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SOME CHARACTERIZATIONS OF INCLINED CURVES IN EUCLIDEAN E^n SPACE

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Abstract. We consider a unit speed curve α in the Euclidean n-dimensional space $\mathbf{E}^n$ and denote the Frenet frame of α by $\{\mathbf{V}_1, \ldots, \mathbf{V}_n\}$. We say that α is a cylindrical helix if its tangent vector $\mathbf{V}_1$ makes a constant angle with a fixed direction U. In this work we give different characterizations of such curves in terms of their curvatures.

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1. Introduction and statement of results

An helix in the Euclidean 3-space $\mathbf{E}^3$ is a curve where the tangent lines make a constant angle with a fixed direction. An helix curve is characterized by the fact that the ratio κ/τ is constant along the curve, where κ and τ are the curvature and the torsion of α, respectively. Helices are well known curves in classical differential geometry of space curves [4] and we refer to the reader for recent works on this type of curves [2, 7]. Recently, Magden [3] have introduced the concept of cylindrical helix in the Euclidean 4-space $\mathbf{E}^4$, saying that the tangent lines make a constant angle with a fixed directions. He characterizes a cylindrical helix in $\mathbf{E}^4$ if and only if the function

$$\Big(\frac{\kappa_1}{\kappa_2}\Big)^2 + \Big(\frac{1}{\kappa_3}\Big(\frac{\kappa_1}{\kappa_2}\Big)'\Big)^2 \tag{1}$$

is constant along the curve, where κ_3 and κ_4 are the third and the fourth curvature of the the curve. See also [5].

In this work we consider the generalization of the concept of general helices in the Euclidean n-space $\mathbf{E}^n$. Let $\alpha : I \subset \mathbb{R} \to \mathbf{E}^n$ be an arbitrary curve in $\mathbf{E}^n$. Recall that the curve α is said to be of unit speed (or parameterized by

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the arc-length function s) if $\langle \alpha'(s), \alpha'(s)\rangle = 1$, where $\langle , \rangle$ is the standard scalar product in the Euclidean space $\mathbf{E}^n$ given by

$$\langle X, Y\rangle = \sum_{i=1}^{n} x_i\, y_i,$$

for each $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n) \in \mathbf{E}^n$.

Let $\{\mathbf{V}_1(s), \ldots, \mathbf{V}_n(s)\}$ be the moving frame along α, where the vectors $\mathbf{V}_i$ are mutually orthogonal vectors satisfying $\langle \mathbf{V}_i, \mathbf{V}_i\rangle = 1$. The Frenet equations for α are given by ([2])

$$\begin{bmatrix} \mathbf{V}_1' \\ \mathbf{V}_2' \\ \mathbf{V}_3' \\ \vdots \\ \mathbf{V}_{n-1}' \\ \mathbf{V}_n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \cdots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \cdots & 0 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \kappa_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_{n-1} \\ \mathbf{V}_n \end{bmatrix}. \tag{2}$$

Recall that the functions $\kappa_i(s)$ are called the i-th curvatures of α. If $\kappa_{n-1}(s) = 0$ for any $s \in I$, then $\mathbf{V}_n(s)$ is a constant vector V and the curve α lies in an $(n-1)$-dimensional affine subspace orthogonal to V, which is isometric to the Euclidean $(n-1)$-space $\mathbf{E}^{n-1}$. We will assume throughout this work that all the curvatures satisfy $\kappa_i(s) \neq 0$ for any $s \in I$, $1 \leq i \leq n-1$.

Definition 1.1. *A unit speed curve $\alpha : I \to \mathbf{E}^n$ is called cylindrical helix if its tangent vector $\mathbf{V}_1$ makes a constant angle with a fixed direction U.*

Our main result in this work is the following characterization of cylindrical helices in the Euclidean n-space $\mathbf{E}^n$.

Theorem 1.2. *Let $\alpha : I \to \mathbf{E}^n$ be a unit speed curve in $\mathbf{E}^n$. Define the functions*

$$G_1 = 1,\ \ G_2 = 0,\ \ G_i = \frac{1}{\kappa_{i-1}}\Big[\kappa_{i-2}G_{i-2} + G_{i-1}'\Big],\ \ 3 \leq i \leq n. \tag{3}$$

Then α is a cylindrical helix if and only if the function

$$\sum_{i=3}^{n} G_i^2 = C \tag{4}$$

is constant. Moreover, the constant $C = \tan^2\theta$, θ being the angle that makes $\mathbf{V}_1$ with the fixed direction U that determines α.

This theorem generalizes in arbitrary dimensions what happens for $n = 3$ and $n = 4$, namely: if $n = 3$, (4) writes $G_3^2 = \kappa_1/\kappa_2 = \kappa/\tau$ and for $n = 4$, (4) agrees with (1).

2. Proof of Theorem 1.2

Let α be a unit speed curve in $\mathbf{E}^n$. Assume that α is a cylindrical helix curve. Let U be the direction with which $\mathbf{V}_1$ makes a constant angle θ and, without loss of generality, we suppose that $\langle U, U\rangle = 1$. Consider the differentiable functions a_i, $1 \leq i \leq n$,

$$U = \sum_{i=1}^{n} a_i(s)\,\mathbf{V}_i(s), \quad s \in I, \tag{5}$$

that is,

$$a_i = \langle \mathbf{V}_i, U\rangle,\ 1 \leq i \leq n.$$

Then the function $a_1(s) = \langle \mathbf{V}_1(s), U\rangle$ is constant, and it agrees with $\cos\theta$:

$$a_1(s) = \langle \mathbf{V}_1, U\rangle = \cos\theta \tag{6}$$

for any s. By differentiating (6) with respect to s and using the Frenet formula (2) we have

$$a_1'(s) = \kappa_1 \langle \mathbf{V}_2, U\rangle = \kappa_1\, a_2 = 0.$$

Then $a_2 = 0$ and therefore U lies in the subspace $Sp(\mathbf{V}_1, \mathbf{V}_3, \ldots, \mathbf{V}_n)$. Because the vector field U is constant, a differentiation in (5), together with (2) gives the following ordinary differential equation system

$$\left.\begin{array}{ll} \kappa_1 a_1 - \kappa_2 a_3 & = 0 \\ a_3' - \kappa_3 a_4 & = 0 \\ a_4' + \kappa_3 a_3 - \kappa_4 a_5 & = 0 \\ \vdots & \\ a_{n-1}' + \kappa_{n-2} a_{n-2} - \kappa_{n-1} a_n & = 0 \\ a_n' + \kappa_{n-1} a_{n-1} & = 0 \end{array}\right\} \tag{7}$$

Define the functions $G_i = G_i(s)$ as follows

$$a_i(s) = G_i(s)\, a_1,\ 3 \leq i \leq n. \tag{8}$$

We point out that $a_1 \neq 0$: on the contrary, (7) gives $a_i = 0$, for $3 \leq i \leq n$ and so, $U = 0$: contradiction. The first $(n-2)$-equations in (7) lead to

$$\left.\begin{array}{l} G_3 = \dfrac{\kappa_1}{\kappa_2} \\ G_4 = \dfrac{1}{\kappa_3} G_3' \\ G_5 = \dfrac{1}{\kappa_4}\left[\kappa_3 G_3 + G_4'\right] \\ \vdots \\ G_{n-1} = \dfrac{1}{\kappa_{n-2}}\left[\kappa_{n-3} G_{n-3} + G_{n-2}'\right] \\ G_n = \dfrac{1}{\kappa_{n-1}}\left[\kappa_{n-2} G_{n-2} + G_{n-1}'\right] \end{array}\right\} \tag{9}$$

The last equation of (7) leads to the following condition;

$$G'_n + \kappa_{n-1}G_{n-1} = 0. \tag{10}$$

We do the change of variables:

$$t(s) = \int^s \kappa_{n-1}(u)du, \qquad \frac{dt}{ds} = \kappa_{n-1}(s).$$

In particular, from the last equation of (9), we have

$$G'_{n-1}(t) = G_n(t) - \Big(\frac{\kappa_{n-2}(t)}{\kappa_{n-1}(t)}\Big)G_{n-2}(t).$$

As a consequence, if α is a cylindrical helix, substituting the equation (10) in the last equation yields

$$G''_n(t) + G_n(t) = \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)}.$$

The general solution of this equation is

$$\begin{aligned} G_n(t) =& \Big(A - \int \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)} \sin t\, dt\Big) \cos t \\ &+ \Big(B + \int \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)} \cos t\, dt\Big) \sin t, \end{aligned} \tag{11}$$

where A and B are arbitrary constants. Then (11) takes the following form

$$\begin{aligned} G_n(s) =& \Big(A - \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1}(s)ds\Big] ds\Big) \cos \int \kappa_{n-1}(s)ds \\ &+ \Big(B + \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1}(s)ds\Big] ds\Big) \sin \int \kappa_{n-1}(s)ds. \end{aligned} \tag{12}$$

From (10), the function G_{n-1} is given by

$$\begin{aligned} G_{n-1}(s) =& \Big(A - \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1}(s)ds\Big] ds\Big) \sin \int \kappa_{n-1}(s)ds \\ &- \Big(B + \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1}(s)ds\Big] ds\Big) \cos \int \kappa_{n-1}(s)ds. \end{aligned} \tag{13}$$

From equation (9), we have

$$\begin{aligned} \sum_{i=3}^{n-2} G_iG'_i &= G_3\kappa_3G_4 + G_4\Big(\kappa_4G_5 - \kappa_3G_3\Big) + \dots \\ &+ G_{n-3}\Big(\kappa_{n-3}G_{n-2} - \kappa_{n-4}G_{n-4}\Big) + G_{n-2}G'_{n-2} \\ &= G_{n-2}\Big(G'_{n-2} + \kappa_{n-3}G_{n-3}\Big) \\ &= \kappa_{n-2}G_{n-2}G_{n-1} \end{aligned}$$

Substituting (13) in the above equation and integrate it, we have

$$\begin{aligned}\sum_{i=3}^{n-2} G_i^2 &= C - \Big(A - \int \Big[\kappa_{n-2}(s)G_{n-2}(s)\sin\int\kappa_{n-1}ds\Big]ds\Big)^2 \\ &\quad -\Big(B + \int \Big[\kappa_{n-2}(s)G_{n-2}(s)\cos\int\kappa_{n-1}ds\Big]ds\Big)^2,\end{aligned} \tag{14}$$

where C is a constant of integration. From Equations (12) and (13), we have

$$\begin{aligned}G_n^2 + G_{n-1}^2 &= \Big(A - \int \Big[\kappa_{n-2}(s)G_{n-2}(s)\sin\int\kappa_{n-1}ds\Big]ds\Big)^2 \\ &\quad +\Big(B + \int \Big[\kappa_{n-2}(s)G_{n-2}(s)\cos\int\kappa_{n-1}ds\Big]ds\Big)^2,\end{aligned} \tag{15}$$

It follows from (14) and (15) that

$$\sum_{i=3}^{n} G_i^2 = C.$$

Moreover, the constant C is calculated as follows. From (8), together with the $(n-2)$-equations (9), we have

$$C = \sum_{i=3}^{n} G_i^2 = \frac{1}{a_1^2}\sum_{i=3}^{n} a_i^2 = \frac{1-a_1^2}{a_1^2} = \tan^2\theta,$$

where we have used (2) and the fact that U is a unit vector field.

We do the converse of Theorem. Assume that the condition (9) is satisfied for a curve α. Let $\theta \in \mathbb{R}$ be so that $C = \tan^2\theta$. Define the unit vector U by

$$U = \cos\theta\Big[\mathbf{V}_1 + \sum_{i=3}^{n} G_i\,\mathbf{V}_i\Big].$$

By taking into account (9), a differentiation of U gives that $\dfrac{dU}{ds} = 0$, which means that U is a constant vector field. On the other hand, the scalar product between the unit tangent vector field $\mathbf{V}_1$ with U is

$$\langle \mathbf{V}_1(s), U\rangle = \cos\theta.$$

Thus α is a cylindrical helix curve. This finishes the proof of Theorem 1.2.

As a direct consequence of the proof, we generalize Theorem 1.2 in Minkowski space and for timelike curves.

Theorem 2.1. *Let $\mathbf{E}_1^n$ be the Minkowski n-dimensional space and let $\alpha : I \to \mathbf{E}_1^n$ be a unit speed timelike curve. Then α is a cylindrical helix if and only if the function $\sum_{i=3}^{n} G_i^2$ is constant, where the functions G_i are defined as in (3).*

Proof. The proof is carried the same steps as above, and we omit the details. We only point out that the fact that α is timelike means that $\mathbf{V}_1(s) = \alpha'(s)$ is a timelike vector field. The other V_i in the Frenet frame, $2 \le i \le n$, are unit spacelike vectors, and the second equation in (2) changes to $\mathbf{V}_2' = \kappa_1\mathbf{V}_1 + \kappa_2\mathbf{V}_3$ ([1, 6]). □

3. Further characterizations of cylindrical helices

In this section we present new characterizations of a cylindrical helix in $\mathbf{E}^n$. The first one is a consequence of Theorem 1.2.

Theorem 3.1. *Let $\alpha : I \subset R \to \mathbf{E}^n$ be a unit speed curve in the Euclidean space $\mathbf{E}^n$. Then α is a cylindrical helix if and only if there exists a C^2-function $G_n(s)$ such that*

$$G_n = \frac{1}{\kappa_{n-1}}\Big[\kappa_{n-2}G_{n-2} + G'_{n-1}\Big], \quad \frac{dG_n}{ds} = -\kappa_{n-1}(s)G_{n-1}(s), \tag{16}$$

where

$$G_1 = 1, G_2 = 0, G_i = \frac{1}{\kappa_{i-1}}\Big[\kappa_{i-2}G_{i-2} + G'_{i-1}\Big], \ 3 \leq i \leq n-1.$$

Proof. Let now assume that α is a cylindrical helix. By using Theorem 1.2 and by the differentiation of the (constant) function given in (4), we obtain

$$\begin{aligned} 0 &= \sum_{i=3}^{n} G_i\, G'_i \\ &= G_3\kappa_3G_4 + G_4\Big(\kappa_4G_5 - \kappa_3G_3\Big) + \dots \\ &\quad \dots + G_{n-1}\Big(\kappa_{n-1}G_n - \kappa_{n-2}G_{n-2}\Big) + G_nG'_n \\ &= G_n\Big(G'_n + \kappa_{n-1}G_{n-1}\Big). \end{aligned}$$

This shows (16). Conversely, if (16) holds, we define a vector field U by

$$U = \cos\theta\Big[\mathbf{V}_1 + \sum_{i=3}^{n} G_i\,\mathbf{V}_i\Big].$$

By the Frenet equations (2), $\frac{dU}{ds} = 0$, and so, U is constant. On the other hand, $\langle \mathbf{V}_1(s), U\rangle = \cos\theta$ is constant, and this means that α is a cylindrical helix. □

At the end, we give an integral characterization of a cylindrical helix.

Theorem 3.2. *Let $\alpha : I \subset R \to \mathbf{E}^n$ be a unit speed curve in the Euclidean space $\mathbf{E}^n$. Then α is a cylindrical helix if and only if the following condition is satisfied*

$$\begin{aligned} G_{n-1}(s) = \ &\Big(A - \int \Big[\kappa_{n-2}G_{n-2}\sin\int\kappa_{n-1}ds\Big]ds\Big)\sin\int^s\kappa_{n-1}(u)du \\ &-\Big(B + \int \Big[\kappa_{n-2}G_{n-2}\cos\int\kappa_{n-1}ds\Big]ds\Big)\cos\int^s\kappa_{n-1}(u)du. \end{aligned} \tag{17}$$

for some constants A and B.

Proof. Suppose that α is a cylindrical helix. By using Theorem 3.1, let define $m(s)$ and $n(s)$ by

$$\phi(s) = \int^s \kappa_{n-1}(u)du,$$

$$(18)\qquad \begin{aligned} m(s) = &\quad G_n(s)\cos\phi + G_{n-1}(s)\sin\phi + \textstyle\int \kappa_{n-2}G_{n-2}\sin\phi\, ds,\\ n(s) = &\quad G_n(s)\sin\phi - G_{n-1}(s)\cos\phi - \textstyle\int \kappa_{n-2}G_{n-2}\cos\phi\, ds. \end{aligned}$$

If we differentiate equations (18) with respect to s, and taking into account of (17) and (16), we obtain $\dfrac{dm}{ds} = 0$ and $\dfrac{dn}{ds} = 0$. Therefore, there exist constants A and B such that $m(s) = A$ and $n(s) = B$. By substituting into (18) and solving the resulting equations for $G_{n-1}(s)$, we get

$$G_{n-1}(s) = \Big(A - \int \kappa_{n-2}G_{n-2}\sin\phi\, ds\Big)\sin\phi - \Big(B + \int \kappa_{n-2}G_{n-2}\cos\phi\, ds\Big)\cos\phi.$$

Conversely, suppose that (17) holds. In order to apply Theorem 3.1, we define $G_n(s)$ by

$$G_n(s) = \Big(A - \int \kappa_{n-2}G_{n-2}\sin\phi\, ds\Big)\cos\phi + \Big(B + \int \kappa_{n-2}G_{n-2}\cos\phi\, ds\Big)\sin\phi.$$

with $\phi(s) = \int^s \kappa_{n-1}(u)du$. A direct differentiation of (17) gives

$$G'_{n-1} = \kappa_{n-1}G_n - \kappa_{n-2}G_{n-2}.$$

This shows the left condition in (16). Moreover, a straightforward computation leads to $G'_n(s) = -\kappa_{n-1}G_{n-1}$, which finishes the proof. □

We end this section with a characterization of cylindrical helices only in terms of the curvatures of α. From the definitions of G_i in (3), one can express the functions G_i in terms of G_3 and the curvatures of α as follows:

$$(19)\qquad G_j = \textstyle\sum_{i=0}^{j-3} A_{ji}G_3^{(i)},\ \ 3 \le j \le n,$$

where

$$G_3^{(i)} = \frac{d^{(i)}G_3}{ds^i},\ \ G_3^{(0)} = G_3 = \frac{\kappa_1}{\kappa_2}.$$

Then

$$\begin{aligned} &G_4 = \kappa_3^{-1}G'_3 = A_{41}G'_3 + A_{40}G_3, A_{41} = \kappa_3^{-1}, A_{40} = 0\\ &G_5 = A_{52}G''_3 + A_{51}G'_3 + A_{50}G_3, A_{52} = \kappa_4^{-1}A_{41}, A_{51} = \kappa_4^{-1}A'_{41}, A_{50} = \kappa_4^{-1}\kappa_3 \end{aligned}$$

and so on. Define the functions $A_{ji} = A_{ij}(s)$, $3 \le j$, $0 \le i \le j-3$ as the following:

$$A_{30} = 1, A_{40} = 0$$

$$A_{j0} = \kappa_{j-1}^{-1}\kappa_{j-2}A_{(j-2)0} + \kappa_{j-1}^{-1}A'_{(j-1)0},\ 5 \le j \le n$$

$$
\begin{aligned}
A_{j(j-3)} &= \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}...\kappa_4^{-1}\kappa_3^{-1}, \text{for } 4 \leq j \leq n \\
A_{j(j-4)} &= \kappa_{j-1}^{-1}\Big(\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}\ldots\kappa_4^{-1}\kappa_3^{-1}\Big)' + \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\Big(\kappa_{j-3}^{-1}\ldots\kappa_4^{-1}\kappa_3^{-1}\Big)' \\
&\quad + \ldots + \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}\ldots\kappa_4^{-1}\Big(\kappa_3^{-1}\Big)', \\
&\qquad \text{for } 5 \leq j \leq n \\
A_{ji} &= \kappa_{j-1}^{-1}\kappa_{j-2}A_{(j-2)i} + \kappa_{j-1}^{-1}\Big(A'_{(j-1)i} + A_{(j-1)(i-1)}\Big) \\
&\qquad \text{for } 1 \leq i \leq j-5, 6 \leq j \leq n
\end{aligned}
$$

and $A_{ji} = 0$ otherwise.

The second equation of (16) leads the following condition:

$$
\begin{aligned}
A_{n(n-3)}G_3^{(n-2)} &+\Big(A'_{n(n-3)} + A_{n(n-4)}\Big)G_3^{(n-3)} \\
&+\textstyle\sum_{i=1}^{n-4}\Big[A'_{ni} + A_{n(i-1)} + \kappa_{n-1}A_{(n-1)i}\Big]G_3^{(i)} \\
&+\Big(A'_{n0} + \kappa_{n-1}A_{(n-1)0}\Big)G_3 = 0, \quad n \geq 3.
\end{aligned}
\tag{20}
$$

As a consequence of (20) and Theorem 1.2, we have the following corollary.

Corollary 3.3. *Let $\alpha : I \to \mathbf{E}^n$ be a unit speed curve in $\mathbf{E}^n$. The next statements are equivalent:*

1. *α is a cylindrical helix.*

2. $$
\begin{aligned}
0 &= A_{n(n-3)}\Big(\frac{\kappa_1}{\kappa_2}\Big)^{(n-2)} + \Big(A'_{n(n-3)} + A_{n(n-4)}\Big)\Big(\frac{\kappa_1}{\kappa_2}\Big)^{(n-3)} \\
&+\textstyle\sum_{i=1}^{n-4}\Big[A'_{ni} + A_{n(i-1)} + \kappa_{n-1}A_{(n-1)i}\Big]\Big(\frac{\kappa_1}{\kappa_2}\Big)^{(i)} \\
&+\Big(A'_{n0} + \kappa_{n-1}A_{(n-1)0}\Big)\Big(\frac{\kappa_1}{\kappa_2}\Big), \qquad n \geq 3.
\end{aligned}
$$

3. *The function*

$$\textstyle\sum_{j=3}^{n}\sum_{i=0}^{j-3}\sum_{k=0}^{j-3} A_{ji}A_{jk}\Big(\frac{\kappa_1}{\kappa_2}\Big)^{(i)}\Big(\frac{\kappa_1}{\kappa_2}\Big)^{(k)} = C$$

is constant, $j - i \geq 3$, $j - k \geq 3$.

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