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NEW EXTREMAL POLYNOMIALS AND THEIR APPROXIMATION PROPERTIES

M. Küçükaslan[1], F.G. Abdullayev[2]

Abstract. Let $G \subset \mathbb{C}$ be a simply connected region whose boundary $L := \partial G$ is a Jordan curve and $z_0 \in G$ be an arbitrary fixed point. Let $w = \varphi(z)$ be the conformal mapping of G onto the disk $D(0, r_0) := \{w : |w| < r_0\}$, satisfying $\varphi(z_0) = 0$, $\varphi'(z_0) = 1$. Let us consider the following extremal problem:

$$\|\varphi - P_n\|_{L_p'(G)} := \left\|\varphi' - P_n'\right\|_{L_p(G)} \to \min,\ p > 0, \tag{1}$$

in the class of all polynomials satisfying $P_n(z_0) = 0$ and $P_n'(z_0) = 1$. There exists a polynomial $\Pi_{n,p}(z)$ furnishing to the (1) and $\Pi_{n,p}(z)$ is determined uniquely when $p > 1$. This kind of polynomials will be called $p-$Bieberbach polynomials.

In this work, we investigate the approximation properties of the polynomials $\{\Pi_{n,p}(z)\}$ to the φ in the L_p^1- and $C-$norms for some regions of the complex plane.

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1. Statement of the Problem and Main Results

Let $G \subset \mathbb{C}$ be a simply connected region whose boundary $L := \partial G$ is a Jordan curve and $z_0 \in G$ be an arbitrary fixed point. Let $w = \varphi(z)$ ($w = \Phi(z)$) be the conformal mapping of G ($\Omega := C\overline{G}$) onto the disk $D(0, r_0) := \{w : |w| < r_0\}$ ($\Delta := C\overline{D(0,1)}$) with normalization $\varphi(z_0) = 0$, $\varphi'(z_0) = 1$ ($\Phi(\infty) = \infty, \Phi'(\infty) > 0$) and let $\psi := \varphi^{-1}$ ($\Psi := \Phi^{-1}$) be an inverse mapping.

Let $0 < p < \infty$. We denote by $L_p^1(G)$ the set of functions $f(z)$ analytic in G and satisfying $f(z_0) = 0$, such that

$$||f||^p_{L_p^1(G)} := ||f'||^p_{L_p(G)} := \iint\limits_G |f'(z)|^p d\sigma_z < \infty,$$

where $d\sigma_z$ denotes two-dimensional Lebesque measure.

[1]Department of Mathematics, Faculty of Science and Literature, Mersin University Mersin, 33343, TURKEY, e-mail: mkucukaslan@mersin.edu.tr

[2]Department of Mathematics, Faculty of Science and Literature, Mersin University, Mersin, 33343, TURKEY, e-mail: fabdul@mersin.edu.tr

Let us consider the following extremal problem:

$$\|\varphi - P_n\|_{L^1_p(G)} \to \min \qquad (2)$$

in the class $\wp_n$ of all polynomials $P_n(z)$, $\deg P_n(z) \leq n$, satisfying $P_n(z_0) = 0$ and $P_n'(z_0) = 1$.

Using a method similar to the one given in [10, p.137], it is seen that there exists a polynomial $\Pi_{n,p}(z) \in \wp_n$ furnishing to the problem (2), and if $p > 1$, these polynomials $\Pi_{n,p}(z)$ are determined uniquely [10, page 142]. We call such polynomials $\Pi_{n,p}(z)$ the $p-$Bieberbach polynomials of degree n for the pair (G, z_0).

The main goal in this work is to investigate the approximation rate of $\Pi_{n,p}(z)$ to the function φ in $C-$norm for some regions of the complex plane, i.e.

$$\|\varphi - \Pi_{n,p}\|_{C(\overline{G})} := \max\left\{|\varphi(z) - \Pi_{n,p}(z)| : z \in \overline{G}\right\} \to 0, \ n \to \infty. \qquad (3)$$

In case of $p = 2$ the solution of the extremal problem (2) coincides with the well known $n-$th Bieberbach polynomial $\pi_n(z) \equiv \Pi_{n,2}(z)$ for the pair (G, z_0) (see, for example, [19], [26] and [14]). The approximation properties in the $C-$norm of $\pi_n(z)$ on $\overline{G}$ was observed first by Keldysh in 1939 [19] for the regions with sufficiently smooth boundary. A considerable progress in this area has been achieved by Mergelyan [21], Suetin [26], Simonenko [24], Andrievskii [6], [7], Gaier [13], [14], Abdullayev [1], [3], [4] Israfilov [17], [18] and the others.

We shall consider the case $p > 1$ in the problem that was explained in (3). For this purpose, first, we will estimate the approximation rate of $\Pi_{n,p}(z)$ to the function φ in L^1_p-norm and then using the well known Simonenko and Andrievski method (see, for example, [6],[13]), the approximation rate of $\Pi_{n,p}(z)$ to the function φ in $C-$norm will be obtained.

Let us give some definitions.

Definition 1.1. *[20, p.97], The Jordan arc (or curve) L is called $K-$ quasiconformal $(K \geq 1)$, if there is a $K-$ quasiconformal mapping f of the region $H \supset L$ such that $f(L)$ is a line segment (or circle).*

$F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $H \supset L$ such that $f(L)$ is a line segment (or circle) and define

$$K_L := \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$ (see [23]).

We say that $\Psi \in Lip\beta$, for some β with $0 < \beta \leq 1$, if

$$|\Psi(w_1) - \Psi(w_2)| \leq c\,|w_1 - w_2|^{\beta}, \ 1 \leq |w_1|, |w_2| \leq 2,$$

where c is an independent constant of w_1, w_2. Similarly, $\varphi \in Lip\alpha$, form some α with $0 < \alpha \leq 1$, if

$$|\varphi(z_1) - \varphi(z_2)| \leq c\,|z_1 - z_2|^{\alpha}, \ z_1, z_2 \in \overline{G}.$$

Definition 1.2. *[14] We say that $G \in Q(\alpha,\beta)$, if L is a quasiconformal curve and $\varphi \in Lip\alpha$, $\Psi \in Lip\beta$ for some α,β with $0<\alpha,\beta \le 1$.*

Theorem 1.3. *Let $G \in Q(\alpha,\beta)$ for some α,β with $0<\alpha\le 1$ and $\frac{1}{2}\le\beta\le 1$. Then, the $p-$Bieberbach polynomials $\Pi_{n,p}(z)$ satisfy*

$$\|\varphi-\Pi_{n,p}\|_{C(\overline{G})} \le \frac{const.}{n^{\gamma}} \tag{4}$$

for any number $n=2,3,...,$ and γ with

$$\gamma \in \begin{cases} \left(0, \frac{2}{p\alpha}-\frac{\alpha\beta}{2}-(\beta-\frac{1}{2})(\frac{2}{p}-1)\right), & 1<p<2, \\ \left(0, \frac{\alpha\beta}{p}-\beta(1-\alpha)(1-\frac{2}{p})\right), & 2\le p<2+\frac{\alpha}{1-\alpha}. \end{cases}$$

Remark 1.4. *If G is a convex region then $\varphi \in Lip1$ [12, p.582] and $\Psi \in Lip1$[22, p.48]. So, (4) is satisfied with $\gamma \in (0,\frac{1}{p})$ for all $p>1$.*

Generally, any region with quasiconformal boundary belongs to the class $Q(\alpha,\beta)$. But quasiconformality coefficient of the curve is not known for this region. Now we can give a similar result that the approximation rate depends on the quasiconformality coefficient of the curve.

Theorem 1.5. *Let L be a $K-$quasiconformal curve. Then, the $p-$Bieberbach polynomials $\Pi_{n,p}(z)$ satisfy*

$$\|\varphi-\Pi_{n,p}\|_{C(\overline{G})} \le \frac{const.}{n^{\gamma}}$$

for any number $n=2,3,...,$ and γ with

$$\gamma \in \begin{cases} \left(0, \frac{1}{pK^2}-\frac{2K^2}{K^2+1}(\frac{2}{p}-1)\right), & 1<p<2, \\ \left(0, \frac{1}{pK^2}-\frac{K^2-1}{K^2(K^2+1)}(1-\frac{2}{p})\right), & 2\le p<2+\frac{K^2+1}{K^2-1}. \end{cases}$$

2. Some Auxiliary Results

Throughout this paper, $c, c_1, c_2, ...,$ are positive, and $\varepsilon, \varepsilon_1, \varepsilon_2, ...,$ sufficiently small positive constants, in general dependent on G. The notation "$a \prec b$" and "$a \asymp b$" will be used instead of "$a \le cb$" and "$c_1 a \le b \le c_2 a$" for some constants c, c_1, c_2, respectively.

The level curve (exterior or interior) can be defined for $t>0$ as,

$$L_t := \{z : |\varphi(z)| = t, \quad if\ t<r_0; |\phi(z)| = t, \quad if\ t>r_0\}$$

and $L_{r_0} := L$, $L_1 := L$ respectively. Let us denote $G_t := intL_t$, $\Omega_t := extL_t$ and $d(z,L) := \inf\{|\zeta - z| : \zeta \in L\}$.

Let L be a $K-$quasiconformal curve. Then there exists a K^2- quasiconformal reflection $y(.)$ across L [5, p.75] such that $y(G)=\Omega$, $y(\Omega)=G$ and the points on L are fixed.

On the other hand, there exists a $C(K)-$quasiconformal reflection $\alpha(.)$ across L (see, [5, p. 75] and [9]) such that

$$|z_1-\alpha(z)|\asymp|z_1-z|\,,\ z_1\in L,\ \varepsilon<|z|<\frac{1}{\varepsilon},$$

$$|\alpha_{\overline{z}}|\asymp|\alpha_z|\asymp 1,\ \varepsilon<|z|<\frac{1}{\varepsilon},$$

(5) $$|\alpha_{\overline{z}}|\asymp|\alpha(z)|^2\,,\ \ |z|<\varepsilon,\ \ |\alpha_{\overline{z}}|\asymp|z|^{-2}\,,|z|>\frac{1}{\varepsilon},$$

and the Jacobian $J_\alpha=|\alpha_z|^2-|\alpha_{\overline{z}}|^2$ of $\alpha(.)$ satisfies $J_\alpha\asymp 1$.

For $R>1$ let us denote $L_R^*:=\alpha(L_R),\ G_R^*=intL_R^*$ and $\Omega_R^*=extL_R^*$. Let $\Phi_R^*:\Omega_R^*\to\Delta$ be a conformal mapping with normalization $\Phi_R^*(\infty)=\infty$ and $\Phi_R^{*\prime}(\infty)>0$. According to [8] we have

(6) $$\begin{array}{rcl} d(z,L) & \asymp & d(t,L_R)\asymp d(z,L_R), \\ |\Phi_R^*(z)| & \leq & |\Phi_R^*(t)|\leq 1+c(R-1) \end{array}$$

for all $z\in L_R^*$ and $t\in L$ such that $d(z,L)=|z-t|$.

Lemma 2.1. *Let G be a quasiconformal curve; $r_*:=\min\left\{|\varphi(\alpha(z))|:z\in L_R\right\}$ and $r^*:=\max\left\{|\varphi(\alpha(z))|:z\in L_R\right\},R>1$. Then,*

(7) $$r_0-r_*\prec r_0-r^*.$$

Proof. Let us define $F(w):=\frac{r_0^2}{\overline{\varphi(\alpha(\Psi(w)))}}$ and extend it to the whole complex plane as follows:

(8) $$z=\widetilde{F}(w):=\begin{cases}\frac{r_0^2}{\overline{\varphi(\alpha(\Psi(w)))}}, & |w|\geq 1,\\ \varphi(\alpha(\Psi(\frac{1}{\overline{w}}))), & |w|<1.\end{cases}$$

Also, let us denote:

$$t:=w(1-\frac{1}{|w|}):\overline{\Delta}\to\left\{t:|t|\geq 0\right\},$$

$$\xi:=\widetilde{F}(w)-\widetilde{F}(\frac{w}{|w|}):\left\{w:|w|\geq r_0\right\}\to\left\{\xi:|\xi|\geq 0\right\},$$

and

$$\xi=\Phi(t):=\widetilde{F}(\frac{|t|+1}{|t|}t)-\widetilde{F}(\frac{t}{|t|}).$$

It is clear that $\Phi:\left\{t:|t|\geq 0\right\}\to\left\{\xi:|\xi|\geq 0\right\}$ quasiconformal and $\Phi(0)=0$, $\Phi(\infty)=\infty$. Taking into account $D-$properties of quasiconformal mapping [9] we have

$$\max_{|t|=R-1}|\Phi(t)|\leq c_1\min_{|t|=R-1}|\Phi(t)|.$$

Since $L = \partial G$ is a quasiconformal curve, then the function $\widetilde{F}$ is a quasiconformal mapping of the plane. So, we have

$$\frac{\max\limits_{|t|=R} |\widetilde{F}(w)| - r_0}{\min\limits_{|t|=R} |\widetilde{F}(w)| - r_0} \leq \frac{1}{c_2} \frac{\max\limits_{|t|=R} |\widetilde{F}(w) - \widetilde{F}(\frac{w}{|w|})|}{\min\limits_{|t|=R} |\widetilde{F}(w) - \widetilde{F}(\frac{w}{|w|})|} \leq \frac{c_1}{c_2} = c_3 \tag{9}$$

From (9) and (8) we have

$$c_3 \geq \frac{\max\limits_{|t|=R} \frac{r_0^2}{|\varphi(\alpha(\Psi(w)))|} - r_0}{\min\limits_{|t|=R} \frac{r_0^2}{|\varphi(\alpha(\Psi(w)))|} - r_0} = \frac{\max\limits_{|t|=R} (r_0 - |\varphi(\alpha(\Psi(w)))|)}{\min\limits_{|t|=R} (r_0 - |\varphi(\alpha(\Psi(w)))|)} =$$

$$= \frac{r_0 - \min\limits_{|t|=R} |\varphi(\alpha(\Psi(w)))|}{r_0 - \max\limits_{|t|=R} |\varphi(\alpha(\Psi(w)))|} = \frac{r_0 - r^*}{r_0 - r_*} \tag{10}$$

The inequality (10) gives the proof. □

Lemma 2.2. *[2] Let $L = \partial G$ be a quasiconformal curve. Then, for every $z \in L$ there exists an arc $\beta(z_0, z)$ in G joining z_0 to z with the following properties.*
i) $d(\xi, L) \asymp |\xi - z|$ for every $\xi \in \beta(z_0, z)$,
ii) If $\widetilde{\beta}(\xi_1, \xi_2)$ is the sub arc of $\beta(z_0, z)$ joining ξ_1 to ξ_2

$$mes\widetilde{\beta}(\xi_1, \xi_2) \prec |\xi_1 - \xi_2|$$

for every pair ξ_1 and $\xi_2 \in \beta(z_0, z)$.

Lemma 2.3. *Let $G \in Q(\alpha, \beta)$ for some α, β with $0 < \alpha, \beta \leq 1$. Then for all polynomials $P_n(z)$, $\deg P_n \leq n$ with $P_n(z_0) = 0$, we have*

$$\|P_n\|_{C(\overline{G})} \prec \|P_n\|_{L^1_p(G)} \begin{cases} 1, & p > 2, \\ \sqrt{\log n}, & p = 2, \\ n^{\frac{2}{p\alpha}}, & p < 2. \end{cases} \tag{11}$$

Proof. The proof for the case $p = 2$ and $p > 2$ was already given in [7], [16] respectively. We will only prove the case $p < 2$.

Let $z \in L$ be an arbitrary point. Since $G \in Q(\alpha, \beta)$ then $L = \partial G$ is quasiconformal, therefore according to Lemma 2.2 there exists $\beta(z_0, z) \subset G$ joining z_0 to z and satisfying the conditions in Lemma 2.2. Using mean-value property of the subharmonic function $|P_n'(\xi)|^p$ (see, for example [11, p.4]) we have

$$|P_n'(\xi)| \leq \frac{1}{(\pi d^2(\xi, L))^{\frac{1}{p}}} \|P_n\|_{L^1_p(G)}, \tag{12}$$

for every arbitrary point $\xi \in \beta(z_0, z)$.

At the same time,

$$|P_n(z)| = \left| \int\limits_{\beta(z_0,z)} P_n'(\xi)d\xi \right| \leq \int\limits_{\beta(z_0,z)} |P_n'(\xi)|\,|d\xi| \tag{13}$$

and combine (12) and (13) we have

$$|P_n(z)| \prec \|P_n\|_{L_p^1(G)} \int\limits_{\beta(z_0,z)} \frac{|d\xi|}{d^{\frac{2}{p}}(\xi,L)}. \tag{14}$$

According to Lemma 2.2 we obtain

$$d(\xi,L) \asymp |\xi - z| \succ |\varphi(\xi) - \varphi(z)|^{\frac{1}{\alpha}} \succ \left(\frac{1}{n}\right)^{\frac{1}{\alpha}} \tag{15}$$

From (14) and (15) we have

$$|P_n(z)| \prec \|P_n\|_{L_p^1(G)} \prec n^{\frac{2}{p\alpha}} \|P_n\|_{L_p^1(G)}\,.$$

Since $z \in L$ is an arbitrary point, taking maximum for $z \in \overline{G}$, we obtained the proof of (11) in the case $p < 2$. □

3. Approximation in the L_p^1−norm

Assume that the region G, bounded by a quasiconformal curve L and $1 < R' < 2$, be fixed. Using quasiconformal reflection $\alpha(.)$, defined as in (5), we can extend φ to the $extL$ as follows:

$$\widetilde{\varphi}(z) := \begin{cases} \varphi(z), & z \in \overline{G}, \\ \varphi(\alpha(z)), & z \in G_{R'} - \overline{G}. \end{cases}$$

Then,

$$\widetilde{\varphi}_{\overline{z}}(z) := \begin{cases} 0, & z \in G, \\ \varphi'(\alpha(z))\alpha_{\overline{z}}(z), & z \in G_{R'} - \overline{G}. \end{cases}$$

From the Cauchy-Pompeiu Formulas [20, p 148], we obtain:

$$\varphi(z) = \frac{1}{2\pi i} \int\limits_{L_{R'}} \frac{\widetilde{\varphi}(\xi)}{\xi - z} d\xi - \frac{1}{\pi} \iint\limits_{G_{R'} - \overline{G}} \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{\xi - z} d\sigma_\xi, \quad z \in G. \tag{16}$$

Let N be a sufficiently large natural number. For $n > N$ and arbitrary $0 < \varepsilon < 1$, let us choose $R = 1 + cn^{\varepsilon - 1}$ such that $1 < R < R'$. Then, $G_{R'} - \overline{G} = (G_{R'} - G_R) \cup \left(G_R - \overline{G}\right)$ and (16) can be shown as follows:

$$\varphi(z) = I_1(z) + I_2(z), \quad z \in G, \tag{17}$$

where

$$I_1(z) := \frac{1}{2\pi i} \int\limits_{L_{R'}} \frac{\widetilde{\varphi}(\xi)}{\xi - z} d\xi - \frac{1}{\pi} \iint\limits_{G_{R'} - G_R} \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{\xi - z} d\sigma_\xi,$$

and

$$I_2(z) := -\frac{1}{\pi} \iint\limits_{G_R - \overline{G}} \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{\xi - z} d\sigma_\xi.$$

Since $I_1(z)$ is analytic function in $\overline{G}$, there exists a polynomial $p_{n-1}(z)$, where $\deg p_{n-1} \leq n-1$ [25, p142], such that

$$|I_1'(z) - p_{n-1}(z)| \leq \frac{c}{n}. \tag{18}$$

Let $Q_n(z) := \int\limits_{z_0}^{z} p_{n-1}(t)dt$. Then, from (17) and (18) we have

$$|\varphi'(z) - Q_n'(z)| \leq \frac{c}{n} + |I_2'(z)|.$$

Taking integral over G of p-th power of above inequality we obtain

$$\iint\limits_G |\varphi'(z) - Q_n'(z)|^p \, d\sigma_z \prec \frac{1}{n^p} + \iint\limits_G |I_2'(z)|^p \, d\sigma_z. \tag{19}$$

The Hilbert transformation

$$(Tf)(z) := -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2} d\sigma_\xi$$

is a bounded linear operator from L_p to L_p for $p > 1$ and Calderun-Zygmund inequality (see [5, p. 98]) gives

$$\iint\limits_G \left| -\frac{1}{\pi} \iint\limits_{G_R - \overline{G}} \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{(\xi - z)^2} d\sigma_\xi \right|^p d\sigma_z \prec \iint\limits_{G_R - \overline{G}} |\varphi'(\alpha(\xi))|^p \, d\sigma_\xi. \tag{20}$$

So, from (19) and (20) we have

$$\iint\limits_G |\varphi'(z) - Q_n'(z)|^p \, d\sigma_z \prec \frac{1}{n^p} + \iint\limits_{G_R - \overline{G}} |\varphi'(\alpha(\xi))|^p \, d\sigma_\xi, \; p > 1. \tag{21}$$

Lemma 3.1. *Let $p > 1$ and $G \in Q(\alpha, \beta)$ for some α and β with $0 < \alpha \leq 1$, $\frac{1}{2} \leq \beta \leq 1$. Then, for any $n = 1, 2, ...$,*

$$\|\varphi - \Pi_{n,p}\|_{L_p^1(G)} \prec n^{-\mu},$$

where

$$\mu \in \begin{cases} \left(0, \frac{\alpha\beta}{2} + (\beta - \frac{1}{2})(\frac{2}{p} - 1)\right), & 1 < p < 2, \\ \left(0, \frac{\alpha\beta}{p} - \beta(1-\alpha)(1 - \frac{2}{p})\right), & 2 \leq p < 2 + \frac{\alpha}{1-\alpha}. \end{cases}$$

Proof. Since $L=\partial G$ is a quasiconformal curve, the estimation (21) is true for $G\in Q(\alpha,\beta)$. For the calculation the integral in the right-hand side in (21) we consider two cases of $p: 1<p<2$ and $p\geq 2$.

Case 1) $1<p<2$. Using Hölder inequality [27, p.105] we obtain

$$\begin{aligned}\iint\limits_{G_R-\overline{G}} \left|\varphi'(\alpha(\xi))\right|^p d\sigma_\xi &\prec \left(\iint\limits_{G_R-\overline{G}} \left|\varphi'(\alpha(\xi))\right|^2 d\sigma_\xi\right)^{\frac{p}{2}} \left(\iint\limits_{G_R-\overline{G}} d\sigma_\xi\right)^{1-\frac{p}{2}}\\ &\prec \left(\iint\limits_{\alpha(G_R-\overline{G})} \left|\varphi'(\xi)\right|^2 d\sigma_\xi\right)^{\frac{p}{2}} \left(\iint\limits_{\alpha(G_R-\overline{G})} d\sigma_\xi\right)^{1-\frac{p}{2}}\\ &= \left[mes\left(\varphi(\alpha(G_R-\overline{G}))\right)\right]^{\frac{p}{2}} \left[mes\left(\alpha(G_R-\overline{G})\right)\right]^{1-\frac{p}{2}} \qquad (22)\end{aligned}$$

Case1-i).

$$mes\left(\varphi(\alpha(G_R-\overline{G}))\right)\leq \pi r_0^2-\pi r_*^2\prec r_0-r_*. \qquad (23)$$

Let us denote points $w_*\in\varphi(\alpha(L_R))$, $|w_*|=r_*$ and w', $|w'|=r_0$ such that $|w_*-w'|=|w_*|-|w'|$ and let $z'=\psi(w')\in L$, $z_*=\psi(w_*)\in L_R^*$, $\widetilde{z}:=\alpha(z_*)$, $\widetilde{w}:=\Phi(\widetilde{z})$. Using boundary properties of the region and (5), we get

$$\begin{aligned} r_0-|w_*| &\leq |\varphi(z')-\varphi(z_*)|\\ &\leq |z'-z_*|^\alpha \asymp |z'-\widetilde{z}|^\alpha\\ &= |\Psi(w')-\Psi(\widetilde{w})|^\alpha \prec |w'-\widetilde{w}|^{\alpha\beta}\\ &\prec (R-1)^{\alpha\beta}\prec \left(\frac{1}{n}\right)^{\alpha\beta}. \qquad (24)\end{aligned}$$

From (24) and (23) we obtain

$$mes\varphi(\alpha(G_R-\overline{G}))\prec\left(\frac{1}{n}\right)^{\alpha\beta}. \qquad (25)$$

Case1-ii). According to (5) we have

$$\begin{aligned} mes\left(\alpha(G_R-G)\right) &= \iint\limits_{\alpha(G_R-\overline{G})} d\sigma_\xi\\ &\asymp \iint\limits_{G_R-\overline{G}} d\sigma_{\alpha(\xi)} = \iint\limits_{1<|w|<R} \left|\Psi'(w)\right|^2 d\sigma_w. \qquad (26)\end{aligned}$$

Let $|w|-1=|w-\widehat{w}|$, $|\widehat{w}|=1$ and $\widehat{z}=\Psi(\widehat{w})$. Then, according to [8] and known properties of quasiconformality we have

$$
\begin{aligned}
|\Psi'(w)| &\asymp \frac{d(\Psi(w),L)}{|w|-1} \\
&\asymp \frac{|\Psi(w)-\Psi(\widehat{w})|}{|w|-1} \\
&\prec \frac{|w-\widehat{w}|^{\beta}}{|w|-1} \prec \left(\frac{1}{|w|-1}\right)^{1-\beta} \qquad (27)
\end{aligned}
$$

Replacing (27) in (26) we obtain

$$
mes\,(\alpha(G_R-G)) \prec \iint\limits_{1<|w|<R} \left(\frac{1}{|w|-1}\right)^{2(1-\beta)} d\sigma_w \prec \left(\frac{1}{n}\right)^{2\beta-1} \qquad (28)
$$

Using (25), (28) and (22) we obtain the proof when $1<p<2$.

Case 2) $p \geq 2$. According to [2] and analogously to (27) we have

$$
\begin{aligned}
\iint\limits_{G_R-G} |\varphi'(\alpha(\xi))|^p\, d\sigma_\xi &\asymp \iint\limits_{\varphi(\alpha(G_R-G))} \left|\psi'(w)\right|^{2-p} d\sigma_w \\
&\asymp \iint\limits_{\varphi(\alpha(G_R-G))} \left(\frac{d(\psi(w),L)}{r_0-|w|}\right)^{2-p} d\sigma_w \\
&\prec \iint\limits_{\varphi(\alpha(G_R-G))} \left(\frac{1}{r_0-|w|}\right)^{(\frac{1}{\alpha}-1)(p-2)} d\sigma_w \\
&\leq \iint\limits_{r_*<|w|<r_0} \left(\frac{1}{r_0-|w|}\right)^{(\frac{1}{\alpha}-1)(p-2)} d\sigma_w \\
&\prec (r_0-r_*)^{1-(\frac{1}{\alpha}-1)(p-2)},
\end{aligned}
$$

where $p<2+\frac{\alpha}{1-\alpha}$. According to (7) in Lemma 2.1 we have $r_0-r_* \prec r_0-r^*$. So, using (6) and the same procedure as in the Case1-i, we have

$$
\begin{aligned}
\iint\limits_{G_R-G} |\varphi'(\alpha(\xi))|^p\, d\sigma_\xi &\prec (r_0-r_*)^{1-(\frac{1}{\alpha}-1)(p-2)} \\
&\prec (r_0-r^*)^{1-(\frac{1}{\alpha}-1)(p-2)} \\
&\prec \left(\frac{1}{n}\right)^{\alpha\beta-\beta(1-\alpha)(p-2)}
\end{aligned}
$$

Let us set,

$$
P_n(z) := Q_n(z) + (\varphi'(z_0) - Q_n'(z_0))(z-z_0).
$$

It is clear that $P_n(z)$ is a polynomial satisfying normalization conditions $P_n(z_0) = 0$, $P_n'(z_0) = 1$, and

$$\|\varphi - P_n\|_{L_p^1(G)} \prec \frac{1}{n} + \left(\frac{1}{n}\right)^{\mu} + |\varphi'(z_0) - Q_n'(z_0)|\,.$$

Using Mean Value Theorem we obtain

$$|\varphi'(z_0) - Q_n'(z_0)| \prec \frac{1}{\pi d^{\frac{2}{p}}(z_0, L)} \|\varphi' - Q_n'\|_{L_p(G)} \prec \frac{1}{n} + \left(\frac{1}{n}\right)^{\mu}.$$

Considering extremal properties of $p-$Bieberbach polynomials the proof is completed. □

Lemma 3.2. *Let $L = \partial G$ be a $K-$quasiconformal curve. Then, for any $n = 1, 2, ...,$*

$$\|\varphi - \Pi_{n,p}\|_{L_p^1(G)} \prec n^{-\mu}, \tag{29}$$

where

$$\mu \in \begin{cases} (0, \frac{1}{pK^2}) & 1 < p < 2, \\ (0, \frac{1}{pK^2} - \frac{K^2-1}{K^2(K^2+1)}(1 - \frac{2}{p})) & 2 \le p < 2 + \frac{K^2+1}{K^2-1}. \end{cases}$$

Proof. We are going to follow the same procedures as in Lemma 3.1 with using the own properties of quasiconformal curve. Then, there is a polynomial $Q_n(z)$, $\deg Q_n \le n$ and $Q_n(z_0) = 0$ satisfying (21).

Case 1) Let $1 < p < 2$. From (22) we have

$$\iint\limits_{G_R - G} |\varphi'(\alpha(\xi))|^p \, d\sigma_\xi \prec [mes\,(\varphi(\alpha(G_R - G)))]^{\frac{p}{2}} \,.\, [mes\,(\alpha(G_R - G))]^{1-\frac{p}{2}} \tag{30}$$

Case 1-i) Let us define $R^* = 1 + 2(R - 1)$ for $R > 1$ and $L_{R^*}^* = \alpha(L_{R^*})$. Let $\Phi_{R^*}^*$ be an appropriate conformal mapping $\Phi_{R^*}^* : \Omega_{R^*}^* \to \Delta$ normalized by $\Phi_{R^*}^*(\infty) = \infty, \Phi_{R^*}^{*\prime}(\infty) > 0$; $\Psi_{R^*}^* := \Phi_{R^*}^{*-1}$ and $S_{\widetilde{R}} := \left\{z : |\Phi_{R^*}^*(z)| = \widetilde{R}\right\}$.

Then,

$$\begin{aligned} mes\,(\varphi(\alpha(G_R - G))) &= mes\,\{[\varphi \circ \Psi_{R^*}^* \circ \Phi_{R^*}^* \circ \alpha]\,(G_R - G)\} \\ &= mes\,\{[\varphi \circ \Psi_{R^*}^*] \circ [(\Phi_{R^*}^* \circ \alpha)(G_R - G)]\}\,. \end{aligned}$$

The function φ can be extended to the $G_R \supset \overline{G}$ using the reflection $y(z)$ as a K^2-quasiconformal mapping as follows:

$$\widehat{\varphi}(z) := \begin{cases} \varphi(z) & z \in \overline{G}, \\ \frac{r_0^2}{\varphi(y(z))} & z \in G_R - \overline{G}. \end{cases}$$

Therefore, φ is a K^2-quasiconformal mapping in $\overline{G}$ and, since $\Psi_{R^*}^*$ is a conformal mapping in $\Omega_{R^*}^*$, then $\varphi \circ \Psi_{R^*}^*$ is a K^2-quasiconformal in $\Omega_{R^*}^* \cap \overline{G}$. From the Goldstein Theorem [15], we have

$$mes\,\{[\varphi \circ \Psi_{R^*}^*] \circ [(\Phi_{R^*}^* \circ \alpha)(G_R - G)]\} \prec \{mes([\Phi_{R^*}^* \circ \alpha]\,(G_R - G)\}^{\frac{1-\epsilon}{K^2}} \tag{31}$$

for an arbitrary small $\varepsilon > 0$.

According to [8] we can choose $\widetilde{R} > 1$ such that $intS_{\widetilde{R}} - intL^*_{R^*} \supset \alpha(G_R - G)$ and $\widetilde{R} - 1 \prec R - 1$. Then,

$$\begin{aligned} mes\,[\Phi^*_{R^*} \circ \alpha]\,(G_R - G) &\leq mes\,[\Phi^*_{R^*}\left(intS_{\widetilde{R}} - \overline{intL^*_{R^*}}\right)] \\ &\prec \widetilde{R} - 1 \prec R - 1 \asymp \frac{1}{n} \end{aligned} \tag{32}$$

From (31)-(32) we obtain

$$mes\,(\varphi(\alpha(G_R - G))) \prec \left(\frac{1}{n}\right)^{\frac{1-\epsilon}{K^2}} \tag{33}$$

Case 1-ii) $\Psi^*_{R^*}$ can be extended to the whole plane as a K^2-quasiconformal mapping and from the Goldstein Theorem [15], we have

$$\begin{aligned} mes\,(\alpha(G_R - G)) &= mes\{[\Psi^*_{R^*} \circ \Phi^*_{R^*} \circ \alpha]\,(G_R - G)\} \\ &\prec \{mes[(\Phi^*_{R^*} \circ \alpha)(G_R - G)]\}^{\frac{1-\epsilon}{K^2}} \\ &\prec \left(\frac{1}{n}\right)^{\frac{1-\epsilon}{K^2}} \end{aligned} \tag{34}$$

If we combine (33) and (34) in (30), Case 1 is obtained.

Case 2) Let $p \geq 2$.

Taking into account Lemma 2.1 in [2] we have

$$\left|\psi'(w)\right|^{2-p} \asymp \left(\frac{1}{r_0 - |w|}\right)^{\vartheta} \tag{35}$$

where $\vartheta := (p-2)\frac{K^2-1}{K^2+1}$. So, according to (5) and (35) we obtain

(36)

$$\iint\limits_{\alpha(G_R - G)} |\varphi'(\xi)|^p\, d\sigma_\xi \asymp \iint\limits_{\varphi(\alpha(G_R - G))} \left(\frac{1}{r_0 - |w|}\right)^{\vartheta} d\sigma_w \leq \iint\limits_{r_* < |w| < r_0} \left(\frac{1}{r_0 - |w|}\right)^{\vartheta} d\sigma_w.$$

Let $w = te^{i\theta}, r_* < t < r_0$ and $0 \leq \theta \leq 2\pi$. From (36) we have

$$\begin{aligned} \iint\limits_{\alpha(G_R - G)} |\varphi'(\xi)|^p\, d\sigma_\xi &\asymp \int\limits_0^{2\pi}\int\limits_{r_*}^{r_0} \left(\frac{1}{r_0 - t}\right)^{\vartheta} t\; dt d\theta \\ &= 2\pi \int\limits_{r_*}^{r_0} \left(\frac{1}{r_0 - t}\right)^{\vartheta} (t - r_0 + r_0)\; dt \\ &= 2\pi r_0 \int\limits_{r_*}^{r_0} \left(\frac{1}{r_0 - t}\right)^{\vartheta} dt - 2\pi \int\limits_{r_*}^{r_0} \left(\frac{1}{r_0 - t}\right)^{\vartheta - 1} dt \\ &\asymp (r_0 - r_*)^{1-\vartheta},\; \theta < 1. \end{aligned}$$

Taking into account Lemma 2.1 and Case 1-ii we have

$$\begin{aligned}
\iint\limits_{\alpha(G_R-G)} |\varphi'(\xi)|^p \, d\sigma_\xi &\prec (r_0 - r^*)^{1-\vartheta} \\
&\prec [mes\{w : r^* < |w| < r_0\}]^{1-\vartheta} \\
&\prec \{mes[(\varphi\circ\alpha)(G_R - G)]\}^{1-\vartheta} \\
&\prec \{mes[\varphi\circ\Psi^*_{R^*}\circ\Phi^*_{R^*}\circ\alpha](G_R - G)\}^{1-\vartheta} \\
&\prec \left(\frac{1}{n}\right)^{\frac{1-\vartheta}{K^2}}.
\end{aligned}$$

This gives Case 2 and if we define $P_n(z)$ as in Lemma 3.1, then using extremal properties of $\Pi_{n,p}(z)$ we obtain (29). □

We use a method similar to the one of Andrievskii and Simonenko employed in the proofs of the analogous theorems for $p = 2$ (see [7], [14] and [24]).

Lemma 3.3. *Let $G \subset \mathbb{C}$ be a simply connected region so that*

$$\|\varphi - \Pi_{n,p}\|_{L^1_p(G)} \prec n^{-\mu}$$

for each $\mu \in (0,1), n = 2, 3, ...,$ and

$$\|P_n\|_{C(\overline{G})} \prec \|P_n\|_{L^1_p(G)} \begin{cases} 1, & p > 2, \\ \sqrt{\log n}, & p = 2, \\ n^\eta, & \eta > 0, 0 < p < 2, \end{cases} \tag{37}$$

for all polynomials $P_n(z)$ of degree $\le n$ and normalized $P_n(z_0) = 0$. Then,

$$\|\varphi - \Pi_{n,p}\|_{C(\overline{G})} \prec n^{\eta-\mu}.$$

Proof. In fact, for each $\varkappa = \mu - \eta$ and natural numbers n, k with $2^k \le n \le 2^{k+1}$, by Lemma 3.1 and Lemma 3.2 we obtain

$$\left\|\Pi_{2^{k+1},p} - \Pi_{n,p}\right\|_{L^1_p(G)} \prec n^{-\mu}$$

and this, for each $j > k$

$$\left\|\Pi_{2^{j+1},p} - \Pi_{2^j,p}\right\|_{L^1_p(G)} \prec 2^{-j\mu}$$

Since,

$$\varphi(z) = \Pi_{2^{k+1},p} + \sum_{j=k+1}^{\infty} \left[\Pi_{2^{j+1},p} - \Pi_{2^j,p}\right], \; z \in G,$$

consequently

$$\begin{aligned}
\|\varphi - \Pi_{n,p}\|_{C(\overline{G})} &\le \left\|\Pi_{2^{k+1},p} - \Pi_{n,p}\right\| + \sum_{j=k+1}^{\infty} \left\|\Pi_{2^{j+1},p} - \Pi_{2^j,p}\right\| \\
&\prec n^{-\varkappa} + \sum_{j=k+1}^{\infty} 2^{(j+1)\eta - j\mu} \prec n^{-\varkappa}.
\end{aligned}$$

□

4. Proof of Theorem 1.3 and Theorem 1.5

Proof. To give the proof of Theorem 1.3 and Theorem 1.5, in the light of analogy given above, it is enough to choose suitable μ and η in (37) for any region.

Therefore, by taking μ from Lemma 3.1 (μ from Lemma 3.2), and η from Lemma 2.3 (η from [4, Lemma 2.4]) the proof of Theorem 1.3 (Theorem 1.5) can be obtained. □

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