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PROBABILISTIC PROOF OF A FIXED POINT THEOREM IN K - CONVEX LINEAR TOPOLOGICAL SPACES

In this paper we shall prove a fixed point theorem in non-archimedean probabilistic locally convex spaces which is analogous to the theorem in [5] for probabilistic locally convex spaces.

First, we shall give some definitions which will be used in the sequel.

Let S be a linear space over a non-archimedean complete field K and for every i in the index set I consider a function $\mathcal{F}^i : S \to \Delta^+$ where Δ^+ is the family of distribution functions F such that $F(0)=0$.

Definition *S is called a non-archimedean probabilistic locally convex spaces if for every $i \in I$ the following conditions are satisfied: ($\mathcal{F}^i(x)$ will be denoted by F^i_x)*

1. $F^i_0 = H$ *where the function H is defined by* $H(\varepsilon) = \begin{cases} 0 & \varepsilon \leqslant 0 \\ 1 & \varepsilon > 0 \end{cases}$

2. $F^i_{\lambda x}(\varepsilon) = F^i_x\left(\dfrac{\varepsilon}{|\lambda|}\right)$ *for every* $\lambda \in K, \lambda \neq 0$, *every* $x \in S$ *and every* $\varepsilon > 0$.

3. $F^i_{x+y}(\varepsilon) \geqslant t\,(F^i_x(\varepsilon),\ F^i_y(\varepsilon))$ *for every* $x, y \in S$ *and every* $\varepsilon > 0$ *where* t *is the so called t-norm* [3].

If we suppose that $t(a, b) = \min\{a, b\}$ it is easy to prove that S becomes a locally K-convex linear topological space [2] if $\mathcal{N} = \{N^i(\varepsilon, \xi)\}$ $i \in I, \varepsilon > 0, \xi \varepsilon (0, 1)$ is the neighborhood system at 0 where $N^i(\varepsilon, \xi) = \{x \mid F^i_x(\epsilon) > 1 - \xi\}$. Also S is a Hausdorff K-convex linear topological space if $\{F^i_x = H$ for every $i \in I\} \Leftrightarrow \{x = 0\}$.

Theorem. *Let $(S, \mathcal{F}, \min)$ be a sequentially complete Hausdorff non-archimedean probabilistic locally convex space and T be a mapping of S into S so that the following conditions are satisfied:*

1. *For every $i \in I$ there exist $q(i) > 0$ and $f(i) \in I$ such that for every $\varepsilon > 0$, $x \in S$ and $y \in S$ the following inequality holds:*

$$F^i_{T^{n(x)}x - T^{n(x)}y}\quad (q(i)\,\varepsilon) \geqslant F^{f(i)}_{x-y}(\varepsilon)$$

where $n(x)$ is a natural number which depends on x.

2. *For every $i \in I$ there exist $n(i) \in N$ and $Q(i) \in (0, 1)$* so that:

$$q(f^n(i)) \leqslant Q(i) \text{ for } every\ n \geqslant n(i)$$

3. *There exists* $x_0 \in S$ *such that for every* $\varepsilon > 0$ *and* $i \in I$:

$$\lim_{p\to\infty} F_{T^k x_0 - x_0}^{f^n(i)}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1$$

uniformly in respect to $n = 0, 1, 2, \ldots$ *for every* $k = 1, 2, \ldots, n(x_0)$.
Then there exists one and only one solution x *of the equation* $x = Tx$ *which also satisfies the condition:*

$$\lim_{p\to\infty} F_{x-x_0}^{f^n(x)}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1 \tag{1}$$

for every $i \in I$, $\varepsilon > 0$ *uniformly in respect to* $n = 0, 1, 2, \ldots$.

Proof: As in [5] we shall prove that for every $i \in I$ and $\varepsilon > 0$:

$$\lim_{p\to\infty} F_{T^m x_0 - x_0}^{f^n(i)}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1 \tag{2}$$

uniformly in respect to $m = 1, 2, \ldots$ and $n \geqslant n(i)$. Let us suppose that $sn(x_0) < m \leqslant (s+1)\, n(x_0)$ where $s \in N$. Then we have:

$$F_{T^m x_0 - x_0}^{f^n(i)}(\varepsilon) \geqslant \min\{F_{T^m x_0 - T^{n(x_0)} x_0}^{f^n}(\varepsilon),\ F_{T^{n(x_0)} x_0 - x_0}^{f^n(i)}(\varepsilon)\} =$$

$$= \min\{F_{T^{m-n(x_0)} T^{n(x_0)} x_0 - T^{n(x_0)} x_0}^{f^n(i)}(\varepsilon),\ F_{T^{n(x_0)} x_0 - x_0}^{f^n(i)}(\varepsilon)\} \geqslant$$

$$\geqslant \min\left\{F_{T^{m-n(x_0)} x_0 - x_0}^{f^{n+1}(i)}\left(\frac{\varepsilon}{q(f^n(i))}\right),\ F_{T^{n(x_0)} x_0 - x_0}^{f^n(i)}\right\} \geqslant$$

$$\geqslant \min\left\{F_{T^{m-2n(x_0)} x_0 - x_0}^{f^{n+2}(i)}\left(\frac{\varepsilon}{q(f^n(i))\, q(f^{n+1}(i))}\right),\ F_{T^{n(x_0)} x_0 - x_0}^{f^{n+1}(i)}\left(\frac{\varepsilon}{q(f^n(i))}\right),\right.$$

$$\left.,\ F_{T^{n(x_0)} x_0 - x_0}^{f^n(i)}(\varepsilon)\right\} \geqslant \ldots \geqslant \min\left\{F_{T^{m-ns(x_0)} x_0 - x_0}^{f^{n+s}(i)}\left(\frac{\varepsilon}{\prod_{r=n}^{n+s-1} q(f^r(i))}\right),\right.$$

$$\left. F_{T^{n(x_0)} x_0 - x_0}^{f^{n+s-1}(i)}\left(\frac{\varepsilon}{\prod_{r=n}^{n+s-2} q(f^r(i))}\right), \ldots,\ F_{T^{n(x_0)} x_0 - x_0}^{f^n(n)}(\varepsilon)\right\}.$$

Because of $\prod_{r=n}^{n+s-t} q(f^r(i)) < 1$ for every $t = 1, 2, \ldots, s$ and every $n \geqslant n(i)$ we have:

$$F_{T^m x_0 - x_0}^{f^n(i)}(\varepsilon) \geqslant \min\{F_{T^{m-sn(x_0)} x_0 - x_0}^{f^{n+s}(i)}(\varepsilon),\ F_{T^{n(x_0)} x_0 - x_0}^{f^{n+s-1}(i)}(\varepsilon), \ldots,\ F_{T^{n(x_0)} x_0 - x_0}^{f^n(i)}(\varepsilon)\}$$

and so:

$$F_{T^m x_0 - x_0}^{f^n(i)}\left(\frac{\varepsilon}{Q(i)^p}\right) \geqslant \min\left\{F_{T^{m-sn(x_0)} x_0 - x_0}^{f^{n+s}(i)}\left(\frac{\varepsilon}{Q(i)^p}\right),\right.$$

$$\left. F_{T^{n(x_0)} x_0 - x_0}^{f^{n+s-1}(i)}\left(\frac{\varepsilon}{Q(i)^p}\right), \ldots,\ F_{T^{n(x_0)} x_0 - x_0}^{f^n(i)}\left(\frac{\varepsilon}{Q(i)^p}\right)\right\}.$$

From the condition 3. of the Theorem it follows that there exists $P(i, \varepsilon, \xi)$ for every $i \in I$, $\varepsilon>0$ and $\xi \in (0, 1)$ such that:

$$F^{f^n(i)}_{T^k x_0 - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) > 1-\xi$$

for every $p \geqslant P(i, \varepsilon, \xi)$, every $n=0, 1, 2, \ldots$ and every $k=1, 2, \ldots, n(x_0)$ and so:

$$F^{f^n(i)}_{T^m x_0 - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) > 1-\xi \text{ for every } p \geqslant P(i, \varepsilon, \xi) \text{ every } m \geqslant n(x_0) \text{ and every } n \geqslant n(i)$$

From this we conclude that the relation (2) holds uniformly in respect to $n \geqslant n(i)$ and $m=1, 2, \ldots,$

As in [5] we define the sequence $\{x_n\}$ in the following way:

$m_0=n(x_0)$, $x_1=T^{m_0}x_0$, $m_i=n(x_i)$, $x_{i+1}=T^{m_i}x_i$. Then we have:

$$F^i_{x_{n+1}-x_n}(\varepsilon)=F^i_{T^{m_n}x_n - T^{m_{n-1}}x_{n-1}}(\varepsilon)=F^i_{T^{m_n}-T^{m_n}x_{n-1}-T^{m_{n-1}}x_{n-1}}(\varepsilon)\geqslant$$

$$\geqslant F^{f(i)}_{T^{m_n}x_{n-1}-x_{n-1}}\left(\frac{\varepsilon}{q(i)}\right)\geqslant F^{f^2(i)}_{T^{m_n}x_{n-2}-x_{n-2}}\left(\frac{\varepsilon}{q(i)\,q(f(i))}\right)\geqslant$$

$$\geqslant F^{f^n(i)}_{T^{m_n}x_0-x_0}\left(\frac{\varepsilon}{\prod\limits_{r=0}^{n-1} q(f^r(i))}\right)\geqslant F^{f^n(i)}_{T^{m_n}x_0-x_0}\left(\frac{Q(i)^{n(i)}}{\prod\limits_{r=0}^{n(i)-1} q(f^r(i))}\,\frac{1}{Q(i)^n}\right)$$

and:

$$F^i_{x_{n+p}-x_n}(\varepsilon)\geqslant \min\left\{F^i_{x_{n+p}-x_{n+p-1}}(\varepsilon), \ldots, F^i_{x_{n+1}-x_n}(\varepsilon)\right\}\geqslant$$

$$\geqslant \min\left\{F^{f^{n+p-1}(i)}_{T^{m_{n+p-1}}x_0-x_0}\left(\frac{\varepsilon(i)}{Q(i)^{n+p-1}}\right) F^{f^{n+p-2}(i)}_{T^{m_{n+p-2}}x_0-x_0}\left(\frac{\varepsilon(i)}{Q(i)^{n+p-2}}\right), \ldots,\right.$$

$$\left. F^{f^n(i)}_{T^{m_n}x_0-x_0}\left(\frac{\varepsilon(i)}{Q(i)^n}\right)\right\}; \quad \varepsilon(i)=\frac{Q(i)^{n(i)}\varepsilon}{\prod\limits_{r=0}^{n(i)-1} q(f^r(i))}.$$

Using the condition (2) we conclude that for every $i \in I$, $\varepsilon>0$ and $\xi \in (0, 1)$ there exists $N(i, \varepsilon, \xi)$ such that:

$$F^i_{x_{n+p}-x_n}(\varepsilon)>1-\xi \text{ for every } n \geqslant N(i, \varepsilon, \xi) \text{ and every } p=1, 2, \ldots$$

i. e. the sequence $\{x_n\}$ is a Cauchy sequence.

The space $(S, \mathcal{F}, \min)$ is sequentially complete and so there exists $x=\lim\limits_{n\to\infty} x_n$. From $F^i_{Tx_n-x_n}(\varepsilon)\geqslant F^{f^n(i)}_{Tx_0-x_0}\left(\frac{\varepsilon}{\prod\limits_{s=0}^{n-1} q(f^s(i))}\right)$ it follows that $\lim\limits_{n\to\infty} Tx_n-x_n=0$ and so $Tx=x$.

Next, we shall show that:

$$\lim_{p\to\infty} F^{f^n(i)}_{x-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)=1 \text{ holds}$$

uniformly in respect to $n=0, 1, 2, \ldots$. For every $n\geqslant n(i)$ we have:

$$F^{f^n(i)}_{x_s-x_{s-1}}(\varepsilon)\geqslant F^{f^{s-1}(f^n(i))}_{T^{m_{s-1}}x_0-x_0}\left(\frac{\varepsilon}{\prod_{r=0}^{s-2} q(f^r(f^n)(i))}\right)\geqslant F^{f^{n+s-1}(i)}_{T^{m_{s-1}}x_0-x_0}(\varepsilon)$$

and from:

$$F^{f^n(i)}_{x_0-x_0}(\varepsilon)\geqslant \min\left\{F^{fu(i)}_{x_s-x_{s-1}}(\varepsilon), F^{f^n(i)}_{x_{s-1}-x_{s-2}}(\varepsilon), \ldots, F^{f^n(i)}_{x_1-x_0}(\varepsilon)\right\}$$

it follows:

$$F^{f^n(i)}_{x_s-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)\geqslant \min\left\{F^{f^{n+s-0}(i)_0}_{T^{m_{s-1}}x_0-x}\left(\frac{\varepsilon}{Q(i)^p}\right), F^{f^{n+s-2}(i)}_{T^{m_{s-9}}x_0-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right), \ldots, F^{f^n(i)}_{x_1-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)\right\}.$$

Because of (2) it follows that for every $i\in I$, $\varepsilon>0$ and $\xi\in(0, 1)$ there exists $P(i, \varepsilon, \xi)$ such that:

$$F^{f^n(i)}_{x_s-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)>1-\xi \tag{3}$$

for every $p\geqslant P(i, \varepsilon, \xi)$, every $n\geqslant n(i)+1$ and every $s=0, 1, 2, \ldots$.

From:

$$F^{f^n(i)}_{x-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)\geqslant \min\left\{F^{f^n(i)}_{x-x_s}\left(\frac{\varepsilon}{Q(i)^p}\right), F^{f^n(i)}_{x_s-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)\right\}$$

and (3) we obtain that there exists $P'(i, \varepsilon, \xi)$ such that:

$$F^{f^n(i)}_{x-x_0}\left(\frac{\varepsilon}{Q(i)^p}\right)>1-\xi \text{ for every } p\geqslant P'(i, \varepsilon, \xi)$$

and every $n=0, 1, 2, \ldots$.

From $x=Tx$ and $y=Ty$ it follows:

$$F^{i}_{x-y}(\varepsilon)=F^{i}_{T^{n(x)}x-T^{n(x)}y}(\varepsilon)\geqslant F^{f(i)}_{x-y}\left(\frac{\varepsilon}{q(i)}\right)\geqslant \ldots \geqslant F^{f^n(i)}_{x-y}\left(\frac{\varepsilon}{\prod_{r=0}^{n-1} q(f^r(i))}\right)$$

$$\geqslant \min\left\{F^{f^n(i)}_{x-x_0}\left(\frac{\varepsilon}{\prod_{r=0}^{n-1} q(f^r(i))}\right), F^{f^n(i)}_{y-x_0}\left(\frac{\varepsilon}{\prod_{r=0}^{n-1} q(f^r(i))}\right)\right\}$$

and so there exists one and only one solution of the equation $x=Tx$ which satisfies the condition (1).

As in [5] it can be shown that $x = \lim_{n\to\infty} T^n x_0$ which concludes the proof of the theorem.

Remark: Suppose that M is a closed subset of S such that:

$$\sup_{\varepsilon} \inf_{x,y\varepsilon M} F^{i}_{x-y}(\varepsilon) = 1 \text{ for every } i \in I \tag{4}$$

and that T is a continuous mapping from M into M. If for every $i \in I$ there exists $g(i)$ such that:

$$F_x^{f^n(i)}(\varepsilon) \geqslant F_x^{g(i)}(\varepsilon) \text{ for every } x \in S,\ n \in N \text{ and } \varepsilon > 0$$

it is easy to show that t can be any continuous T-norm. This follows from:

$$F^{i}_{x_{n+p}-x_n}(\varepsilon) \geqslant F^{f^n(i)}_{\sum\limits_{Tr=n}^{n+p-1} n(x_r)\, x_0 - x_0}\left(\frac{\varepsilon(i)}{Q(i)^n}\right) \geqslant F^{g(i)}_{\sum\limits_{Tr=n}^{n+p-1} n(x_r)\, x_0 - x_0}\left(\frac{\varepsilon(i)}{Q(i)^n}\right) \geqslant$$

$$\geqslant \inf_{x,y\varepsilon M} F^{g(i)}_{x-y}\left(\frac{\varepsilon(i)}{Q(i)^n}\right)$$

and if we use (4) we conclude that $\{x_n\}$ is a Cauchy sequence. In this case $(S, \mathcal{F}, t)$ is a non-archimedean linear topological space.

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PROBABILISTIČKI DOKAZ JEDNE TEOREME O NEPOKRETNOJ TAČKI U K-KONVEKSNIM LINEARNIM TOPOLOŠKIM PROSTORIMA

Rezime

U ovom radu je dokazana sledeća teorema:

Neka je $(S, \mathcal{F}, \min)$ *sekvencijalno kompletan Hauzdorfov nearhimedovski probabilistički lokalno konveksan prostor i* T *preslikavanje* S *u* S, *tako da su zadovoljeni sledeći uslovi:*

1. *Za svako* $i \in I$ *postoii* $q(i) > 0$ *i* $f(i) \in I$ *tako da je za svako* $\varepsilon > 0$, $x \in S$ i $y \in S$ *zadovoljena sledeća nejednakost:*

$$F^{i}_{T^{n(x)}x - T^{n(x)}y}(q(i)\varepsilon) \geqslant F^{f(i)}_{x-y}(\varepsilon),$$

gde je $n(x)$ *prirodan broj koji zavisi od* x.

2. *Za svako $i \in I$ postoji $n(i) \in N$ i $Q i) \in (0, 1)$ tako da je:*

$$q(f^n(i)) \leqslant Q(i) \text{ za svako } \varepsilon \geqslant n(i).$$

3. *Postoji $x_0 \in S$ tako da je za svako $\varepsilon > 0$ i $i \in I$:*

$$\lim_{p \to \infty} F^{f^n(i)}_{T^k x_0 - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1$$

uniformno u odnosu na $n = 0, 1, 2, \ldots$. za svako $k = 1, 2, \ldots, n(x_0)$. Tada postoji jedno i samo jedno rešenje x jednačine $x = Tx$ koje takođe zadovoljava i uslov:

$$\lim_{p \to \infty} F^{f^n(i)}_{x - x_0}\left(\frac{\varepsilon}{Q(i)^p}\right) = 1,$$

za svako $i \in I$, $\varepsilon > 0$ uniformno u odnosu na $n = 0, 1, 2, \ldots$