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CYCLIC GROUP ACTIONS ON Spin^c BUNDLES[1]

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Abstract. Let L be a complex line bundle over a closed, oriented Riemannian 4-manifold X with $c_1(L) = \omega_2(TX)$ mod 2. Let $\mathbf{Z}_p$ be a cyclic group of order p(p is prime) that acts on X as orientation preserving isometry with an oriented, connected, compact 2-dimensional submanifold Σ as a fixed point set and on L such that the projection $L \to X$ is a $\mathbf{Z}_p$-map. In this paper, we investigate the action of $\mathbf{Z}_p$ on the Seiberg-Witten equations, and obtain a relation of the dimension of the moduli space of the quotient bundle and its pull-back bundle. Also, we discuss the Seiberg-Witten invariant of the quotient bundle when X is a Kähler manifold.

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1. Introduction

Let X be a closed, oriented Riemannian 4-manifold and let $L \to X$ be a complex line bundle on X with $c_1(L) = \omega_2(TX)$ mod 2. Then there is a principal Spin$^c(4)$-bundle ξ on X with $\det \xi = L$ and the twisted $(\pm\frac{1}{2})$-spinor bundles $W^{\pm}$ (associated to L).

In [8], Seiberg and Witten introduced a new kind of differential-geometric equation on the unitary connections on L and the sections of W^+. The space of solution of the Seiberg-Witten equations defines an invariant on X, the so-called Seiberg-Witten invariant. In [10], Witten showed that Kähler surfaces have nontrivial Seiberg-Witten invariants. In [9], Wang showed that the Seiberg-Witten invariant vanishes on the quotient manifold X/σ if $\sigma : X \to X$ is a free, anti-holomorphic involution on a Kähler surface X with $b_2^+(X) > 3$ and $K_X^2 > 0$. In [4], Cho showed that when $p = 2$, for a Kähler surface X with $b_2^+(X) > 3$ and $H_2(X; Z)$ has no 2-torsion and on which an anti-holomorphic involution acts with fixed point set Σ, a Lagrangian surface with genus greater than 0 and $[\Sigma] \in 2H_2(X, Z)$, if $K_X^2 > 0$ or $K_X^2 = 0$ and the genus $g(\Sigma) > 1$, then the Seiberg-Witten invariant of the quotient manifold X' vanishes. When $K_X^2 = 0$ and the genus $g(\Sigma) = 1$, if there is a $\mathbf{Z}_2$-equivariant Spin^c-structure ξ on X

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whose virtual dimension of the Seiberg-Witten moduli space is zero, then there is a Spin^c-structure ξ' on X' such that the Seiberg-Witten invariant is ± 1.

In this paper, we investigate a finite group $\mathbf{Z}_p$(p is prime) action on X with an embedded Riemannian surface Σ as fixed point set. Let $\mathbf{Z}_p$ act on L in which $L \to X$ is a $\mathbf{Z}_p$-map. In section 2, we give some preliminaries to prove the main theorem and calculate the dimension of the $\mathbf{Z}_p$-invariant moduli space of pull-back bundle of the quotient bundle. In section 3, we calculate the dimension of the moduli space of the quotient bundle and obtain the relation of the dimension of the moduli space of the quotient bundle and the dimension of the $\mathbf{Z}_p$-invariant moduli space of pull-back bundle of the quotient bundle. Besides, if X is a Kähler manifold, we have $SW(L/h) = \pm 1$ under some conditions.

2. Cyclic group actions on the Spin^c-bundles

Let X be a closed, oriented Riemannian 4-manifold and let $L \to X$ be a complex line bundle on X with $c_1(L) = \omega_2(TX)$ mod 2. Then there is a principal Spin$^c(4)$-bundle ξ on X with $\det \xi = L$ and the twisted $(\pm\frac{1}{2})$-spinor bundles $W^{\pm}$ (associated to L).

Let P_L be the principal $U(1)$-bundle associated to L and $P_{SO(4)}$ be the orthonormal frame bundle associated to the tangent bundle TX of X.

Let $\mathcal{A}(L)$ be the set of all Riemannian connections on L and $\Gamma(W^+(\xi))$ be the space of all sections of $W^+(\xi) \to X$. The gauge group $\mathcal{G}(L)$ of all bundle automorphisms on L, acts on the space $\mathcal{A}(L) \times \Gamma(W^+(\xi))$.

For a positive spinor field $\varphi \in \Gamma(W^+(\xi))$ and a unitary connection A on L, the Seiberg-Witten equations are defined by $F_A^+ = q(\varphi)$, $D_A(\varphi) = 0$, where $D_A : \Gamma(W^+(\xi)) \to \Gamma(W^-(\xi))$ is the Dirac operator associated to the connection A. It is the composition of the covariant derivative ∇_A on $\Gamma(W^+(\xi))$ and the Clifford multiplication. $q : C^\infty(W^+(\xi)) \to \Omega_X^+(i\mathbf{R})$ is a quadratic map defined by $q(\varphi) = \varphi \bigotimes \varphi^* - \frac{\|\varphi\|^2}{2} Id$.

Let $SW(\xi)$ be the set of all solutions of the Seiberg-Witten equations. Then the gauge group $\mathcal{G}(L)$ acts on $SW(\xi)$ and defines the moduli space $M(\xi)$ by $SW(\xi)/\mathcal{G}(L)$ of the gauge equivalence classes of all solutions of the Seiberg-Witten equations. We consider perturbed Seiberg-Witten equations such as $F_A^+ + i\delta = q(\varphi)$, $D_A(\varphi) = 0$, where δ is a smooth, real valued, self-dual two-form on X.

For a generic self-dual two-form δ, the perturbed moduli space $M_\delta(\xi)$ of the gauge equivalence classes of all irreducible solutions of the perturbed Seiberg-Witten equations is a smooth manifold with its dimension $\frac{1}{4}(c_1(L)^2 - (2\chi + 3\mathrm{sign})_X)$. If the metric on X is chosen so that the perturbed Seiberg-Witten equations admit no reducible solutions, which can be achieved in a path-connected subset of metrics if $b_2^+(X) > 1$, then $M_\delta(\xi)$ will be compact. In this situation if $\dim M_\delta(\xi) = 2d \geq 0$, $d \in \mathbf{Z}$ then we define the Seiberg-Witten invariant $SW(\xi)$ such as

$$SW(\xi) = \int_{M_\delta(\xi)} c_1(M_\delta(\xi)_0)^d$$

where $M_\delta(\xi)_0$ is the framed moduli space. For detail see [6].

In general, there are infinitely many elements $c_1(L) \in H^2(X;Z)$ satisfying $c_1(L) = \omega_2(TX)$ mod 2. Each such element induces a Spin^c-structure on X. However, there are only finitely many elements in $H^2(X;Z)$ such that their Seiberg-Witten invariants are non-zero. Such an element in $H^2(X;Z)$ is called a basic class. So the set of basic classes is finite. Furthermore, X is said to be of simple type if all basic classes satisfy $c_1(L)^2[X] = (2\chi + 3\text{sign})_X$. In particular, if $H^2(X;Z)$ has no 2-torsion then there is a one-to-one correspondence between the characteristic line bundle L satisfying $c_1(L) \equiv \omega_2(TX)$ mod 2 and the Spin^c-structure ξ.

Suppose that a cyclic group $\mathbf{Z}_p$ (p is prime) acts on X by an orientation preserving isometry. The induced action of $\mathbf{Z}_p$ on the frame bundle $P_{SO(4)}$ commutes with the right action of $SO(4)$ on $P_{SO(4)}$. Choose a $\mathbf{Z}_p$ action on $P_L \to X$ which is compatible with the $\mathbf{Z}_p$ action on X, and commutes with the canonical right action of $U(1)$ on P_L. If the induced $\mathbf{Z}_p$ action on the product $P_{SO(4)} \times P_L$ lifts to an action on ξ then we say that ξ is preserved by $\mathbf{Z}_p$ action. Note that the induced action on ξ might have to form a larger group. Thus we say that ξ is $\mathbf{Z}_p$-equivariant if the $\mathbf{Z}_p$ action on $P_{SO(4)} \times P_L$ lifts to a $\mathbf{Z}_p$ action on ξ.

Since Spin$^c(4)$ is a 2-fold covering space of $SO(4) \times U(1)$, by Bredon [2] the $\mathbf{Z}_p$ action on $P_{SO(4)} \times P_L$ can be lifted to an action of some group $\triangle$ on ξ which is an extension as follows

$$0 \to \mathbf{Z}_2 \to \triangle \to \mathbf{Z}_p \to 0. \tag{2.1}$$

If p is odd prime, then $\triangle \cong \mathbf{Z}_{2p} \cong \mathbf{Z}_2 \times \mathbf{Z}_p$ and so there is a subgroup of $\triangle$ which is isomorphic to $\mathbf{Z}_p$ and then ξ is $\mathbf{Z}_p$-equivariant. If $p = 2$, we can not always get a $\mathbf{Z}_p$-equivariant Spin^c-structure ξ because the exact sequence (1) is non-trivial and $\triangle \cong \mathbf{Z}_4$ is not isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

From now on we will assume that $\mathbf{Z}_p$ acts on X as an orientation preserving isometry with an embedded surface Σ as fixed point set and the Spin^c-structure ξ is $\mathbf{Z}_p$-equivariant. For simplicity we choose a metric on X which is $\mathbf{Z}_p$-invariant. Since ξ is $\mathbf{Z}_p$-equivariant, there are $\mathbf{Z}_p$ actions $\widetilde{h}$ on ξ and h on L. Then there is a $\mathbf{Z}_p$ action $h^* : \mathcal{A}(L) \times \Gamma(W^+(\xi)) \to \mathcal{A}(L) \times \Gamma(W^+(\xi))$ defined by

$$h^*(A, \varphi) = (A - h^{-1}dh, \tilde{h}^{-1} \circ \varphi \circ \sigma),$$

where $\sigma : X \to X$ is the $\mathbf{Z}_p$ action on X.

Since $\mathbf{Z}_p$ acts on X and L as orientation-preserving isometries, for each point $x \in \Sigma$ there exists a number $\alpha = \frac{m}{p}$, $m = 0, \cdots, p-1$, such that for the $\mathbf{Z}_p$ action h on L

$$h = \exp 2\pi i(\alpha) : L|_x \to L|_x.$$

Let $\mathcal{A}(L) \times \Gamma(W^+(\xi)))^{\mathbf{Z}_p}$ be the fixed point set of the $\mathbf{Z}_p$ action h^* on $\mathcal{A}(L) \times \Gamma(W^+(\xi)))$. Since $h = \exp 2\pi i(\frac{m}{p})$ on the restriction $L|_\Sigma$ of L over the fixed point set Σ, we can consider p different $\mathbf{Z}_p$-invariant spaces $\mathcal{A}(L)\times\Gamma(W^+(\xi)))^{\mathbf{Z}_p}$ denoted by

$$\mathcal{A}(L)\times\Gamma(W^+(\xi)))^{\mathbf{Z}_p}\equiv\mathcal{A}(L)\times\Gamma(W^+(\xi)))^{h(\frac{m}{p})}$$

on which $h=\exp 2\pi i(\frac{m}{p})$ on $L|_\Sigma$, $m=0,\cdots,p-1$.

Consider the $\mathbf{Z}_p$-invariant solution set

$$SW(\xi)^{h(\frac{m}{p})}\subset\mathcal{A}(L)\times\Gamma(W^+(\xi)))^{h(\frac{m}{p})}$$

of the Seiberg-Witten equations. Similarly, let $\mathcal{G}(L)^{h(\frac{m}{p})}$ be the $\mathbf{Z}_p$-invariant gauge group depending on the action $h=\exp 2\pi i(\frac{m}{p})$ on $L|_\Sigma$. Thus we can consider p different $\mathbf{Z}_p$-invariant moduli spaces

$$M(\xi)^{\mathbf{Z}_p}=M(\xi)^{h(\frac{m}{p})}=SW(\xi)^{h(\frac{m}{p})}/\mathcal{G}(L)^{h(\frac{m}{p})}, m=0,\cdots,p-1.$$

In [5], Cho calculates the dimension of the $\mathbf{Z}_p$-invariant moduli space $M(\xi)^{\mathbf{Z}_2}$ by the Lefschetz theorem of Atiyah-Segal [7] and Atiyah-Singer G-index theorem.

Theorem 2.1. *[5] In the above notations, the virtual dimension of the $\mathbf{Z}_p$-invariant moduli space $dim M(\tilde{P})_{\tau'}$, $\tau^{'}=s\tau\in\mathbf{Z}_p$, is*

$$\begin{aligned}\dim M(\tilde{P})_{\tau'} &= \frac{1}{p}[\dim M(\tilde{P})-\frac{p-1}{2}\chi(\Sigma)\\ &+\frac{1}{2}\sum_{k=1}^{p-1}\frac{-1+cos\frac{k\theta_L}{2}cos\frac{k\theta_N}{2}}{sin^2\frac{k\theta_N}{2}}\Sigma\cdot\Sigma+\frac{1}{2}\sum_{k=1}^{p-1}\frac{sin\frac{k\theta_L}{2}}{sin\frac{k\theta_N}{2}}c_1(L)[\Sigma]],\end{aligned}$$

and $\sum_{k=1}^{p-1}\frac{cos\frac{k\theta_L}{2}}{sin^2\frac{k\theta_N}{2}}c_1(L)[\Sigma]=\sum_{k=1}^{p-1}\frac{sin\frac{k\theta_L}{2}cos\frac{k\theta_N}{2}}{sin^2\frac{k\theta_N}{2}}\Sigma\cdot\Sigma$, *where $k\theta_L$ and $k\theta_N$ are determined by the action* $\det\tau^{'k}=\exp k\theta_L i$ *on $L|_\Sigma$ and* $\sigma_*^k=\exp k\theta_N i$ *on N_Σ respectively, $k=1,\cdots,p-1$.*

Now we consider the projection map $\pi: X\to X/\sigma\equiv X'$ with $\pi(\Sigma)\equiv\Sigma'$ and the quotient bundle $L/h\to X'$ where $\sigma: X\to X$ and $h: L\to L$ are the $\mathbf{Z}_p$ actions as before. We can always find a smooth manifold structure on the quotient manifold X' (for details see [10] and [3]).

By Section 3 in [1] and Proposition 1.4 in [10] we have a relation

$$\omega_2(TX)=\begin{cases}\pi^*\omega_2(TX')-PD^{-1}[\Sigma] & \text{if } p=2\\ \pi^*\omega_2(TX') & \text{if } p\neq 2\end{cases}$$

of the second Stieffel-Whitney class, where $PD: H^2(X;\mathbf{Z}_2)\to H_2(X;\mathbf{Z}_2)$ is the Poincaré duality map and $[\Sigma]\in H_2(X;\mathbf{Z}_2)$ is the class represented by the fixed point set Σ. Under the above assumption, we have the following result

Lemma 2.2. *If $[\Sigma]\in H_2(X;\mathbf{Z}_2)$ and $H_2(X';\mathbf{Z})$ has no 2-torsion then there is a $Spin^c$-structure $\tilde{\xi}$ associated with the pull-back bundle $\pi^*(L/h)\to X$.*

Proof. If $p=2$ and $[\Sigma]\in H_2(X;\mathbf{Z})$ then $c_1(L/h)\equiv\omega_2(TX')$ mod 2. If $p>2$ is a prime then $c_1(L/h)\equiv\omega_2(TX')$ mod 2. In this condition, since $H_2(X';\mathbf{Z})$ has no 2-torsion, there is a Spin^c-structure $\xi'\to X'$ whose determinant bundle is $L/h\to X'$.

So if $p>2$, $c_1(\pi^*(L/h))=\pi^*(c_1(L/h))\equiv\pi^*\omega_2(TX')=\omega_2(TX)$ mod 2.

If $p=2$, $c_1(\pi^*(L/h))\equiv\pi^*\omega_2(TX')=\omega_2(TX)+PD^{-1}[\Sigma]$ mod 2. Since $[\Sigma]\in 2H_2(X;\mathbf{Z})$ we have $c_1(\pi^*(L/h))\equiv\omega_2(TX)$ mod 2. Since $H_2(X;\mathbf{Z})$ has no 2-torsion then there is a one-to-one correspondence between the characteristic line bundle $\pi^*(L/h)$ and the Spin^c-structure ξ. □

From now we assume the Z_p-action h on L acts trivially on the restriction $L|_\Sigma$. Then we have the following result.

Lemma 2.3. *If the Z_p-action h on $L|_\Sigma$ satisfies that $h=Id$ then the dimension of the $\mathbf{Z}_p$-invariant moduli space $dimM(\tilde\xi)^{h(0)}$ is*

$$\dim M(\tilde\xi)^{h(0)}=\frac{1}{p}[\dim M(\tilde\xi)-\frac{p-1}{2}\chi(\Sigma)-\frac{1}{4}\sum_{k=1}^{p-1}\csc^2\frac{k\theta_N}{4}\Sigma\cdot\Sigma] \tag{2.2}$$

and $c_1(\pi^*(L/h))[\Sigma]=0$ where $k\theta_N$ is defined as Theorem 2.1.

Proof. Since the Z_p-action $h=Id$ on $L|_\Sigma$ then it is easy to know that $\pi^*(L/h)|_\Sigma\cong L|_\Sigma$, so $h=Id$ on $\pi^*(L/h)|_\Sigma$. Besides, we can also show that $\pi^*(L/h)/h\cong L/h$. Thus from Theorem 2.1 we obtain the result. □

3. Dimension of the moduli space of the quotient bundle

Let A' be the connection of the quotient bundle L/h and A is the connection of the pull-back bundle $\pi^*(L/h)$. Then

(3.1)
$$c_1^2(\pi^*(L/h))[X]=-\frac{1}{4\pi^2}\int_X F_A\wedge F_A=-\frac{p}{4\pi^2}\int_X' F'_A\wedge F'_A=pc_1^2(L/h)[X'].$$

Besides,

$$\chi(X')=\frac{1}{p}[\chi(X)+(p-1)\chi(X^g)]=\frac{1}{p}[\chi(X)+(p-1)\chi(\Sigma)]. \tag{3.2}$$

When $p\neq 2$, we have

$$\begin{aligned}\mathrm{sign}(X') &= \frac{1}{p}[\mathrm{sign}(X)+2\mathrm{sign}(g,X)+\cdots+\mathrm{sign}(g^{\frac{p-1}{2}},X)]\\ &= \frac{1}{p}[\mathrm{sign}(X)+2\sum_{K=1}^{\frac{p-1}{2}}\mathrm{sign}(g^k,X)]\\ &= \frac{1}{p}[\mathrm{sign}(X)+2\sum_{K=1}^{\frac{p-1}{2}}\csc^2\frac{k\pi}{p}\Sigma\cdot\Sigma]\end{aligned}$$

Then, from (2.2), (3.1), (3.2) and the above equation we can calculate the dimension of the moduli space of the quotient bundle L/h as follows

$$\begin{aligned}\dim M(\xi') &= \frac{1}{4}[c_1(L/h)^2 - 2\chi(X') - 3\mathrm{sign}(X')]\\ &= \frac{1}{4}[\frac{1}{p}c_1^2(\pi^*(L/h)) - \frac{2}{p}(\chi(X) + (p-1)\chi(\Sigma)) -\\ &\quad -\frac{3}{p}(\mathrm{sign}(X) + 2\sum_{K=1}^{\frac{p-1}{2}} \csc^2\frac{k\pi}{p}\Sigma\cdot\Sigma)]\\ &= \frac{1}{p}[\dim M(\tilde{\xi}) - \frac{p-1}{2}\chi(\Sigma) - \frac{3}{2}\sum_{K=1}^{\frac{p-1}{2}}\csc^2\frac{k\pi}{p}\Sigma\cdot\Sigma]\\ &= \dim M(\tilde{\xi})^h + \frac{1}{p}[\frac{1}{4}\sum_{k=1}^{p-1}\csc^2\frac{k\theta_N}{4} - \frac{3}{2}\sum_{K=1}^{\frac{p-1}{2}}\csc^2\frac{k\pi}{p}]\Sigma\cdot\Sigma\end{aligned}$$

and $c_1(\pi^*(L/h))[\Sigma] = 0$.

Thus, if $\Sigma\cdot\Sigma = 0$ then we have

$$\dim M(\xi') = \dim M(\tilde{\xi})^h.$$

Besides, since $h = Id$ on $L|_\Sigma$ then by Lemma 2.6 of [3] we have

$$\dim M(\xi') = \dim M(\xi)^h.$$

Thus,

$$\dim M(\xi') = \dim M(\tilde{\xi})^h = \dim M(\xi)^h.$$

When $p = 2$, we can also obtain this result in the same way.

Theorem 3.1. *Let X be a closed, oriented, smooth 4-manifold, σ is a $\mathbf{Z}_p$-action (p is prime) with fixed point set Σ which is a 2-dimension, compact submanifold and $[\Sigma] \in 2H_2(X;\mathbf{Z})$. We assumed that $H_2(X';\mathbf{Z})$ has no 2-torsion. If $\Sigma\cdot\Sigma = 0$ and $h = Id$ on $L|_\Sigma$, then we have*

$$\dim M(\xi') = \dim M(\tilde{\xi})^h = \dim M(\xi)^h$$

where ξ, ξ' and $\tilde{\xi}$ are the Spin^c-structure defined as before.

Moreover, if X is a Kähler surface and $\dim M(\tilde{\xi})^h = 0$, then for a generic σ-invariant metric g the invariant moduli space $M(\tilde{\xi})^h$ is a point over a Kähler surface and hence the moduli space $\dim M(\xi')$ is a point. Thus the Seiberg-Witten invariant for the Spin^c-structure ξ' is ± 1 on X'.

Corollary 3.2. *Let X be a Kähler surface and $\dim M(\tilde{\xi})^h = 0$, then under the same assumption as Theorem 2.3, we have $SW(L/h) = \pm 1$.*

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