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ON THE CONVERGENCE OF THE SERIES OF RATIONAL OPERATORS

Abstract. *Our topic is the convergence of one class of the series of rational operators in the field M of Mikusiński operators. Using Ditkin's result* [2] *which connects the operators with Laplace transformations and following the ideas of Erdélyi,* [3] *we will give the representation in the field M of the following convergent series of rational operators*

$$\left(1+\frac{x}{ks+p-x}\right)^{\nu}=\sum_{n=0}^{\infty}\frac{(\nu)_n x^n}{n!\,(ks+p)^n} \qquad (Re\,\nu>0)$$

where p is an arbitrary complex number and $k>0$.

Series of rational operators constitute an important class of the operators in the field M of Mikusiński's operators.

Proposition 1. *Suppose that the following conditions are satisfied.*

$$\text{(i)}\ \ a_n=0\,(n^k) \qquad (n\to\infty)$$

for some positive integer k.

(ii) *Let the real functions* $b_n(x)$ *and the positive continuous function* $b(x)$ *be such that*

$$b_n(x)=b_n\,n^{-1}\,b(x) \qquad (n=1, 2, \ldots\ \alpha\leqslant x\leqslant\beta)$$

for some positive sequence b_n.

Then the series

$$(1) \qquad \sum_{n=1}^{\infty}\frac{a_n\,n!\,b_1 b_2\ldots b_n}{(b_1(x)\,s+b_1)\,(b_2(x)\,s+b_2)\ldots(b_n(x)\,s+b_n)}$$

is operationally convergent in the interval $[\alpha, \beta]$ *and defines the continuous operational function in this interval.*

Proof. Multiplying series (1) by $l=\frac{1}{s}\in C$, we get

(2) $$\sum_{n=1}^{\infty}\frac{a_n\, n!\, b_1\ldots b_n}{s\,(b_1(x)\,s+b_1)\,(b_2(x)\,s+b_2)\ldots(b_n(x)\,s+b_n)}=$$

$$\sum_{n=1}^{\infty}a_n\left[\frac{1}{s}-\binom{n}{1}\frac{1}{s+(b(x))^{-1}}+\ldots+(-1)^n\frac{1}{s+n\,(b(x))^{-1}}\right]=$$

$$=\left\{\sum_{n=1}^{\infty}a_n\left(1-exp\left(-\frac{t}{b(x)}\right)\right)^n\right\}.$$

If (i) holds the last series converges uniformly in $o\leqslant t\leqslant T$, $\alpha\leqslant x\leqslant\beta$ since we have

$$o<1-exp\left(-\frac{t}{b(x)}\right)\leqslant q(T)<1,\quad \forall x\in[\alpha,\beta]\quad\text{and}$$

$$\frac{n^k}{\Gamma(k+1)}\sim\binom{n+k}{n}\quad(n\leftarrow\infty)$$

namely, there exists $n_o\in N$ such that

$$(n>n_o)\Rightarrow\left(|a_n|\left(1-exp\left(-\frac{t}{b(x)}\right)\right)^n\leqslant Kn^k\left(1-exp\left(-\frac{t}{b(x)}\right)\right)^n\right.$$

$$\leqslant M\binom{n+k}{n}\left(1-exp\left(\frac{t}{b(x)}\right)\right)^n$$

$$\left.\leqslant M\binom{n+k}{n}(q(T))^n\right)$$

and the series

$$\sum_{n=0}^{\infty}\binom{n+k}{n}q^n(T)=\frac{1}{(1-q(T))^{k+1}}$$

converges.

This means that the series (2) converges almost uniformly in $o\leqslant t<\infty$, $\alpha\leqslant x\leqslant\beta$, and the series (1) defines the continuous operational function in $\alpha\leqslant x\leqslant\beta$.

Proposition 2. *If a sequence of positive number* $b_1, b_2,\ldots$ *satisfies conditions*

(3) $$\sum_{n=1}^{\infty}\frac{1}{b_n}=\infty$$

(4) $$b_{n+1}-b_n>\delta>0\quad(n=1,2,\ldots)$$

and $a_n(x)$ is a sequence of real, continuous and positive functions in the interval $I=$ $=[\alpha, \beta]$ then the series

$$\sum_{n=1}^{\infty} \frac{1}{b_n^k (a_n(x)\, s + b_n)} \qquad (k>0) \tag{5}$$

is operationally convergent in the interval I and defines the continuous operational function in I.

Proof. Multiplying series (5) by $l=\frac{1}{s}$ we obtain

$$\sum_{n=1}^{\infty} l \frac{1}{b_n^k (a_n(x)s+b_n)} = \sum_{n=1}^{\infty} l \left\{ \frac{1}{b_n^k\, a_n(x)}\, exp\left(-\frac{b_n\, t}{a_n(x)}\right)\right\}$$

$$=\left\{\sum_{n=1}^{\infty} \frac{1}{b_n^{k+1}}\left(1-exp\left(-\frac{b_n t}{a_n(x)}\right)\right)\right\}.$$

Since

$$\{f_n(x, t)\}=\left\{\frac{1}{b_n^{k+1}}\left(1-exp\left(-\frac{b_n}{a_n(x)}\, t\right)\right)\right\}$$

is a parametric function for $x \in I$, $0 \leqslant t < \infty$ and

$$|f_n(x, t)| < \frac{1}{b_n^{k+1}}$$

it follows the statement of Proposition 2.

Remark. Condition (4) is a stronger one than

$$\sum_{n=1}^{\infty} \frac{1}{b_n^{k+1}} < \infty \quad (k>0) \tag{4.1}$$

obviously (4) implies (4.1), but not conversely.

The consequence of the Proposition 2 are

(A) *If* (3) *and* (4) *hold, the series*

$$\sum_{n=1}^{\infty} \frac{1}{b_n^k (s+b_n)} \tag{5.a}$$

converges operationally for every positive k.

(B) *If* (3) *and* (4) *hold, the series*

$$\sum_{n=1}^{\infty} \frac{1}{s+b_n} \tag{5.b}$$

diverges operationally.

Namely,

$$\frac{1}{s+b_n} = \frac{1}{b_n} - \frac{s}{b_n(s+b_n)}$$

and the statement (B) follows from (3) and (A).

Proposition 3. *If* (3) *and* (4) *hold, the series*

$$\sum_{n=1}^{\infty} b_n \frac{1}{s-b_n} \tag{6}$$

is not convergent in the field M.

Proof. Indeed, if (6) was convergent, there would exist a function $f \in C$ ($f \neq 0$) such that the series

$$f \sum_{n=1}^{\infty} \frac{b_n}{s-b_n} = \sum_{n=1}^{\infty} \left\{ b_n \int_0^t e^{b_n u} f(t-u)\, du \right\}$$

would be uniformly convergent in every interval $0 \leqslant t \leqslant T$.

Then the sequence

$$\int_0 e^{b_n u} f(T-u)\, du$$

would be bounded for all $n=1, 2, \ldots$ Hence it would follow by the theorem on moment [4] (see ch. VII § 7) that $f(T-u)=0$ for $0 \leqslant u \leqslant T$ i. e. that $f(t)=0$ for $0 \leqslant t \leqslant T$. Since T can be fixed arbitrarily $f(t)=0$ for all $t \geqslant 0$, which contradicts $f \neq 0$.

Similarly, if (3) and (4) hold and coefficients a_n are arbitrary real numbers such that $a_n \to \infty$, $n \to \infty$, then the series

$$\sum_{n=1}^{\infty} \frac{a_n}{s-b_n}$$

is not convergent operationally.

Following the ideas of Erdelyi [3] we know that the operator

$$\frac{s^{a-b}}{(s-x)^a} = \left\{ \frac{t^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b; xt) \right\} \quad (Re\, b > 0)$$

where ${}_1F_1(a, b; z)$ denotes the confluent hypergeometric function. As usual ${}_1F_1(a, b; z)$ i defined by the series

$$ {}_1F_1(a, b; z)=\sum_{n=0}^{\infty}\frac{(a)_n z^n}{(b)_n n!}, \quad (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}. $$

Using the following known relation [1] (see. p. 272)

$$ \sum_{n\geqslant 0}\gamma_n t^{n+c-1}{}_1F_1(a, c+n; t)= $$

$$ =\frac{1}{\Gamma(c-\nu)}\int_0^t e^{ux}u^{\nu-1}(t-u)^{c-\nu-1}{}_1F_1(a, c-\nu; t-u)\,du. \quad Re\, c>Re\,\nu>0 $$

where $\gamma_n=\dfrac{\Gamma(\nu+n)\,x^n}{n!\,\Gamma(c+n)}$

we obtain

$$ \sum_{n=0}^{\infty}\frac{\Gamma(\nu+n)\,x^n}{n!}\frac{s^{a-c-n}}{(s-1)^a}=\frac{\Gamma(\nu)}{(s-x)^\nu}\frac{s^{a-c+\nu}}{(s-1)^a} $$

or

(7)
$$ \sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\frac{x^n}{s^n}=\left(\frac{s}{s-x}\right)^{\nu}. $$

Since the convergence radius R of series

$$ \sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}x_n $$

is a positive, $R=1$, then the series (7) regarded as a series of two variables x and t is uniformly convergent in every domain

$$ 0\leqslant x\leqslant x_0,\ 0\leqslant t\leqslant T $$

namely, the series (7) is operationally convergent.

By means of the operator transformation T^{-p} where p may be an arbitrary complex number, we can easily deduce from (7) that is

(8)
$$ \sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\frac{x^n}{(s+p)^n}=\left(\frac{s+p}{s+p-x}\right)^{\nu} $$

$$ =\left(1+\frac{x}{s+p-x}\right)^{\nu} $$

since

$$T^{-p}\sum_{n=0}^{\infty}\frac{(\nu)_n x^n}{n!\, s^n}=\sum_{n=0}^{\infty}\frac{(\nu)_n x^n}{n!\,(s+p)^n}$$

Applying the operator transformation U_k ($k>0$) [4] to formula (8) we obtain more general formula

$$\sum_{n\leq 0}^{\infty} U_k\frac{(\nu)_n x^n}{n!\,(s+p)^n}=\sum_{n\geq 0}^{\infty}\frac{(\nu)_n x^n}{n!\,(k s+p)^n}=\left(\frac{sk+p}{sk+p-x}\right)^{\nu} \tag{9}$$

$$=\left(1+\frac{x}{sk+p-x}\right)^{\nu}$$

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O KONVERGENCIJI REDOVA RACIONALNIH OPERATORA

Rezime

Ispitana je konvergencija u polju operatora Mikusińskog jedne klase redova čiji su opšti članovi racionalni operatori po operatoru diferenciranja s. Koeficijenti b_n tih redova zadovoljavaju sledeće uslove

$$\sum_{n=1}^{\infty}\frac{1}{b_n}=\infty \tag{1}$$

$$b_{n+1}-b_n>\delta>0 \quad (n=1,\ 2,\ldots). \tag{2}$$

Dokazano je da u polju operatora M redovi

$$\sum_{n=1}^{\infty} \frac{1}{s+b_n} \quad \text{i} \quad \sum_{n=1}^{\infty} \frac{b_n}{s-b_n}$$

divergiraju.

Sem toga, koristeći se rezultatima Ditkina [2], koji povezuje operatore sa Laplasovim transformacijama, a prema Erdelyiu [3], data je u polju operatora M reprezentacija sledećeg konvergentnog reda racionalnih operatora

$$\left(1+\frac{x}{ks+p-x}\right)^{\nu} = \sum_{n=0}^{\infty} \frac{(\nu)_n \, x^n}{n!\,(ks-p)^n} \qquad (Re\,\nu>0)$$

gde je p proizvoljan kompleksan broj i $k>0$.