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# FIXED POINT THEOREMS FOR A CLASS OF $A$-CONTRACTIONS ON A 2-METRIC SPACE

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**Abstract.** M.Akram et al. ([1],[2]) have introduced a larger class of mappings called $A$-contraction, which is a proper superclass of Kannan's [7], Bianchini's [3] and Reich's [8] type contractions. In the present paper, we have proved some fixed point theorems for $A$-contraction mappings in a 2-metric space.

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## 1. Introduction and Preliminaries.

The concept of 2-metric spaces has been initiated by Gähler ([4],[5]) and these spaces have subsequently been studied by many authors like Iseki [6], Rhoades [9], Saha and Dey [10], investigating the existence of fixed point and common fixed point for various contractive mappings. Gähler [4] defined 2-metric space as follows:

Let $X$ be a non-empty set. A real valued function $d$ on $X \times X \times X$ is said to be a 2-metric on $X$, if

(I) given distinct elements $x$,$y$ of $X$, there exists an element $z$ of $X$ such that $d(x, y, z) \neq 0$

(II) $d(x, y, z) = 0$ when at least two of $x, y, z$ are equal,

(III) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z$ in $X$, and

(IV) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w$ in $X$.

When $d$ is a 2-metric on $X$, then the ordered pair $(X, d)$ is called a 2-metric space.

A sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if for each $a \in X$, $\lim d(x_n, x_m, a) = 0$ as $n, m \to \infty$.

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A sequence $\{x_n\}$ in X is said to be convergent to an element $x \in X$, if for each $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$

A 2-metric space $X$ is said to be complete, if every Cauchy sequence in $X$ is convergent to an element of X.

On the otherhand, Akram et al. ([1], [2]) defined $A$-contractions as follows: Let a nonempty set $A$ consisting of all functions $\alpha : R_+^3 \to R_+$ satisfying

(i) $\alpha$ is continuous on the set $R_+^3$ of all triplets of nonnegative reals (with respect to the Euclidean metric on $R^3$).

(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$, for all $a, b$.

**Definition 1.1.** A self-map $T$ on a metric space $X$ is said to be $A$-contraction, if it satisfies the condition

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in A$.

Using the notion of $A$-contraction, we are now going to prove the following main results in a setting of 2-metric space.

## 2. Main Results

Before stating our first main result, we formulate the following analogue of $A$-contractions for 2-metric space as follows.

**Definition 2.1.** A self-map $T$ on a 2-metric space $X$ is said to be $A$-contraction, if for each $u \in X$,

$$d(Tx, Ty, u) \leq \alpha(d(x, y, u), d(x, Tx, u), d(y, Ty, u)) \quad \text{holds}$$

for all $x, y \in X$ and for some $\alpha \in A$.

An important fixed point result can be obtained through this analogue of $A$-contraction in 2-metric space as follows.

**Theorem 2.1.** *Let $(X, d)$ be a complete 2-metric space and let $T : X \to X$ be an $A$-contraction. Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0$ be an arbitrary element of $X$ and consider the sequence $\{x_n\}$ of iterates $x_n = T^n x_0$; $n = 1, 2, .....$ Also, we note that $x_{n+1} = T^{n+1}x_0 = T^n(Tx_0) = T^n x_1$ and $x_{n+1} = T(T^n x_0) = Tx_n$. Now

$$\begin{aligned} d(x_1, x_2, u) &= d\left(Tx_0, T^2x_0, u\right) \\ &= d(Tx_0, T(Tx_0), u) \\ &\leq \alpha\left(d(x_0, Tx_0, u), d(x_0, Tx_0, u), d\left(Tx_0, T^2x_0, u\right)\right) \\ &= \alpha(d(x_0, x_1, u), d(x_0, x_1, u), d(x_1, x_2, u)) \end{aligned}$$

implies

$$d(x_1, x_2, u) \leq kd(x_0, x_1, u) \tag{2.1}$$

for some $k \in [0, 1)$, because $\alpha \in A$. Again

$$\begin{aligned} d(x_2, x_3, u) &= d\left(T^2x_0, T^3x_0, u\right) \\ &= d\left(T(Tx_0), T(T^2x_0), u\right) \\ &\leq \alpha\left(d\left(Tx_0, T^2x_0, u\right), d\left(Tx_0, T^2x_0, u\right),\right. \\ &\quad \left. d\left(T^2x_0, T^3x_0, u\right)\right) \\ &= \alpha\left(d(x_1, x_2, u), d(x_1, x_2, u), d(x_2, x_3, u)\right) \\ &\leq kd(x_1, x_2, u) \\ &\leq k^2 d(x_0, x_1, u) \quad \text{by (2.1)} \end{aligned}$$

Proceeding in this way, we get

$$d(x_n, x_{n+1}, u) \leq k^n d(x_0, x_1, u). \tag{2.2}$$

Next

$$\begin{aligned} d(x_n, x_{n+2}, u) &\leq d(x_n, x_{n+2}, x_{n+1}) + d(x_n, x_{n+1}, u) + d(x_{n+1}, x_{n+2}, u) \\ &\leq d(x_n, x_{n+2}, x_{n+1}) + \sum_{r=0}^{1} d(x_{n+r}, x_{n+r+1}, u) \end{aligned} \tag{2.3}$$

Now

$$\begin{aligned} d(x_n, x_{n+2}, x_{n+1}) &= d(x_{n+1}, x_{n+2}, x_n) \\ &= d\left(T^{n+1}x_0, T^{n+2}x_0, x_n\right) \\ &= d\left(T(T^n x_0), T(T^n x_1), x_n\right) \\ &\leq \alpha\left(d(T^n x_0, T^n x_1, x_n), d\left(T^n x_0, T^{n+1}x_0, x_n\right),\right. \\ &\quad \left. d\left(T^n x_1, T^{n+1}x_1, x_n\right)\right) \\ &= \alpha\left(d(x_n, x_{n+1}, x_n), d(x_n, x_{n+1}, x_n),\right. \\ &\quad \left. d(x_{n+1}, x_{n+2}, x_n)\right) \\ &\leq kd(x_n, x_{n+1}, x_n) \end{aligned}$$

So it follows that,

$$d(x_n, x_{n+2}, x_{n+1}) = 0. \tag{2.4}$$

So from (2.3) and (2.4) we get,

$$d(x_n, x_{n+2}, u) \leq \sum_{r=0}^{1} d(x_{n+r}, x_{n+r+1}, u) \tag{2.5}$$

Again, by repeated use of property (IV) in the definition of 2-metric space, we get,

$$d(x_n, x_{n+3}, u) \leq \sum_{r=0}^{1} d(x_{n+3}, x_{n+r}, x_{n+r+1}) + \sum_{r=0}^{2} d(x_{n+r}, x_{n+r+1}, u)$$

Similarly, we can show that $d(x_{n+3}, x_n, x_{n+1}) = 0$ and $d(x_{n+3}, x_{n+1}, x_{n+2}) = 0$. Hence $d(x_n, x_{n+3}, u) \leq \sum_{r=0}^{2} d(x_{n+r}, x_{n+r+1}, u)$. Proceeding in the same manner, we get for any integer $p > 0$,

$$d(x_n, x_{n+p}, u) \leq \sum_{r=0}^{p-1} d(x_{n+r}, x_{n+r+1}, u).$$

So by (2.2), we have for any integer $p > 0$,

$$d(x_n, x_{n+p}, u) \leq \frac{k^n}{1-k} d(x_0, x_1, u) \to 0 \quad \text{as } n \to \infty, \text{ since } k \in [0, 1).$$

Hence $\{x_n\}$ is a Cauchy sequence in $X$ and so by completeness of $X$, $\{x_n\}$ converges to a point $z \in X$. Again

$$\begin{aligned} d(x_{n+1}, Tz, u) &= d(T(T^n x_0), Tz, u) \\ &\leq \alpha\left(d(T^n x_0, z, u), d\left(T^n x_0, T^{n+1} x_0, u\right), d(z, Tz, u)\right) \\ &= \alpha\left(d(x_n, z, u), d(x_n, x_{n+1}, u), d(z, Tz, u)\right) \end{aligned}$$

Taking limit as $n \to \infty$, we get,

$$d(z, Tz, u) \leq \alpha\left(d(z, z, u), d(z, z, u), d(z, Tz, u)\right) \leq kd(z, z, u) = 0$$

implying that $Tz = z$.

To prove the uniqueness of $z$, let $w$ be another fixed point of $T$. Then

$$\begin{aligned} d(z, w, u) &= d(Tz, Tw, u) \\ &\leq \alpha\left(d(z, w, u), d(z, Tz, u), d(w, Tw, u)\right) \\ &= \alpha\left(d(z, w, u), d(z, z, u), d(w, w, u)\right) \\ &= \alpha\left(d(z, w, u), 0, 0\right) \\ &\leq k.0 \\ &= 0, \end{aligned}$$

which gives $z = w$ and thus the uniqueness is proved. □

**Remark.** If the 2-metric space is not complete and the mapping is not an $A$-contraction, then there is no guarantee to have a fixed point for the mapping. To support our contention, we cite an example.

**Example 2.1.** Let $X = \{1, 2, 3, 4\}$ be a finite set with a function $d : X \times X \times X \to R$ defined as follows

$d(x, y, z) = 0;$ if at least any two of $x, y, z$ are equal.

$d(x, y, z) = d(y, x, z) = d(z, y, x)$ for $x \neq y \neq z$ be such that
$d(1, 2, 3) = 6,\ d(1, 2, 4) = 7,\ d(1, 3, 4) = 8,\ d(2, 3, 4) = 9$

Clearly, $(X, d)$ is an incomplete 2-metric space. Next we define $T : X \to X$ by $T(1) = 2,\ T(2) = 3,\ T(3) = 4,\ T(4) = 1$
Take $x = 1,\ y = 2,\ u = 4$. Then $d(T(1), T(2), 4) = d(2, 3, 4) = 9 = d(2, T(2), 4)$ and $d(1, 2, 4) = 7 = d(1, T(1), 4)$.
Now $d(T(1), T(2), 4) \leq \alpha(d(1, 2, 4), d(1, T(1), 4), d(2, T(2), 4))$ implies $d(2, 3, 4) \leq \alpha(d(1, 2, 4), d(1, 2, 4), d(2, 3, 4))$, but $d(2, 3, 4) \leq kd(1, 2, 4)$ implies $9 \leq k.7$, which is impossible as $k \in [0, 1)$. So $T$ is not an $A$-contraction. Also, it is very clear that $T$ has no fixed point in $X$.

**Corollary 2.1.** *Let $(X, d)$ be a complete 2-metric space and let $T : X \to X$ be such that there exists an integer $n$ and some $\alpha' \in A$,*

$$d(T^n x, T^n y, u) \leq \alpha'(d(x, y, u), d(x, T^n x, u), d(y, T^n y, u)) \quad \textit{holds}$$

*for all $x, y, u \in X$. Then $T$ has a unique fixed point.*

*Proof.* Let us take $S = T^n$. Then by Theorem 2.1, $S$ has a unique fixed point and so $T^n$ has a unique fixed point. Let $x_0$ be a unique fixed point of $T^n$. So $T^n x_0 = x_0$. We have to prove that $x_0$ is also a unique fixed point of $T$. Since $T^n(Tx_0) = T(T^n x_0) = Tx_0$, therefore $Tx_0$ is a fixed point of $T^n$. If $Tx_0 \neq x_0$, then it is a contradiction to the fact that $x_0$ is a unique fixed point of $T^n$. So $Tx_0 = x_0$. □

**Theorem 2.2.** *Let $(X, d)$ be a complete 2-metric space and let $T, S : X \to X$ be such that*

$$d(Tx, Sy, u) \leq \alpha'(d(x, y, u), d(x, Tx, u), d(y, Sy, u)) \quad \textit{holds}$$

*for all $x, y, u \in X$ and for some $\alpha' \in A$. Then there exists a unique common fixed point of $S$ and $T$.*

*Proof.* Let $x_0 \in X$ and define $x_{2n+1} = Tx_{2n},\ x_{2n+2} = Sx_{2n+1}$. Then

$$\begin{aligned}
& d(x_{2n+1}, x_{2n+2}, u) \\
&= d(Tx_{2n}, Sx_{2n+1}, u) \\
&\leq \alpha'(d(x_{2n}, x_{2n+1}, u), d(x_{2n}, Tx_{2n}, u), d(x_{2n+1}, Sx_{2n+1}, u)) \\
&= \alpha'(d(x_{2n}, x_{2n+1}, u), d(x_{2n}, x_{2n+1}, u), d(x_{2n+1}, x_{2n+2}, u)) \\
&\leq kd(x_{2n}, x_{2n+1}, u)
\end{aligned}$$

for some $k \in [0,1)$ as $\alpha^{'} \in A$. Similarly, $d(x_{2n}, x_{2n+1}, u) \leq kd(x_{2n-1}, x_{2n}, u)$ and so $d(x_{2n+1}, x_{2n+2}, u) \leq k^2 d(x_{2n-1}, x_{2n}, u)$. Then for arbitrary $n$, $d(x_n, x_{n+1}, u) \leq k^n d(x_0, x_1, u)$. Proceeding in a similar manner, used in the proof of Theorem 2.1, we claim that $\{x_n\}$ is a Cauchy sequence. Then by the completeness of $X$, $\{x_n\}$ converges to a point $z \in X$. Now

$$\begin{aligned} d(z, Tz, u) &\leq d(z, Tz, x_{2n+2}) + d(z, x_{2n+2}, u) + d(x_{2n+2}, Tz, u) \\ &= d(z, Tz, x_{2n+2}) + d(z, x_{2n+2}, u) + d(Sx_{2n+1}, Tz, u) \\ &\leq d(z, Tz, x_{2n+2}) + d(z, x_{2n+2}, u) + \alpha^{'}(d(x_{2n+1}, z, u), \\ &\quad d(x_{2n+1}, x_{2n+2}, u), d(z, Tz, u)) \end{aligned}$$

Taking limt as $n \to \infty$ on both sides of the inequality, we get

$$d(z, Tz, u) \leq \alpha^{'}(0, 0, d(z, Tz, u))$$

implying $Tz = z$. Similarly, we can show that $Sz = z$. So, $z$ is a common fixed point and uniqueness of $z$ is also very clear. □

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## References

[1] Akram, M., Siddiqui, A. A., A fixed point theorem for $A$-contractions on a class of generalised metric spaces. Korean J. Math. Sciences, 10(2) (2003), 1-5.

[2] Akram, M., Zafar, A. A., Siddiqui, A. A., A general class of contractions: $A$-contractions. Novi Sad J. Math., 38(1) (2008), 25-33.

[3] Bianchini, R., Su un problema di S.Reich riguardante la teori dei punt i fissi. Boll. Un. Math. Ital., 5 (1972), 103-108.

[4] Gähler, S., 2-metric Raume and ihre topologische strucktur. Math. Nachr., 26 (1963), 115-148.

[5] Gähler, S., Uber die unifromisieberkeit 2-metrischer Raume. Math. Nachr., 28 (1965), 235 - 244.

[6] Iseki, K., Fixed point theorems in 2-metric space. Math. Seminar. Notes, Kobe Univ., 3 (1975), 133 - 136.

[7] Kannan, R., Some results on fixed points-II. Amer. Math. Monthly, 76(45) (1969), 405-408.

[8] Reich, S., Kannan's fixed point theorem. Boll. Un. Math. Ital., 4 (1971), 1-11.

[9] Rhoades, B.E., Contractive type mappings on a 2-metric space, Math. Nachr., 91 (1979), 151 - 155.

[10] Saha, M., Dey, D., On the theory of fixed points of contractive type mappings in a 2-metric space. Int. Journal of Math. Analysis, 3(6) (2009), 283 - 293.

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