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ON ϕ-RICCI SYMMETRIC KENMOTSU MANIFOLDS

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Abstract. The present paper deals with the study of ϕ-Ricci symmetric Kenmotsu manifolds. An example of a three-dimensional ϕ-Ricci symmetric Kenmotsu manifold is constructed for illustration.

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1. Introduction

In [15], T. Takahashi introduced the notion of locally ϕ-symmetric Sasakian manifolds as a weaker version of local symmetry of such manifolds. In [9], U.C. De studied ϕ-symmetric Kenmotsu manifolds with several examples. In 1971, K. Kenmotsu [11] introduced a class of contact metric manifolds, called Kenmotsu manifold, which is not Sasakian. Kenmotsu manifolds have been studied by several authors such as Pitis [14], Binh, Tamassy, De and Tarafdar [3], De and Pathak [5], Özgür [12], Özgür and De [13] and many other geometricians. Recently, U.C. De, in [8] introduced the notion of ϕ-Ricci symmetric Sasakian manifolds and obtained some interesting results of this manifold.

The present paper is organized as follows: Section 2 is equipped with some prerequisites about Kenmotsu manifolds. In section 3, we study ϕ-Ricci symmetric Kenmotsu manifolds. Three-dimensional ϕ-Ricci symmetric Kenmotsu manifolds are studied in section 4. In the last section, we have construct an example of three-dimensional Kenmotsu manifold which supports the results obtained in section 3 and 4.

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact Riemannian manifold with structure tensors (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field; ξ is the structure vector field; η is a 1-form and g is the Riemannian metric. It is well known that (ϕ, ξ, η, g)-structure satisfies the following conditions ([1],[2]):

(2.1) $$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

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(2.2) $$\phi^2 X = -X + \eta(X)\xi, \qquad g(X,\xi) = \eta(X),$$

(2.3) $$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M.

In addition, if

(2.4) $$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,$$

(2.5) $$\nabla_X \xi = X - \eta(X)\xi,$$

where ∇ denotes the operator of covariant differentiation with respect to g, then (M, ϕ, ξ, η, g) is called a Kenmotsu manifold [11].

In a Kenmotsu manifold, the following relations hold [6]:

(2.6) $$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X),$$

(2.7) $$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

(2.8) $$S(X,\xi) = -2n\eta(X), \quad Q\xi = -2n\xi,$$

for any vector fields X, Y, Z, where R is the Riemannian curvature tensor and S is the Ricci tensor.

From (2.6), we have

(2.9) $$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X,$$

(2.10) $$R(X,\xi)\xi = \eta(X)\xi - X.$$

Also, since $S(X,Y) = g(QX,Y)$, we have

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y),$$

where Q is the Ricci operator.

Using the properties $g(X, \phi Y) = -g(\phi X, Y)$, $Q\phi = \phi Q$, (2.2) and (2.7), we get

(2.11) $$S(\phi X, \phi Y) = S(X,Y) + 2n\eta(X)\eta(Y).$$

3. ϕ-Ricci symmetric Kenmotsu manifolds

At first, we recall

Definition 3.1. *[15] A Kenmotsu manifold M is said to be locally ϕ-symmetric, if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

for any vector fields X, Y, Z, W orthogonal to ξ.

Definition 3.2. *[8] A Kenmotsu manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2(\nabla_X Q)(Y) = 0,$$

for any vector fields X, Y on M and $S(X,Y) = g(QX,Y)$.

If X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

Definition 3.3. *[8] A Kenmotsu manifold M is said to be Einstein manifold if its Ricci tensor S is of the form*

$$S(X,Y) = \alpha g(X,Y),$$

where α is a constant and X, Y are any vector fields on M.

Theorem 3.1. *A $(2n+1)$-dimensional ϕ-Ricci symmetric Kenmotsu manifold is an Einstein manifold.*

Proof. Let us assume that the manifold is ϕ-Ricci symmetric. Then we have

$$\phi^2(\nabla_X Q)(Y) = 0.$$

Using (2.2) in the above, we get

$$-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{3.1}$$

From (3.1), it follows that

$$-g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0, \tag{3.2}$$

which on simplifying gives

$$-g(\nabla_X Q(Y), Z) + S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{3.3}$$

Replacing Y by ξ in (3.3), we get

$$-g(\nabla_X Q(\xi), Z) + S(\nabla_X \xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{3.4}$$

By using (2.5) and (2.9) in (3.4), we obtain

$$2n[g(X,Z)-\eta(X)\eta(Z)]+S(X,Z)-\eta(X)S(\xi,Z)+\eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{3.5}$$

Replacing X by ϕX and Z by ϕZ in (3.5), we have

$$S(\phi X, \phi Z) = -2ng(\phi X, \phi Z). \tag{3.6}$$

In view of (2.3) and (2.11), (3.6) becomes

$$S(X,Z) = -2ng(X,Z),$$

which implies that the manifold is an Einstein manifold.

Now, since a ϕ-symmetric Riemannian manifold is ϕ-Ricci symmetric, we have

Corollary 3.2. *A ϕ-symmetric Kenmotsu manifold is an Einstein manifold.*

Theorem 3.3. *If a $(2n+1)$-dimensional Kenmotsu manifold is an Einstein manifold, then it is ϕ-Ricci symmetric.*

Proof. Let us suppose that the manifold is an Einstein manifold. Then

$$S(X,Y) = \alpha g(X,Y),$$

where $S(X,Y) = g(QX,Y)$ and α is a constant. Hence $QX = \alpha X$. So, we have

$$\phi^2((\nabla_Y Q)(X)) = 0.$$

This completes the proof. □

In view of Theorem 3.1 and Theorem 3.3, we have

Theorem 3.4. *A $(2n+1)$-dimensional Kenmotsu manifold is ϕ-Ricci symmetric if and only if it is an Einstein manifold.*

4. Three-dimensional ϕ-Ricci symmetric Kenmotsu manifolds

Theorem 4.1. *If the scalar curvature r of a 3-dimensional Kenmotsu manifold is equal to -6, then the manifold is ϕ-Ricci symmetric.*

Proof. The curvature tensor of a 3-dimensional Kenmotsu manifold is of the form [5]

$$\begin{aligned} R(X,Y)Z &= \tfrac{r+4}{2}[g(Y,Z)X - g(X,Z)Y] - \tfrac{r+6}{2}[g(Y,Z)\eta(X)\xi \\ &\quad - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.1}$$

From (4.1), we obtain

$$S(X,Y) = \frac{r+2}{2}g(X,Y) - \frac{r+6}{2}\eta(X)\eta(Y), \tag{4.2}$$

which implies that

$$QX = (\frac{r+2}{2})X - \frac{r+6}{2}\eta(X)\xi. \tag{4.3}$$

Taking the covariant differentiation of (4.3) with respect to W, we get

$$\begin{aligned} (\nabla_W Q)X &= (\tfrac{1}{2})dr(W)X - (\tfrac{1}{2})dr(W)\eta(X)\xi \\ &\quad - (\tfrac{r+6}{2})g(X,-\phi W)\xi - (\tfrac{r+6}{2})\eta(X)(\nabla_W \xi). \end{aligned} \tag{4.4}$$

Now, applying ϕ^2 on both sides of (4.4) and using (2.2), we have

$$\phi^2((\nabla_W Q)(X)) = (\frac{1}{2})[dr(W)(-X + \eta(X)\xi) - (r+6)\eta(X)\phi^2(\nabla_W \xi)]. \tag{4.5}$$

This completes the proof of the theorem. □

Theorem 4.2. *A 3-dimensional Kenmotsu manifold is locally ϕ-Ricci symmetric if and only if the scalar curvature r is constant.*

Proof. Taking X orthogonal to ξ in (4.5), we obtain

$$\phi^2((\nabla_W Q)(X)) = -\frac{1}{2}dr(W)X. \tag{4.6}$$

The proof follows from (4.6) and Theorem 4.1.

5. Example

In this section we construct an example of ϕ-Ricci symmetric Kenmotsu manifold which supports Theorem 4.1.

We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3 : x \neq 0\}$, where (x,y,z) are the standard coordinates in R^3. We choose the vector fields

$$e_1 = x\frac{\partial}{\partial z}, \qquad e_2 = x\frac{\partial}{\partial y}, \qquad e_3 = -x\frac{\partial}{\partial x},$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0,$$

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any vector field Z on M. We define the (1,1) tensor field ϕ as $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$ and $\phi(e_3) = 0$. The linearity property of ϕ and g yields that

$$\eta(e_3) = 1, \phi^2 X = -X + \eta(X)e_3,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. Thus for $e_3 = \xi$, $M(\phi,\xi,\eta,g)$ defines an almost contact metric manifold.

Let ∇ be the Levi-Civita connection with respect to g. Then we have

$$[e_1,e_2] = 0, \quad [e_1,e_3] = e_1, \quad [e_2,e_3] = e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y). \end{aligned}$$

Taking $e_3 = \xi$ and using the Koszul's formula, we obtain

$$\nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1, \quad \nabla_{e_1} e_1 = -e_3,$$
$$\nabla_{e_2} e_3 = e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0,$$
$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.$$

We observe that the manifold satisfies the condition $\nabla_X \xi = X - \eta(X)\xi$, for $e_3 = \xi$. Hence $M(\phi, \xi, \eta, g)$ is a 3-dimensional Kenmotsu manifold. By using above results, we can easily obtain the following:

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_1)e_1 = -e_2,$$
$$R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_2 = -e_3,$$
$$R(e_1, e_2)e_3 = 0, \quad R(e_3, e_2)e_3 = e_2, \quad R(e_3, e_1)e_2 = 0.$$

The definition of Ricci tensor in 3-dimensional manifold implies that

$$S(X, Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i). \tag{5.1}$$

Using the components of the curvature tensor in (5.1), we get the following results:

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2,$$
$$S(e_1, e_2) = 0, \quad S(e_1, e_3) = 0, \quad S(e_2, e_3) = 0.$$

In view of above relations, it follows that the scalar curvature of the manifold is equal to -6 and the Ricci tensor $S(X, Y) = -2g(X, Y)$. Hence $QX = -2X$, which implies that $\phi^2((\nabla_W Q)(X)) = 0$. Thus we observe that the scalar curvature of the manifold under consideration is -6, and it is ϕ-Ricci symmetric. So this example verifies Theorem 4.1. As the above manifold is ϕ-Ricci symmetric and Einstein, the example also agrees with Theorem 3.4, for 3-dimensional case.

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