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QUASI-IDEALS AND MINIMAL QUASI-IDEALS IN Γ-SEMIRINGS

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Abstract. In this paper we introduce the concept of minimal quasi-ideal in a Γ-semiring. Some properties of minimal quasi-ideals in Γ-semirings are provided.

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1. Introduction

The notion of quasi-ideal was first introduced for semigroups [8] and then for rings by Steinfeld [10]. The general properties of quasi-ideals for semigroups and rings are proved in [9]. Iseki [5] introduced the concept of quasi-ideal for a semiring without zero and gave some characterizations of it. Using quasi-ideals Shabir, Ali, Batool [7] characterized semirings. Quasi-ideal is a generalization of a left and a right ideal.

As a generalization of a Γ-ring and a semiring the notion of Γ-semiring was introduced by Rao [6]. It is natural to extend the concept of quasi-ideals in Γ-semirings and this is done by Chinram [2] as a generalization of quasi-ideals in Γ-semigroups. The Γ-semirings introduced by Chinram [2] and Dutta [4] are different (see Remark 2.2). In this paper we study quasi-ideals in Γ-semirings introduced by Rao [6]. Minimal quasi-ideals for Γ-semigroups are studied by Chinram [3] and for semirings by Iseki [5]. On this line we introduce the notion of minimal quasi-ideals in Γ-semirings. Some properties of minimal quasi-ideals are furnished. Also, we introduce the concept of quasi-simple Γ-semiring.

2. Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in the sequel. For this we follow Dutta [4].

Definition 2.1. *Let S and Γ be two additive commutative semigroups. S is called Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ (images to be denoted by $a\alpha b$;for all $a, b \in S$ and for all $\alpha \in \Gamma$) satisfying the following conditions:*

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(i) $a\alpha\,(b+c) = (a\ \alpha b) + (a\ \alpha c)$
(ii) $(b+c)\,\alpha a = (b\ \alpha a) + (c\ \alpha a)$
(iii) $a(\alpha+\beta)c = (a\ \alpha c) + (a\ \beta c)$
(iv) $a\alpha\,(b\beta c) = (a\ \alpha b)\,\beta c$ *; for all* $a, b, c \in S$ *and for all* $\alpha, \beta \in \Gamma$

Obviously, every semiring S is a Γ-semiring but not conversely. For this, let us consider the following example.

Example 1. Let Q be the set of rational numbers. $(S, +)$ be the commutative semigroup of all 2×3 matrices over Q and $(\Gamma, +)$ be commutative semigroup of all 3×2 matrices over Q. Define $A\alpha B=$ usual matrix product of A, α and B; for all $A, B \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semiring but not a semiring.

Remark 2.2. *Let N be the set of natural numbers and $\Gamma = \{1, 2, 3\}$. Define the mapping $N\times\Gamma\times N \longrightarrow N$ by $a\alpha b$=usual product of a, α, b; for all $a, b \in N, \alpha \in \Gamma$. Then N is a Γ-semiring by the definition of Chinram [2]. But Γ is not an additive semigroup, hence it is not a Γ-semiring according to Dutta [4].*

Example 2. Let N be the set of natural numbers and $\Gamma = \{1, 2, 3\}$. $(N$, max.) and (Γ,max.) are commutative semigroups. Define the mapping $N \times \Gamma\times N \rightarrow N$ by, $a\alpha b = \min\{a, \alpha, b\}$; for all $a, b \in N, \alpha \in \Gamma$. Then N is a Γ-semiring.

Example 3. Let Q be the set of rational numbers and $\Gamma = N$ be the set of natural numbers (Q,+) and (N,+) are commutative semigroups. Define the mapping $Q\times \mathrm{N}\times Q \longrightarrow Q$ by $a\alpha b$ =usual product of a, α, b ; $a, b \in Q, \alpha \in \Gamma$.Then Q is a Γ-semiring.

Definition 2.3. *An element $0 \in S$ is said to be an absorbing zero if*

$$0\alpha a = 0 = a\alpha 0, a + 0 = 0 + a = a \ ;\text{for all } a \in S \textit{ and for all } \alpha \in \Gamma.$$

From onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

Definition 2.4. *A nonempty subset T of S is said to be a sub-Γ-semiring of S if (T,+) is a subsemigroup of (S,+) and $a\alpha b \in T$;* for all $a, b \in T$ *and for all* $\alpha \in \Gamma$.

Definition 2.5. *A nonempty subset T of S is called a left (respectively right) ideal of S if T is a subsemigroup of (S,+) and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T$, $x \in S$ and for all $\alpha \in \Gamma$*

Definition 2.6. *If T is both left and right ideal of S, then T is known as an ideal of S.*

Definition 2.7. *If M and N are two nonempty subsets of S, then we define $M + N = \{m + n/m \in M, n \in N\}$ and*

$$M\Gamma N = \left\{ \sum_{i=1}^{n} x_i\alpha_i y_i |\ x_i \in M, \alpha_i \in \Gamma, \mathrm{y_i \in N} \right\}.$$

If N *is the set of natural numbers, then*

$$NA = \left\{ \sum_{i=1}^{n} n_i a_i \mid n_i \in N, a_i \in A \right\}.$$

Definition 2.8. *Let* X *be a nonempty subset of* S. *By* $(X)_l$ *we mean the left ideal of* S *generated by* X *(that is intersection of all left ideal of* S *containing* X*).*

Similarly, $(X)_r$, $(X)_t$ *denote the right and two-sided ideal generated by* X *respectively.*

Definition 2.9. *A left (right, two-sided) ideal* I *of a* Γ*-semiring* S *is said to be left (right, two-sided) k-ideal of* S *if* a , $a + x \in I$,*then* $x \in I$ *for any* $x \in S$.

3. Quasi-ideals

We start with the proofs of two basic results which we will use quite often.

Result 3.1. *For each nonempty subset* X *of* S *the following statements hold.*
(i) $S\Gamma X$ *is a left ideal.*
(ii) $X\Gamma S$ *is a right ideal.*
(iii) $S\Gamma X\Gamma S$ *is an ideal of S.*

Proof.
(i)

$$S\Gamma X = \left\{ \sum_{i=1}^{n} a_i \alpha_i x_i | \ a_i \in S, \alpha_i \in \Gamma, \, x_i \in X \right\}$$

Let $a, b \in S\Gamma X$. Then

$$a + b = \sum_{i=1}^{n} a_i \alpha_i x_i + \sum_{j=1}^{m} b_j \beta_j y_j$$

implies $a + b$ is a finite sum. Hence $a + b \in S\Gamma X$ and this shows $S\Gamma X$ is a subsemigroup of (S,+). For $t \in S, a \in S\Gamma X, and\ \beta \in \Gamma$, then

$$t\beta a = t\beta \left(\sum_{i=1}^{n} a_i \alpha_i x_i \right) = \sum_{i=1}^{n} t\beta \left(a_i \alpha_i x_i \right) = \sum_{i=1}^{n} \left(t\beta a_i \right) \alpha_i x_i \in S\Gamma X.$$

Therefore $S\Gamma X$ is a left ideal of S.
(ii) As in (i) we can prove that $X\Gamma S$ is a right ideal of S.
(iii) By (i) $S\Gamma X$ is a left ideal of S. Hence $S\Gamma X\Gamma S$ is right a ideal of S by(ii). Similarly, by (ii) $X\Gamma S$ is a right ideal of S. Hence $S\Gamma X\Gamma S$ is a left ideal of S by (i). Therefore, $S\Gamma X\Gamma S$ is an ideal of S. □

Result 3.2. *For any nonempty subset X of S we have*
(i) If S has right unit element 1, then $(X)_l = S\Gamma X$.
(ii) If S has right unit element 1, then $(X)_r = X\Gamma S$.
(iii) If S has right unit element 1, then $(X)_t = S\Gamma X\Gamma S$.

Proof.
(i) Let S contain left unit element 1. Then $1\alpha a = a$,for every $a \in S$ *and* $\alpha \in \Gamma$. For any $x \in X$, $x = 1\alpha x \in S\Gamma X$. Hence X is a subset of $S\Gamma X$. As $S\Gamma X$ is a left ideal of S, $NX \subseteq S\Gamma X$. But then we have $(X)_l = NX + S\Gamma X$ (see [2]). This implies $(X)_l \subseteq S\Gamma X + S\Gamma X \subseteq S\Gamma X$. As $(X)_l$ is the smallest left ideal of S containing X. This shows that $(X)_l = S\Gamma X$. Similarly, we can prove that $(X)_r = X\Gamma S$ and$(X)_t = S\Gamma X\Gamma S$. □

Chinram [2] has defined a quasi-ideal Q in a Γ-semiring S as follows.

Definition. *A subsemigroup Q of $(S,+)$ is a quasi-ideal of S if*

$$(S\Gamma Q) \cap (Q\Gamma S) \subseteq Q.$$

Example 4. Let N be the set of natural numbers and $\Gamma = 2N$. Then N is a Γ-semiring and $A = 3N$ is a quasi-ideal of a Γ-semiring N.

Example 5. Consider a Γ-semiring $S = M_{2x2}(N_0)$, where N_0 denotes the set of natural numbers with zero and Γ= S. Define $A\alpha B$= usual matrix product of A, α and B; for all $A, \alpha, B \in S$. Then
$Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N_0 \right\}$ is a quasi-ideal of a Γ-semiring S.

Properties

(1) By a quasi-ideal Q in a semiring S we mean an additive subsemigroup of S such that $SQ \bigcap QS \subseteq Q$ (see Iseki [5]). As every semiring is a Γ-semiring the two definitions given in [2] and [5] of quasi-ideals coincide in a semiring.
(2) Every quasi-ideal of S is a sub Γ-semiring of S.
(3) Every one-sided ideal or two-sided ideal of S is a quasi-ideal of S but converse need not be true. For this consider Γ-semiring given in Example (5). Here

$$Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N_0 \right\}$$

is a quasi-ideal but neither a left ideal nor a right ideal of S.
(4) If Q_1 and Q_2 are quasi-ideals of S, then $Q_1\Gamma\, Q_2$ need not be a quasi-ideal of S. For this consider the following example.

Example 6. If $T = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^+ \right\}$, then T is a semigroup with respect to usual matrix multiplication.

If $S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^+ \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $\Gamma = S$, then S is a Γ-semiring with usual matrix multiplication. Define $+$ in S by $A + B = 0$ if $A,\ B \in\ S$ and $A + 0 = 0 + A = A$, for all $A \in S$. If

$$Q_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^+, 0 < a < b \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},\ \text{and}$$

$$Q_2 = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^+, a > 0, q > 5 \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then Q_1 is a right ideal and Q_2 is a left ideals of S and hence Q_1 and Q_2 are quasi-ideals of S. But $Q_1 \Gamma Q_2$ is a not a quasi-ideal of S.

(5) The sum of two quasi-ideals of S need not be a quasi-ideal of S. We illustrate this by the following example.

Example 7. Let $S = M_{2x2}(N_0)$ be a semiring. If Γ= S, then S forms a Γ-semiring with $A\alpha B=$ usual matrix product of $A, \alpha\, B$; for all $A, \alpha, B \in S$. $Q_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N_0 \right\}$ and $Q_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mid b \in N_0 \right\}$ are quasi-ideals of S,
but $Q_1 + Q_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in N_0 \right\}$ is not a quasi-ideal of S.

(6) Arbitrary intersection of quasi-ideals of S is either empty or a quasi-ideal of S.
Proof. Let $T = \bigcap_{i \in \triangle} \{Q_i / Q_i \text{ is a quasi-ideal of } S\}$, where $\triangle$ denotes any indexing set, be a nonempty set. T is a subsemigroup of $(S, +)$. Further $(S\Gamma T) \cap (T\Gamma S) = (\ S\Gamma(\bigcap_{i \in \triangle} Q_i)\ ) \cap (\ (\ \bigcap_{i \in \triangle} Q_i)\Gamma S)\ \subseteq\ (Q_i \Gamma S) \cap (S\Gamma Q_i) \subseteq Q_i$, for all $i \in \triangle$. $(S\Gamma T) \cap (T\Gamma S) \subseteq \bigcap_{i \in \triangle} Q_i = T$.This shows that T is a quasi-ideal of S.
(7) The set of all quasi-ideals of S forms a Moore family and hence a complete lattice (see Birkhoff [1]).
(8) If Q is a quasi-ideal of S, then $Q^2 = Q\Gamma Q \subseteq Q$.
Proof. As Q is a quasi-ideal of S, $(S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$. We have $Q^2 = Q\Gamma Q \subseteq Q\Gamma S$ and $Q^2 = Q\Gamma Q \subseteq S\Gamma Q$. Hence $Q^2 \subseteq (S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$. Thus $Q^2 = Q\Gamma Q \subseteq Q$.
(9) For each nonempty subset X of S, $(S\Gamma X) \cap (X\Gamma S)$ is a quasi-ideal of S.
Proof. $S\ \Gamma\,(S\Gamma X) \cap (X\Gamma S)\,\Gamma S = (S\Gamma S)\Gamma X \cap X\Gamma(S\Gamma S) \subseteq\ (\ S\Gamma X) \cap (X\Gamma S)$. Therefore $(S\Gamma X) \cap (X\Gamma S)$ is a quasi-ideal of S.
(10) If S has an identity element 1, then every quasi-ideal of S is expressed as an intersection of a left ideal and a right ideal of S.
Proof. Let S be a Γ-semiring with an identity element 1. Let Q be a quasi-ideal of S. Then $S\Gamma Q$ is a left ideal and $Q\Gamma S$ is a right ideal of S (see Result 3.1). As S contains an identity element 1, by Result (3.2) we have $(Q)_l = S\Gamma Q$ and $(Q)_r = Q\Gamma S$. Therefore $Q \subseteq (Q)_l = S\Gamma Q$ and $Q \subseteq (Q)_r = Q\Gamma S$ imply $Q \subseteq$

$(S\Gamma Q)\cap(Q\Gamma S)$. But Q being a quasi-ideal of S, $(S\Gamma Q)\cap(Q\Gamma S) \subseteq Q$. Therefore $Q = (S\Gamma Q) \cap (Q\Gamma S)$. Thus every quasi-ideal of S is an intersection of a left ideal and a right ideal of S.

(11) Intersection of a right ideal and a left ideal of S is a quasi-ideal of S

Proof. Let R be a right ideal and L be a left ideal of S. Then $R \cap L$ is a subsemigroup of $(S,+)$. Further $(S\Gamma(R\cap L))\cap((R\cap L)\Gamma S) \subseteq (S\Gamma L)\cap(R\Gamma S) \subseteq L \cap R$. Hence $R\cap L$ is a quasi-ideal of S. □

Recall that an element e of S is an idempotent element if $e^2 = e\alpha e = e$, for all $\alpha \in \Gamma$. With the help of idempotent elements in S we obtain quasi-ideals in S. This we prove in the following theorem.

Theorem 3.3. *Let L be a left of S. Then for any idempotent elements e of S, $e\Gamma L$ is a quasi-ideal of S.*

Proof. First we prove that $e\Gamma L = L \cap (e\Gamma S)$. We know $(e\Gamma S) + (e\Gamma S) = e\Gamma(S+S) \subseteq e\Gamma S$. Hence $e\Gamma S$ is a subsemigroup of $(S,+)$. As $(e\Gamma S)\Gamma S = e\Gamma(S\Gamma S) \subseteq e\Gamma S$, $e\Gamma S$ is a right ideal of S. As $e \in S$ and L is left ideal of S, $e\Gamma L \subseteq L$. Further $e\Gamma L \subseteq e\Gamma S$. These will imply $e\Gamma L \subseteq L\cap(e\Gamma S)$. For the reverse inclusion let $a \in L\cap(e\Gamma S)$.

$$\text{Then } a = \sum_{i=1}^{n} e\alpha_i x_i \quad , \quad \text{for } x_i \in S \ , \ \alpha_i \in \Gamma \ .$$

Thus $a = \sum_{i=1}^{n} e^2\alpha_i x_i = \sum_{i=1}^{n} (e\alpha e)\,\alpha_i x_i = e\alpha \sum_{i=1}^{n} e\alpha_i x_i = e\alpha a \in e\Gamma L$. This shows that $L\cap(e\Gamma S) \subseteq e\Gamma L$. Hence $L\cap(e\Gamma S) = e\Gamma L$.

As L is a left ideal and $e\Gamma S$ is a right ideal of S we get $e\Gamma L$ is a quasi-ideal of S (see Property (11)). □

As in Theorem 3.3 we can prove the following theorem.

Theorem 3.4. *Let R be a right ideal of S. Then for any idempotent elements e of S. $R\Gamma e$ is a quasi-ideal of S.*

Theorem 3.5. *Let R be a right ideal and L be a left of S. Then for any idempotent elements e, f of S, $e\Gamma S\Gamma f$ is a quasi-ideal of S.*

Proof. First we prove that $e\Gamma S\Gamma f = (e\Gamma S)\cap(S\Gamma f)$. $e\Gamma S\Gamma f = (e\Gamma S)\,\Gamma f \subseteq e\Gamma S$ and $e\Gamma S\Gamma f = e\Gamma(S\Gamma f) \subseteq S\Gamma f$. Thus $e\Gamma S\Gamma f \subseteq (e\Gamma S)\cap(S\Gamma f)$. Let $a \in (S\Gamma f)\cap(\mathrm{e}\Gamma\mathrm{S})$. Then

$$a = \sum_{i=1}^{n} x_i\alpha_i f = \sum_{i=1}^{n} x_i\alpha_i\,(f\alpha f) = \left(\sum_{i=1}^{n} x_i\alpha_i f\right)\alpha f = \left(\sum_{j=1}^{m} e\beta_j y_j\right)\alpha f$$

Thus $a = a\alpha f$, for all $\alpha \in \Gamma$. As

$$a \in e\Gamma S \ , \ \alpha \in \Gamma \implies a = a\alpha f \in e\Gamma S\Gamma f.$$

We get $(e\Gamma S)\cap(S\Gamma f)\subseteq e\Gamma S\Gamma f$. Thus $(e\Gamma S)\cap(S\Gamma f)=e\Gamma S\Gamma f$. As $S\Gamma f$ is a left ideal and $e\Gamma S$ is a right ideal of S we get $(e\Gamma S)\cap(S\Gamma f)=e\Gamma S\Gamma f$ is a quasi-ideal of S (see Property (11)). □

Intersection of a quasi-ideal and a sub Γ-semiring of S is a quasi-ideal of that sub Γ-semiring of S. We prove this in the following theorem.

Theorem 3.6. *If Q is a quasi-ideal and T is a sub Γ-semiring of S then $Q\cap T$ is a quasi-ideal of T.*

Proof. As $Q\cap T$ is a subsemigroup of $(S,+)$ and $Q\cap T\subseteq T$, we get $Q\cap T$ is a subsemigroup of $(T,+)$. Further,

$$T\Gamma(T\cap Q)\cap(T\cap Q)\Gamma T\subseteq(T\Gamma Q)\cap(Q\Gamma T)\subseteq(S\Gamma Q)\cap(Q\,\Gamma S)\subseteq Q.$$

And $T\Gamma\,(T\cap Q)\cap(T\cap Q)\,\Gamma T\subseteq(T\,\Gamma T)\cap(T\,\Gamma T)\subseteq T\cap T=T$. Imply $T\Gamma(T\cap Q)\cap(T\cap Q)\Gamma T\subseteq Q\cap T$. This shows that $Q\cap T$ is a quasi-ideal of T.□

Now we define a quasi-simple Γ-semiring as follows.

Definition 3.7. *A Γ-semiring S is said to be a quasi-simple Γ-semiring if S is the unique quasi-ideal of S, i.e. S has no proper quasi-ideal.*

A characterization of quasi-simple Γ-semiring is furnished in the following theorem.

Theorem 3.8. *If S is a Γ-semiring, then S is quasi-simple Γ-semiring if and only if* $(S\Gamma a)\cap(a\Gamma S)=S$, for all $a\in S$.

Proof. Suppose S is a quasi-simple Γ-semiring. For any $a\in S\;S\Gamma a$ *and* $a\Gamma S$ are left and right ideals of S respectively. Therefore $(S\Gamma a)\cap(a\Gamma S)$ is a quasi-ideal of S(see Property (11)). Further $S\Gamma a\subseteq S$ and $a\Gamma\mathrm{S}\subseteq S$ imply $(S\Gamma a)\cap(a\Gamma\mathrm{S})\subseteq S$. As S is a quasi-simple Γ-semiring, $S=(S\Gamma a)\cap(a\Gamma S)$. Conversely, suppose that $S=(S\Gamma a)\cap(a\Gamma S)$. Let Q be quasi-ideal of S. For any $q\in Q$, by assumption we have, $S=(S\Gamma q)\cap(q\Gamma S)\subseteq(S\Gamma Q)\cap(Q\Gamma S)\subseteq Q$. Therefore $S\subseteq Q$. Thus $S=Q$. Hence S is a quasi-simple Γ-semiring. □

4. Minimal Quasi-ideal

In this section we introduce the concept of a minimal quasi-ideal of a Γ-semiring. We define

Definition 4.1. *Let Q be a quasi-ideal of S. Q is said to be minimal quasi-ideal of S if Q does not contain any other proper quasi-ideal of S.*

Properties of minimal quasi-ideals of a Γ-semiring S are proved in the following theorems.

Theorem 4.2. *The intersection of a minimal right ideal and a minimal left ideal of a* Γ*-semiring* S *is a minimal quasi-ideal of* S*.*

Proof. Let R and L denote minimal right ideal and minimal left ideal of S respectively. Define $Q = R \cap L$. Then Q is a quasi-ideal of S(see Property (11)). Let Q_1be a quasi-ideal of S such that $Q_1 \subseteq Q$. By Result (3.1) $S\,\Gamma Q_1$ is a left ideal and $Q_1\Gamma$S is a right ideal of S. $Q_1 \subseteq L$ implies $S\Gamma Q_1 \subseteq S\Gamma L \subseteq L$. Also $Q_1 \subseteq R$ implies $Q_1\Gamma$S $\subseteq R\Gamma S \subseteq R$. By the minimality of R and L we have, $S\,\Gamma Q_1 = L$ and $Q_1\Gamma S = R$. Therefore, $Q = R \cap L = (S\,\Gamma Q_1) \cap (Q_1\Gamma\, S) \subseteq Q_1$. Hence $Q = Q_1$. This shows that Q is a minimal quasi-ideal of S. □

Theorem 4.3. *If* Q *is a minimal quasi-ideal of* S *then any two nonzero elements of* Q *generate the same left (right) ideal of* S*.*

Proof. Let Q be a minimal quasi-ideal of S and x be a nonzero element of Q. Then $(x)_l$, the left ideal generated by x,is a quasi-ideal of S. Hence $(x)_l \cap Q$ is a quasi-ideal of S. As $(x)_l \cap Q \subseteq Q$ and Q is a minimal quasi-ideal of S we get $(x)_l \cap Q = Q$. Thus $Q \subseteq (x)_l$. For any nonzero element y of Q, $y \in Q$ implies $y \in (x)_l$.Therefore $(y)_l \subseteq (x)_l$. Similarly, we can show that
$(x)_l \subseteq (y)_l$. Hence $(x)_l = (y)_l$.

In the same way we can prove that any two nonzero elements of Q generate the same right ideal of S. □

Theorem 4.4. ***Theorem 4.4:-*** *Let* Q *be a quasi-ideal of* S*. If* Q *itself is a quasi-simple* Γ*-semiring, then* Q *is a minimal quasi-ideal of* S*.*

Proof. As Q is a quasi-ideal of S, Q is a sub Γ-semiring of S(see Property (2)). Suppose Q is a quasi-simple Γ-semiring. Let Q_1 be a quasi-ideal of S such that $Q_1 \subseteq Q$. Then $(Q\Gamma Q_1)\,(Q_1\Gamma Q) \subseteq (S\Gamma Q_1)\,(Q_1\Gamma S) \subseteq Q_1$. Therefore Q_1 is a quasi-ideal of Q. $Q_1 \subseteq Q$, Q_1 is a quasi-ideal of Q and Q is a quasi-simple Γ-semiring imply $Q_1 = Q$. Therefore Q is a minimal quasi-ideal of S. □

Any minimal quasi-ideal Q of S can be represented as an intersection of a minimal left ideal and a minimal right ideal of S. We prove this in the following theorem.

Theorem 4.5. *Every minimal quasi-ideal* Q *of* S *is represented as* $Q = (S\Gamma a) \cap (a\Gamma S)$*, where* a *is any element of* Q*,* $S\Gamma a$ *and* $a\Gamma S$ *be a minimal left ideal and a minimal right ideal of* S *respectively.*

Proof. Let Q be a minimal quasi-ideal of S and $a \in Q$.Then $S\Gamma a$ *and* $a\Gamma S$ be left ideal and right ideal of S respectively. Therefore $(S\Gamma a) \cap (a\Gamma S)$is a quasi-ideal of S (see Property (11)). $(S\Gamma a) \cap (a\Gamma S) \subseteq (S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$. By the minimality of Q we get

$$Q = (S\Gamma a) \cap (a\Gamma S)\,.$$

Now to show that $S\Gamma a$ is a minimal left ideal.

Let L be a left ideal of S such that $L \subseteq S\Gamma a$. Then $S\Gamma L \subseteq L \subseteq S\Gamma a$. $(S\Gamma L) \cap (a\Gamma S) \subseteq (S\Gamma a) \cap (a\Gamma S) = Q$. As $S\Gamma L$ is a left ideal of S(see result 3.1) and $a\Gamma$S is a right ideal of S, we get $(S\Gamma L) \cap (a\Gamma \mathrm{S})$ is a quasi-ideal of S (see Property (11)). Further as $(S\Gamma L) \cap (a\Gamma \mathrm{S}) \subseteq Q$ and Q is minimal quasi-ideal of S we have $Q = (S\Gamma L) \cap (a\Gamma \mathrm{S}) \subseteq S\Gamma L$. Now $S\Gamma a \subseteq S\Gamma Q \subseteq S\Gamma\,(S\Gamma L) = (S\Gamma S)\,\Gamma L \subseteq S\Gamma L \subseteq L$ shows that $S\Gamma a \subseteq L$. Therefore, $S\Gamma a = L$. Hence $S\Gamma a$ is a minimal left ideal of S. Similarly, we can prove that $a\Gamma$S is a minimal right ideal of S. □

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