepsilon$ *is the fractional derivative stochastic process only if* $G_\varepsilon^{(j)}(0) = 0$, *for all* $j = 1, 2, \ldots$. *For* $\alpha \in \mathbb{N}$ *this is true for any stochastic process* G.

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. In case when $\alpha \in \mathbb{N}$, the estimate (5.1) holds for every $G(t) \in \mathcal{G}^\Omega([0,\infty))$. This follows from the first part of Proposition 3.1 and the fact that, for a natural number α, the derivative ${}_0^cD_t^\alpha G(t) = \partial_t^\alpha G(t)$ is a Colombeau generalized stochastic process.

If $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $\beta \in \mathbb{N}$, then

$${}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_\varepsilon(t)\right) = \left({}_0^cD_t^\alpha G_\varepsilon\right)^{(\beta)}(t).$$

If $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $k-1 < \beta < k$, $k \in \mathbb{N}$, one has

$${}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_\varepsilon(t)\right) = \frac{1}{\Gamma(k-\beta)} \int_0^t \frac{\left({}_0^cD_t^\alpha G_\varepsilon\right)^{(k)}(\tau)}{(t-\tau)^{\beta+1-k}}\, d\tau.$$

Therefore, for $k-1 < \beta \leq k$, $k \in \mathbb{N}$, $\sup_{t\in[0,T]} \left| {}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_\varepsilon(t)\right)\right|$ has a moderate bound if $\sup_{\tau\in[0,T]} \left| \left({}_0^cD_t^\alpha G_\varepsilon\right)^{(k)}(\tau)\right|$ has a moderate bound.

This means that, in case when $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, one needs to put the same restrictions on G as in Theorem 3.1 in order to provide that (5.1) holds, i.e., that ${}_0^cD_t^\alpha G_\varepsilon(t)$ is a fractional derivative stochastic process. More precisely, the condition $G_\varepsilon^{(j)}(0) = 0$, for all $j = 1, 2, \ldots$, has to be satisfied. $\square$

Lemma 5.1. *Let* $G(t) \in \mathcal{G}^\Omega([0,\infty))$ *be a Colombeau generalized stochastic process satisfying* $G_\varepsilon^{(j)}(0) = 0$, *for all* $j = 1, 2, \ldots$ *and let* $(G_{1\varepsilon}(t))_\varepsilon$ *and* $(G_{2\varepsilon}(t))_\varepsilon$ *be two different representatives of* G. *Then, for every* $\alpha, \beta > 0$,

$$\sup_{t\in[0,T]} \left| {}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_{1\varepsilon}(t)\right) - {}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_{2\varepsilon}(t)\right)\right| \ \textit{is negligible.}$$

Proof. Fix $\omega \in \Omega$ and $\varepsilon \in (0,1)$. First, note that for $\alpha \in \mathbb{N}$, ${}_0^cD_t^\alpha G_{1\varepsilon}(t)$ and ${}_0^cD_t^\alpha G_{2\varepsilon}(t)$ are the usual derivatives of order α and thus, elements of $\mathcal{E}_M^\Omega([0,\infty))$. According to the second part of Proposition 3.1, the assertion immediately follows.

If $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and $\beta \in \mathbb{N}$ then

$${}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_{1\varepsilon}(t)\right) = \partial_t^\beta \left({}_0^cD_t^\alpha G_{1\varepsilon}(t)\right) \text{ and } {}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_{2\varepsilon}(t)\right) = \partial_t^\beta \left({}_0^cD_t^\alpha G_{1\varepsilon}(t)\right),$$

and, since $G_\varepsilon^{(j)}(0) = 0$, for all $j = 1, 2, \dots$, the derivatives from the right-hand sides are defined and have moderate for every $\beta \in \mathbb{N}$. Now it is not difficult to prove the assertion.

Finally, if $m - 1 < \alpha < m$ and $k - 1 < \beta < k$, $m, k \in \mathbb{N}$, then

$$\begin{aligned}
&\sup_{t\in[0,T]} | \,{}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_{1\varepsilon}(t)\right) - {}_0^cD_t^\beta \left({}_0^cD_t^\alpha G_{2\varepsilon}(t)\right) | \\
&= \frac{1}{\Gamma(k-\beta)} \sup_{t\in[0,T]} \int_0^t \frac{\left({}_0^cD_t^\alpha G_{1\varepsilon}\right)^{(k)}(\tau) - \left({}_0^cD_t^\alpha G_{2\varepsilon}\right)^{(k)}(\tau)}{(t-\tau)^{\beta+1-k}}\, d\tau \\
&\leq \frac{1}{\Gamma(k-\beta)} \frac{T^{k-\beta}}{k-\beta} \sup_{\tau\in[0,T]} \left| \left({}_0^cD_t^\alpha G_{1\varepsilon}\right)^{(k)}(\tau) - \left({}_0^cD_t^\alpha G_{2\varepsilon}\right)^{(k)}(\tau) \right| .
\end{aligned}$$

Since $G_\varepsilon^{(j)}(0) = 0$, for $j = 1, 2, \dots$,

$$\sup_{\tau\in[0,T]} \left| \left({}_0^cD_t^\alpha G_{1\varepsilon}\right)^{(k)}(\tau) - \left({}_0^cD_t^\alpha G_{2\varepsilon}\right)^{(k)}(\tau) \right|$$

is negligible and the assertion immediately follows. $\square$

Now, we can define the Colombeau αth fractional derivative stochastic process, as follows.

Definition 5.2. *Let $G(t) \in \mathcal{G}^\Omega([0,\infty))$ and $\alpha > 0$. We say that the Caputo αth fractional derivative ${}_0^cD_t^\alpha G$ is the Colombeau αth fractional derivative stochastic process if it satisfies (5.1).*

Thus, the following assertion holds:

Theorem 5.2. *Let $G(t) \in \mathcal{G}^\Omega([0,\infty))$ be a Colombeau generalized stochastic process and let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, the Caputo αth fractional derivative ${}_0^cD_t^\alpha G(t)$ is the Colombeau αth fractional derivative stochastic process in case when $G_\varepsilon^{(j)}(0) = 0$, for all $j = 1, 2, \dots$. For $\alpha \in \mathbb{N}$ this is satisfied for any stochastic process $G(t)$.*

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