Zbornik radova Prirodno-matematičkog fakulteta — Univerzitet u Novom Sadu, knjiga 7 (1977)
Review of Research of Science — University of Novi Sad, Volume 7 (1977)

Olga Hadžić

A FIXED POINT THEOREM IN RANDOM NORMED SPACES

In his paper: Statistical Metrics, Proc. Acad. Sci. USA, 28, 1942, 535—537 K. Menger has generalized the notion of metric space attaching to each pair (p, q) in the cartesian product SxS a distribution function $F_{p, q}$. Thus was obtained the notion of probabilistic metric space (briefly PM-space). The notion of random normed space was introduced by A. N. Šerstnev [13] and in this paper we shall prove a fixed point theorem for mapping $\Phi z=(H(x, y), K(x, y))$, $z=(x, y)$, where S_1 and S_2 are random normed space, $U \subset S_1$, $V \subset S_2$, $H: Ux\, V \to U$ and $K: Ux\, V \to V$.

Important contributions in the study of PM-spaces are due to Wald [5], B. Schweizer and A. Sklar [3], O. Onicescu [6] and in [1], [2], [7] and [8] are given some fixed point theorems for contraction mapping in *PM*-spaces. In [9] the notion of the Kuratowski function is introduced and based on it in [11] are defined the functions probabilistic densifying and α-contractive and some fixed point theorems for such functions are given.

First, we shall give some definitions and theorems to be used in the sequel.

Definition 1. *The mapping* $t: [0, 1] \, x \, [0, 1] \to [0, 1]$ *is a t-norm if the following conditions are satisfied:*

1. $t(a, 1)=a$ *for every* $a \in [0, 1]$ *and* $t(0, 0)=0$
2. $t(a, b)=t(b, a)$ *for every* $a, b \in [0, 1]$
3. If $c \geqslant a$ *and* $d \geqslant b$ *then* $t(c, d) \geqslant t(a, b)$
4. $t(t(a, b), c)=t(a, t(b, c))$ *for every* $a, b, c \in [0, 1]$ Let Δ^+ be the family of all distribution functions F such that $F(0)=0$

Definition 2. *Probabilistic metric space is an ordered pair* $(S, \mathcal{F})$ *where* S *is an abstract set of elements and* $\mathcal{F}$ *is a mapping of* $S \times S$ *into a collection* Δ^+ (*the value of* $\mathcal{F}$ *at* $(u, v) \in S \times S$ *will be denoted by* $F_{u, v}$) *where the functions* $F_{u, v}$ *are assumed to satisfy the following conditions:*

(a) $F_{u, v}(x)=1$ *for all* $x>0$ *if and only if* $u=v$.

(b) $F_{u, v}(0)=0$ *for every* $(u, v) \in SxS$

(c) $F_{u, v}=F_{v, u}$ *for every* $(u, v) \in SxS$

(d) $F_{u, v}(x)=1$ *and* $F_{v, w}(y)=1$ *imply* $F_{u, w}(x+y)=1$

Definition 3. *A Menger space is a triplet* $(S, \mathcal{F}, t)$ *where* $(S, \mathcal{F})$ *is a PM-space and t-norm t is such that Menger's triangle inequality:*

$$F_{u,w}(x+y) \geqslant t(F_{u,v}(x), F_{v,w}(y))$$

is satisfied for all $u, v, w \in S$ *and for all* $x>0, y>0$.

The (ε, λ)-topology in $(S, \mathcal{F}, t)$ is introduced by the family of (ε, λ)-neighborhood of $v \in S$:

$$U_v(\varepsilon, \lambda)=\{u \in S: F_{u,v}(\varepsilon)>1-\lambda\}, \ \varepsilon>0, \lambda \in (0, 1)$$

Definition 4. *A random normed space is a triplet* $(S, \mathcal{F}, t)$ *with the following properties:*

1. *t-norm t is stronger than* $T_m(a, b)=\max\{a+b-1, 0\}$

2. $F_u=H$ *for* $u=0$ *where* $H(x)=\begin{cases} 0 & x \leqslant 0 \\ 1 & x>0 \end{cases}$

3. *For every* $u \in S, x \in R, \lambda \in K, \lambda \neq 0$.

$F_{\lambda u}(x)=F_u\left(\frac{x}{|\lambda|}\right)$ *where S is vector space over the field K.*

4. $F_{u+v}(x+y) \geqslant t(F_u(x), F_v(y))$ *for every* $x>0$ *and* $y>0$. It is easy to see that every random normed space is a Menger space if we take $F_{u,v}=F_{u-v}$.

Definition 5. *A mapping T on PM-space* $(S, \mathcal{F})$ *will be called a generalized contraction iff there exists a constant q,* $0<q<1$, *such that for every* $u, v \in S$:

$$F_{Tu,Tv}(qx) \geqslant \min\{F_{u,v}(x), F_{u,Tu}(x), F_{v,Tv}(x), F_{u,Tv}(2x), F_{v,Tu}(2x)\}, \ x>0$$

Probabilistic diameter of the set $A \subset S$ is defined in the following way:

$$D_A(x)=\sup_{t<x} \inf_{u,v \in A} F_{u,v}(t)$$

The set A is probabilistic bounded if $\sup\limits_x D_A(x)=1$. The Kuratowski function of a probabilistic set A is of the form:

$$\alpha_A(x)=\sup\{\varepsilon>0, \exists A_j \subset S, j=1, 2, \ldots, n, \ A \subset \bigcup_{j=1}^{n} A_j, \ D_{A_j}(x) \geqslant \varepsilon\}$$

The Kuratowski function has the following properties:

1. $\alpha_A \in \Delta$
2. $\alpha_A(x) \geqslant D_A(x)$
3. $\Phi \neq A \subset B \subset S \Rightarrow \alpha_A(x) \geqslant \alpha_B(x)$
4. $\alpha_{A \cup B}(x)=\min\{\alpha_A(x), \alpha_B(x)\}$
5. $\alpha_A(x)=\alpha_{\overline{A}}(x)$ where $\overline{A}$ is adherence in (ε, λ)-topology

Definition 6. *If T is a mapping of random normed space* S *into itself so that for every subset* A *of* S *such that* $\alpha_A < H$ *we have:*

$$\alpha_{T(A)} > \alpha_A$$

T *is called probabilistic densifying.*

Theorem 1 [1]. *Let* $(S, \mathcal{F}, t)$ *be a Menger space,* $t = \min$ *and* $T: S \to S$ *is a generalized contraction on* S *and* S *is* T*-orbitally complete. Then* T *has a unique fixed point* $v \in S$ *and* $\lim_{n\to\infty} T^n u = v$ *for every* $u \in S$.

Theorem 2 [1]. *Let* $\{T_i\}$ $i \in N$ *be a sequence of mappings on a Menger space* $(S, \mathcal{F}, \min)$ *and let* $T: S \to S$ *be a generalized contraction on* S *which is* T*-orbitally complete. If each* T_i $(i = 1, 2, \ldots)$ *has at least one fixed point* v_i *and the sequence* $\{T_i\}_i$ $i \in N$ *on the subset:*

$I = \{u | \; u \in S$, *there is some* T_i *such that* $u = T_i \, u\}$ *converges uniformly to* T *then the sequence* $\{u_i\}$ $i \in N$ *converges to a unique fixed point* v *of* T.

Theorem 3 [11]. *Let* $(S, \mathcal{F}, t)$ *be a complete random normed space,* A *a probabilistic bounded, closed, convex subset of* S *and* T *a continuous probabilistic densifying selfmapping of* S *such that* $TA \subset A$. *If* $t = \min$, *then there exists at least one fixed point of the mapping* T *in the set* A.

Theorem 4. *Suppose that* $(S_1, \mathcal{F}_1, \min)$ *and* $(S_2, \mathcal{F}_2, \min)$* *are complete random normed spaces. Further, let* U *be a closed, convex subset of* S_1, V *be a closed, convex and probabilistic bounded subset of* S_2 *and* $H: U\text{x}V \to U$, $K: U\text{x}V \to V$ *such that the following conditions are satisfied:*

1. *The mapping* H *is uniformly continuous and for every* $x > 0$ *the following inequality holds:*

$$F_{H(u,\,w)-H(v,\,w)}(qx) \geqslant \min\{F_{u-v}(x), F_{u-H(u,\,w)}(x), F_{v-H(v,\,w)}(x),$$
$$F_{u-H(v,\,w)}(2x), F_{v-H(u,\,w)}(2x)\}$$

for every $u, v \in U$ *and every* $w \in V$, $o < q < 1$.

2. *The mapping* K *is continuous and for every set* Q *such that* $\alpha_Q < H$ *we have:* $\alpha_{K(U,Q)} > \alpha_Q$

Then there exists at least one element $z_0 \in U\text{x}V$ *such that:*

$$\Phi \, z_0 = z_0 \;\; (\Phi z = (H(x, y), K(x, y)), \; z = (x, y))$$

Proof: For every $w \in V$ we shall define the mapping H_w of the set U into itself such that:

$$H_w(u) = H(u, w) \text{ for every } u \in V, w \in V$$

From the condition 1. it follows that the mapping H_w satisfies the following inequality:

$$F_{H_w(u)-H_w(v)}(qx) \geqslant \min\{F_{u-v}(x), F_{u-H_w(u)}(x), F_{v-H_w(u)}(x), F_{u-H_w(v)}(2x)$$
$$F_{v-H_w(v)}(2x)\}$$

for every u, $v \in U$ and $w \in V$.

* If $K(U, V)$ is compact and (S_2, F_2, t) is admissible [14] in (ε, λ) topology from [14] it follows the existence of element $z_0 \in U \times V$ such that $\Phi \, z_0 = z_0$.

From Theorem 1 it follows that for every $w \in V$ there exists one and only one element $Rw \in U$ such that: $H_w(Rw)=Rw$. We shall prove that the mapping $R: V \to U$ is a continuous mapping. Since (ε, λ) topology is a metrizable topology we shall prove that from $\lim_{n \to \infty} w_n = w_o$, where $\{w_n\}_{n \in N}$ is a sequence from U it follows:

$$\lim_{n \to \infty} Rw_n = Rw_o$$

Let $\tilde{H}_n$ be the mapping H_{w_n} for every $n \in N$ and $\tilde{H}_o = H_{w_o}$. Since the mapping H is uniformly continuous we have:

$$\lim_{n \to \infty} H(u, w_n) = H(u, w_o) \text{ for every } u \in U,$$

namely that:

$$\lim_{n \to \infty} H_n(u) = H_o(u)$$

uniformly in respect to $u \in U$. From Theorem 2 it follows that $\lim_{n \to \infty} Rw_n = Rw_o$ and so the mapping R is continuous.

Now, we shall define the mapping T of the set V into itself in the following way:

$$Tw = K(Rw, w) \quad \text{for every } w \in V$$

It is obvious that the mapping T is continuous. We shall show that the mapping T is densifying.

From the definition of the mapping T it follows:

$$TQ = \{Tw; w \in Q\} = \{K(Rw, w); w \in Q\} \subset \{K(u, v); u \in U, v \in V\} = K(U, Q)$$

Suppose now that $\alpha_A < H$, $A \subset V$. From the property 3. of the Kuratowski function and the condition 2. of the Theorem we have:

$$\alpha_{T(Q)}(x) \geqslant \alpha_{K(U, Q)}(x) \quad \text{for every } x > 0$$

and so:

$$\alpha_{TQ} \geqslant \alpha_{K(U, Q)} > \alpha_Q$$

which means that the mapping T is probabilistic densifying. From the Theorem 3 it follows that there exists $w_o \in V$ such that $Tw_o = w_o$. If we take for element z_o the element $(Rw_o, w_o) \in U \mathrm{x} V$ we have:

$$\Phi z_o = z_o$$

which completes the proof.

Remark: Every Banach space is complete random normed space if $F_u(x) = H(x - \|u\|)$ for every $u \in S$. If we take $t = \min$ and $\mathcal{F}: S \to \Delta^+$ is defined by $\mathcal{F}(u) = F_u$ for every $u \in S$ then $(S, \mathcal{F}, \min)$ is a random normed space. It is easy to see that (ε, λ) topology induced on S the norm topology and that every bounded subset is probabilistic bounded. If $K(U, V)$ is a compact set it is obvious that:

$$\alpha_Q < H \quad \text{implies } \alpha_{K(U,Q)} > \alpha_Q$$

and so the Theoremi of Avramescu from [12] follows from our theorem.

REFERENCES

[1] Ljubomir B. Ćirić, *On fixed points of generalized contractions on probabilistic metric spaces*, Publ. Inst. Math., tome 18 (32), 1975, 71–78.

[2] V. Sehgal and A. Bharucha-Reid, *Fixed points of contractions mappings on probabilistic metric spaces*, Math. System theory, 6 (1972), 97–102.

[3] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. 10, 1960, 313–334.

[4] B. Schweizer, A. Sklar and E. Thorp, *The metrization of statistical metric spaces*, Pacific J. Math., 10 (1960), 673–675.

[5] A. Wald, *On a Statistical Generalization of Metric Spaces*, Proc. Nat. Acad. Sci. USA, 29 (1943), 196–197.

[6] O. Onicescu, *Nombres et systémes aléatoires*, Editions de l'Academie de la R. P. Roumaine, Bucarest, 1968.

[7] V. Istratescu and I. Sacuiu, *Fixed Points Theorems for Contraction Mappings on Probalbilistic Metric Spaces*, Rev. Roum. Math. Pures et Appl., XVIII, 9, 1973, 1375–1381.

[8] H. Sherwood, *Complete Probabilistic Metric Space*, Z. Wahrscheinlichkeitstheorie verw. Geb. 20, 1971.

[9] Gh. Bocsan and Gh. Constantin, *The Kuratowski function and some applications to the probabilistic metric spaces*, Universitatea din Timisoara, Facultatea de mathematica-mecanica, Seminarul de teoria functiilor si matematici aplicate, A. Spatii metrice probabiliste, № 1.

[10] Gh. Bocsan and Gh. Constantin, The Kuratowski *function and some applications to the probabilistic metric spaces*, Atti Acad. Naz. Lincei, LV, 1973, Fasc. 3–4, 236–240.

[11] Gh. Bocsan, *On some fixed point theorem in random normed spaces*, A. Spatii metrice probabiliste, № 13, 1974.

[12] C. Avramescu, *Asupra unei teoreme de punct fix*. Studii si Cercetari mat. Acad. R. S. R. 1970, 22 №, 215–221.

[13] A. N. Šerstnev, *The notion of Random Normed Space*, DAN USSR, 149 (2), 1963, 280–283.

[14] Siegfried Hahn und Karl-Friedrich Pötter, *Über Fixpunkte kompakter Abbildungen in topologischen Vektorräumen*, Studia Math. T. L. 1974, 1–16.

Olga Hadžić

TEOREMA O NEPOKRETNOJ TAČKI U SLUČAJNIM NORMIRANIM PROSTORIMA

Rezime

U ovom radu je dokazana sledeća teorema:

Teorema: *Pretpostavimo da su* $(S_1, \mathcal{F}_1, \min)$ *i* $(S_2, \mathcal{F}_2, \min)$ *kompletni slučajni normirani prostori. Neka je dalje U zatvoren i konveksan podskup od* S_1, *V zatvoren, konveksan i probabilistički ograničen podskup od* S_2, *H preslikavanje proizvoda* $U \times V$ *u U i K preslikavanje proizvoda* $U \times V$ *u V tako da su zadovoljeni sledeći uslovi:*

1. *Preslikavanje H je neprekidno i za svako* $x>0$ *je:*

$$F_{H(u,w)-H(v,w)}(qx) \geqslant \min\{F_{u-v}(x), F_{u-H(u,w)}(x), F_{v-H(v,w)}(x), F_{u-H(v,w)}(2x), F_{v-H(u,w)}(2x)\}$$

za svako $u, v \in U$ *i svako* $w \in V$ *gde je* $0<q<1$.

2. *Preslikavanje K je neprekidno i za svaki skup Q takav da je* $\alpha_Q < H$ *važi:* $\alpha_{(KU,Q)} > \alpha_Q$. *Tada postoji bar jedan elemenat* $z_0 \in U \times V$ *takav da je* $\Phi z_0 = z_0$ *gde je* $\Phi z = (H(x, y), K(x, y))$ $z=(x, y)$.