Novi Sad J. Math.
Vol. 39, No. 2, 2009, 43-49

p-VALENT HOLOMORPHIC FUNCTIONS WITH COMPLEX ORDER BY USING SUBORDINATION

Sh. Najafzadeh[1], A. Ebadian[2], M. Eshaghi Gordji[3]

Abstract. A new class of multivalent holomorphic functions with complex order in terms of subordination is introduced. We find some properties of this class like, sufficient coefficient bound, integral operator and Feketo-Szego problem.

AMS Mathematics Subject Classification (2000): Primary 30C45; Secondary 30C80

Key words and phrases: Convolution, Univalent functions, Subordination, Feketo-Szego problem

1. Introduction and preliminaries

Let

$$A_{p,n} = \{f(z) |\ \ f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k\} \tag{1}$$

be the class of analytic and p-valent functions in the unit disc $\mathcal{D} = \{z : |z| < 1\}$. For the functions f and g, analytic in $\mathcal{D}$, we say f is subordinate to g denoted by $f \prec g$ if for some analytic function $\omega(z)$ with conditions $\omega(0) = 0$ and $|\omega(z)| < 1, \quad f(z) = g(\omega(z)), \quad z \in \mathcal{D}$. We define $\sum_{p,n}^{\lambda}(\alpha, \gamma, \eta, b)$ to be the subclass of $A_{p,n}$ consisting of functions of the form (1.1) which satisfies

$$\frac{U_z^{(\lambda,p)} f(z)}{z^p} \prec \frac{1 + [\gamma + b(\alpha - \gamma)(p - \eta)]z}{1 + \gamma z}, \tag{2}$$

where $-1 \leq \gamma < \alpha \leq 1, \quad 0 < \eta < p, \quad 0 \neq b \in \mathbb{C}, \quad 0 \leq \lambda \leq p$ and $U_z^{\lambda,p)} f(z)$ as a fractional differential operator defined by

$$U_z^{(\lambda,p)} f(z) = \frac{\Gamma(p + 1 - \lambda)}{\Gamma(p + 1)}\ z^\lambda\ D_z^\lambda f(z),$$

$$U_z^{(0,p)} f(z) = f(z), \quad U_z^{(1,p)} f(z) = \frac{zf'}{p}$$

[1]Department of Mathematics, University of Maragheh, Maragheh, Iran, e-mail: najafzadeh1234@yahoo.ie
[2]Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran, e-mail: ebadian.ali@gmail.com
[3]Department of Mathematics, Faculty of Natural Sciences Semnan, University, Semnan, Iran, e-mail: majid.eshaghi@gmail.com

and $D_z^{\lambda} f(z)$ is the fractional derivative of order λ (see [2]). The special cases of this class were studied by Mogra [4], Patel [5] and M.K. Aouf [1]. It is easy to see that if $f(z) \in A_{p,n}$ then

$$U_z^{(\lambda,p)} f(z) = z^p + \sum_{k=n+p}^{\infty} a_k \delta_p(k,\lambda) z^k \tag{3}$$

where $\delta_p(k,\lambda) = \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}$. In special case when $\lambda = 1$ we obtain the same class that was studied by J. Patel and A.K. Mohanty [6] and when $b = \cos e^{-i\theta}$, $\lambda = 1$ and the same values of α, γ, η we obtain many well-known subclasses that have been investigated previously. (See [4] and [7])

2. Main Results

In this section we give the sufficient coefficient bounds for $f(z) \in \sum_{p,n}^{\lambda}(\alpha,\gamma,\eta,b)$.

Theorem 2.1. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=n+p}^{\infty} \delta_p(k,\lambda)|a_k| \le \frac{|b|(\alpha-\gamma)(p-\eta)}{1+|\gamma|}, \tag{4}$$

then $f(z) \in \sum_{p,n}^{\lambda}(\alpha,\gamma,\eta,b)$. The result is sharp.

Proof. We have to show that (1.2) holds. However, (1.2) is equivalent to

$$\frac{U_z^{(\lambda,p)} f(z)}{z^p} = \frac{1+[\gamma+b(\alpha-\gamma)(p-\eta)]\omega(z)}{1+\gamma\omega(z)}$$

or

$$U_z^{(\lambda,p)} f(z) + \gamma\omega(z) U_z^{(\lambda,p)} f(z) = z^p + z^p\{\gamma + b(\alpha-\gamma)(p-\eta)\}\omega(z)$$

or

$$U_z^{(\lambda,p)} f(z) - z^p = \{b(\alpha-\gamma)(p-\eta)z^p - \gamma(U_z^{(\lambda,p)} f(z) - z^p)\}\omega(z)$$

or

$$|U_z^{(\lambda,p)} f(z) - z^p| < |b(\alpha-\gamma)(p-\eta)z^p - \gamma(U_z^{(\lambda,p)} f(z) - z^p)|.$$

By (1.3) and putting $|z| = 1$ we have

$$|\textstyle\sum_{k=n+p}^{\infty} a_k\delta_p(k,\lambda)z^k| - |b(\alpha-\gamma)(p-\eta)z^p - \gamma\sum_{k=n+p}^{\infty} a_k\delta_p(k,\lambda)z^k| \le$$
$$\textstyle\sum_{k=n+p}^{\infty} \delta_p(k,\lambda)|a_k| - \{|b|(\alpha-\gamma)(p-\eta)z^p - |\gamma|\sum_{k=n+p}^{\infty} a_k\delta_p(k,\lambda)|a_k|\}$$
$$= \textstyle\sum_{k=n+p}^{\infty}(1+|\gamma|)\delta_p(k,\lambda)|a_k| - |b|(\alpha-\gamma)(p-\eta).$$

By (2.1), the value of the above is less than 0, so $f(z) \in \sum_{p,n}^{\lambda}(\alpha,\gamma,\eta,b)$. The result is sharp for the function $H(z)$ defined in $\mathcal{D}$ by

$$H(z) = z^p + \frac{b(\alpha-\gamma)(p-\eta)}{(1+|\gamma|)\delta_p(k,\lambda)} z^k, \quad k \ge n+p. \tag{5}$$

□

In the next theorem we obtain the coefficient estimate for $\sum_{p,n}^{\lambda}(\alpha,\gamma,\eta,b)$.

Theorem 2.2. *Let* $f(z)$ *belong to* $\sum_{p,n}^{\lambda}(\alpha,\gamma,\eta,b)$*. Then*

$$|a_m| \leq \frac{|b|(\alpha-\gamma)(p-\eta)}{\delta_p(k,\lambda)}, \quad m \geq n+p. \tag{6}$$

Proof. We have

$$\frac{U_z^{(\lambda,p)}f(z)}{z^p} = \frac{1+[\gamma+b(\alpha-\gamma)(p-\eta)]\omega(z)}{1+\gamma\omega(z)},$$

where $\omega(z)=\sum_{j=1}^{\infty}\omega_j\ z^j, \quad z\in\mathcal{D}$. So
(7)

$$\sum_{j=n+p}^{\infty}\delta_p(j,\lambda)a_j\ z^j = \left[b(\alpha-\gamma)(p-\eta)z^{p+n-1}-\gamma\sum_{j=n+p}^{\infty}\delta_p(j,\lambda)a_j\ z^j\right]\sum_{j=1}^{\infty}\omega_j\ z^j.$$

If we equalize the coefficients of the same power of z in both sides we have

$$\begin{aligned}&\sum_{j=n+p}^{\infty}\delta_p(j,\lambda)a_j\ z^j+\sum_{j=m}^{\infty}d_jz^j\\ &= \left[b(\alpha-\gamma)(p-\eta)z^{p+n-1}-\gamma\sum_{j=n+p}^{m-1}\delta_p(j,\lambda)a_jz^j\right]\omega(z),\end{aligned} \tag{8}$$

where d_j's are suitable constants. If each side of (2.5) multiplying by conjugate and $z\to 1$ we have

$$\sum_{j=n+p}^{m}\delta_p(j,\lambda)^2|a_j|^2 \leq [|b|(\alpha-\gamma)(p-\eta)]^2+\gamma^2\sum_{j=n+p}^{m-1}\delta_p(j,\lambda)^2|a_j|^2$$

or

$$\delta_p(m,\lambda)^2|a_m|^2 \leq [|b|(\alpha-\gamma)(p-\eta)]^2-(1-\gamma^2)\sum_{j=n+p}^{m-1}\delta_p(j,\lambda)^2|a_j|^2$$

since $-1\leq\gamma\leq\alpha\leq 1$ we have $\delta_p(m,\lambda)^2|a_m|^2 \leq [|b|(\alpha-\gamma)(p-\eta)]^2$ and this completes the proof.
In the next section we consider the special case $\lambda=1$, and verify the behavior of the operator

$$G_s(z)=(1-s)z^p+\frac{s(p+1)}{z}\int_0^z f(t)dt, \quad 0<s<1, \quad f(z)\in A_{p,n}, \tag{9}$$

under condition (1.2). □

3. Integral operator

If we put $\lambda = 1$ in (1.2) we obtain

$$\frac{f'(z)}{pz^{p-1}} \prec \frac{1+[\gamma+b(\alpha-\gamma)(p-\eta)]z}{1+\gamma z}, \tag{10}$$

and denoted by $\sum_{(p,n)}^{(1)}(\alpha,\gamma,\eta,b)$. With a simple calculation we have

$$G_s(z) = z^p + \sum_{k=n+p}^{\infty} b_k\ z^k \quad where \quad b_k = \frac{(p+1)s}{k+1}a_k. \tag{11}$$

Let $\sum_{p,n}^{s}(\alpha,\gamma,\eta,b)$ be the class of functions $G_s(z)$ analytic in $\mathcal{D}$ and defined by (2.6) where $f(z) \in \sum_{(p,n)}^{(1)}(\alpha,\gamma,\eta,b)$. From (3.1) and definition of subordination we have:

Theorem 3.1. *$G_s(z) \in \sum_{p,n}^{s}(\alpha,\gamma,\eta,b)$ if and only if*

$$\frac{[zG_s'(z)]'}{z^{p-1}} = \frac{p(s+p)+[sp(p+1)b(\alpha-\gamma)(p-\eta)+p\gamma(s+p)]\omega(z)}{1+\gamma\omega(z)}. \tag{12}$$

Proof. The conditions (25) and (3.1) give

$$G_s'(z) = pz^{p-1}(1-s) + s(p+1)\frac{f(z)}{z}$$

or

$$z\ G_s'(z) = p(1-s)z^p + s(p+1)f(z)$$

or

$$f(z) = \frac{z\ G_s'(z)}{s(p+1)} - \frac{p(1-s)}{s(p+1)}z^p$$

hence

$$f'(z) = \frac{1}{s(p+1)}(G_s'(z) + z\ G_s''(z)) - \frac{p^2(1-s)}{s(p+1)}z^{p-1}.$$

By putting $f'(z)$ in (3.1) we obtain

$$\begin{aligned}\frac{f'(z)}{p\ z^{p-1}} &= \frac{1}{s(p+1)}\left(\frac{G_s'(z)+z\ G_s''(z)}{z^{p-1}}\right) - \frac{p(1-s)}{s(p+1)} \\ &= \frac{1+[\gamma+b(\alpha-\gamma)(p-\eta)]\omega(z)}{1+\gamma\omega(z)}\end{aligned}$$

or

$$\begin{aligned}&\frac{G_s'(z)+z\ G_s''(z)}{z^{p-1}} = \\ &= \frac{[z\ G_s'(z)]'}{z^{p-1}} \\ &= p^2(1-s) + \frac{sp(p+1)+sp(p+1)[\gamma+b(\alpha-\gamma)(p-\eta)]\omega(z)}{1+\gamma\omega(z)}\end{aligned}$$

if $z\ G_s'(z)=\Phi(z)$ then we have

$$\frac{\Phi'(z)}{z^{p-1}}=\frac{p(s+p)+[sp(p+1)b(\alpha-\gamma)(p-\eta)+p\gamma(s+p)]\omega(z)}{1+\gamma\omega(z)}$$

and the proof is complete. □

In the next theorem we obtain the distortion bounds for $G_s'(z)$.

Theorem 3.2. *Let $G_s(z)\in\sum_{p,n}^{s}(\alpha,\gamma,\eta,b)$. Then for $|z|=r<1$ we have*

$$|G_s(z)|\leq\frac{(s+p)}{p(1-|\gamma|)}r^p+[sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)]\int_0^r\int_0^r\frac{t^p}{u(1-|\gamma|)}dtdu$$

and

$$|G_s(z)|\geq\frac{(s+p)}{p(1-|\gamma|)}r^p-[sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)]\int_0^r\int_0^r\frac{t^p}{u(1+|\gamma|)}dtdu.$$

Proof. If $z\ G_s'(z)=\Phi(z),\quad |z|=r<1$ by (3.3) we have

$$|\frac{\Phi'(z)}{z^{p-1}}|\leq\frac{p(s+p)+[sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)]r}{1-|\gamma|r}$$

or

$$\begin{aligned}|(z\ G_s'(z))'| &= |\Phi'(z)|\\ &< \frac{p(s+p)}{1-|\gamma|r}|z|^{p-1}+\frac{[sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)]r}{1-|\gamma|r}|z|^{p-1}.\end{aligned}$$

By choosing positive γ's $(1-|\gamma|r>1-|\gamma|)$ we obtain

$$\begin{aligned}|(z\ G_s'(z))'| &= |\Phi'(z)|\\ &< \frac{p(s+p)}{1-|\gamma|}r^{p-1}+\frac{[sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)]r}{1-|\gamma|}r^p\end{aligned}$$

then by the integration

$$\begin{aligned}&|z\ G_s'(z)|\\ \leq\ &\int_0^r|(tG_s'(t))'|dt\\ \leq\ &\frac{(s+p)}{1-|\gamma|}r^p+\int_0^r\frac{sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)}{1-|\gamma|t}t^pdt\\ =\ &\frac{(s+p)}{1-|\gamma|}r^p+[sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)]\int_0^r\frac{t^p}{1-|\gamma|t}dt,\end{aligned}$$

therefore

$$|G_s'(z)|\leq\frac{(s+p)}{1-|\gamma|}r^{p-1}+[\frac{sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)}{r}]\int_0^r\frac{t^p}{1-|\gamma|t}dt.$$

By integration

$$|G_s(z)| \leq \frac{(s+p)}{p(1-|\gamma|)} r^p + [sp(p+1)|b|(\alpha-\gamma)(p-\eta)+p|\gamma|(s+p)] \int_0^r \int_0^r \frac{t^p}{u(1-|\gamma|)} dt du$$

in the same way we can prove the other part, and so the proof is complete. □

Since $G_s(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k$ where $b_k = \frac{(p+1)s}{k+1} a_k$ we can estimate the Feketo-Szego problem for the coefficient of $G_s(z)$ in the last theorem.

Theorem 3.3. *Let $G_s(z)$ given by (2.5) be in the class $\sum_{p,n}^{s}(\alpha, \gamma, \eta, b)$ and d be any complex number, then*

$$|b_{n+p+1} - d b_{n+p}^2| \leq \frac{(p+1)sb(\alpha-\gamma)(p-\eta)}{B(n+p+1)} max\{1, |h|\}$$

where $h = \gamma + d\frac{s(p+1)b(\alpha-\gamma)(p-\eta)B}{A^2(n+p+1)}$, $A = \delta_p(n+p, \lambda)$, $B = \delta_p(n+p+1, \lambda)$.

Proof. By equating the coefficient of z^{n+p} and z^{n+p-1} in (2.4), we have $Aa_{n+p} = b(\alpha-\gamma)(p-\eta)\omega_1$ where $A = \delta_p(n+p, \lambda) = \frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}$ therefore

$$b_{n+p} = \frac{(p+1)s}{n+p+1} a_{n+p} = \frac{s(p+1)b(\alpha-\gamma)(p-\eta)\omega_1}{A(n+p+1)},$$

also we have

$$\delta_p(n+p+1, \lambda) a_{n+p+1} = b(\alpha-\gamma)(p-\eta)\omega_2 - \gamma \delta_p(n+p, \lambda) a_{n+p} \omega_1$$

or

$$Ba_{n+p+1} = b(\alpha-\gamma)(p-\eta)\omega_2 - \gamma A a_{n+p} \omega_1$$

where $B = \delta_p(n+p+1, \lambda) = \frac{\Gamma(n+p+2)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+2-\lambda)}$ or

$$a_{n+p+1} = \frac{b(\alpha-\gamma)(p-\eta)\omega_2}{B} - \frac{\gamma A}{B} a_{n+p} \omega_1$$

or

$$b_{n+p+1} = \frac{(p+1)s}{n+p+1} [\frac{b(\alpha-\gamma)(p-\eta)\omega_2}{B} - \frac{\gamma A}{B} \frac{n+p+1}{(p+1)s} b_{n+p} \omega_1]$$

or

$$b_{n+p+1} = \frac{(p+1)sb(\alpha-\gamma)(p-\eta)}{B(n+p+1)} [\omega_2 - \gamma \omega_1^2].$$

Then we have

$$\begin{aligned} &|b_{n+p+1} - d b_{n+p}^2| \\ &= |\frac{(p+1)sb(\alpha-\gamma)(p-\eta)}{B(n+p+1)} (\omega_2 - \gamma \omega_1^2) - d[\frac{(p+1)sb(\alpha-\gamma)(p-\eta)}{A(n+p+1)} \omega_1]^2| \\ &= |\frac{(p+1)sb(\alpha-\gamma)(p-\eta)}{B(n+p+1)} [\omega_2 - (\gamma + d\frac{s(p+1)b(\alpha-\gamma)(p-\eta)B}{A^2(n+p+1)}) \omega_1^2|. \end{aligned}$$

By using the fact that for every complex number h we have $|\omega_2 - h\omega_1^2| \leq max\{1, |h|\}$ (see [3]). We have

$$|b_{n+p+1} - db_{n+p}^2| \leq \frac{(p+1)sb(\alpha-\gamma)(p-\eta)}{B(n+p+1)} max\{1, |h|\}$$

where $h = \gamma + d\frac{s(p+1)b(\alpha-\gamma)(p-\eta)B}{A^2(n+p+1)}$. So, the proof is complete. □

References

[1] Aouf, M.K., On a new criterion for univalent functions of order alpha. Rendiconti de Mathematica, Roma, Serie VII, (11) (1991), 47-59.

[2] Benerji, P.K., Shenan, G. M., Certain generalized subclass of analytic and multivalent functions in terms of fractional calculus. Proc. Nat. Acad. Sci. India, 71(A), IV (2001), 321-338.

[3] Keogh, F.R., Merkes, E.P., A coefficient inequality for certain classes of analytic functions. Proc. Amer. Math. Soc., 20 (1969), 8-12.

[4] Mogra, M.L., On a special class of p-valent analytic functions. Bull. Inst. Math. Academia Sinica, 14 (1986), 51-65.

[5] Patel, J., On a class of p-valent functions with negative and missing coefficients. Kyungpook Math. J, 36 (1996), 29-40.

[6] Patel, J., Mohanty, A.K., On a class of p-valent analytic functions with complex order. Kyunpook Math. J, 43 (2003), 199-209.

[7] Sohi, N.S., A class of p-valent analytic functions, India J. Pure Appl. Math, 10 (1979), 826-834.

Received by the editors September 1, 2008