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HARMONIC ANALYSIS OF FUNCTIONS

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Abstract. Sato's hyperfunctions are known to be represented as the boundary values of harmonic functions as well as those of holomorphic functions. The author obtains a bijective Poisson mapping

$$P : \mathcal{S}^{*\prime}(\mathbb{R}^n) \longrightarrow \mathcal{S}^{*\prime}(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n)$$

where $H(S^*\mathbb{R}^n)$ is a kind of Hardy subspace of $\mathcal{B}(S^*\mathbb{R}^n)$. Moreover, the author has an isomorphism between Sobolev spaces

$$P : W^s(\mathbb{R}^n) \longrightarrow W^{s+(n-1)/4}(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n).$$

There are some similar results in case of other functions.

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0. Introduction

The purpose of this paper is to characterize the regularity of a certain type of functions including ultradistributions by their harmonic prolongation. This kind of work can be traced back to the classical Fourier Analysis. However, the role of the work will never have finished as long as one makes new functions to be applied to the spectral analysis. In order to understand this, we have only to mention that M. Sato described hyperfunctions as boundary values of harmonic functions in his earliest paper. We treat here some classes of hyperfunctions defined on the unit sphere S^n of $\mathbb{R}^{n+1}$ and those defined on $\mathbb{R}^n$. And we can see here a useful characterization of the singular spectrum and harmonic expansions. These results can be used for the elementary treatment of the theory of algebraic analysis. Moreover, we are interested in how much regularity of those functions can be changed by Poisson and Hörmander operators from the viewpoint of Sobolev spaces. G. Lebeau [9] treated hyperfunctions on the unit sphere. The author applies his method to the functions on $\mathbb{R}^n$ and he obtains a bijective Poisson mapping for the tempered ultradistributions and for Sobolev spaces.

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1. Ultradistributions

Let Ω be an open subset of $\mathbb{R}^n$ whose point is denoted by $x = (x_1, \ldots, x_n)$. We use $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \partial_j = \frac{\partial}{\partial x_j}$, $j = 1, \cdots, n$.

Definition 1.1. Let $u \in C^\infty(\Omega)$. Then we say that u is in $\mathcal{E}^{\{s\}}(\Omega)$(the space of ultradifferentiable functions of class $\{s\}$) ($s \geqq 1$) if for any compact subset K of Ω there are positive constants C and h such that

$$\sup\{|\partial^\alpha u(x)|; x \in K\} \leqq Ch^{|\alpha|}\alpha!^s.$$

Also we say that u is in $\mathcal{E}^{(s)}(\Omega)$ (the space of ultradifferentiable functions of class (s)) ($s \geqq 1$) if for any compact subset K of Ω and for any positive number h there is a constant C such that the above inequality holds. We denote by $\mathcal{D}^{\{s\}}(\Omega)$ and $\mathcal{D}^{(s)}(\Omega)$ the subspaces of $\mathcal{E}^{\{s\}}(\Omega)$ and $\mathcal{E}^{(s)}(\Omega)$ respectively, which consist of functions with compact support in Ω.

We remark that $\mathcal{E}^{\{1\}}(\Omega)$ (or $\mathcal{A}(\Omega)$) is the space of all real analytic functions defined in Ω. We have the inclusion

$$\mathcal{E}^{(s)}(\Omega) \subset \mathcal{E}^{\{s\}}(\Omega) \subset \mathcal{E}^{(t)}(\Omega) \quad 1 \leqq s < t,$$

$$\mathcal{D}^{(s)}(\Omega) \subset \mathcal{D}^{\{s\}}(\Omega) \subset \mathcal{D}^{(t)}(\Omega) \quad 1 < s < t.$$

The topology of the spaces $\mathcal{E}^{\{s\}}(\Omega)$, $\mathcal{E}^{(s)}(\Omega)$ for $s \geqq 1$ and $\mathcal{D}^{\{s\}}(\Omega)$, $\mathcal{D}^{(s)}(\Omega)$ for $s > 1$ is defined as follows:

(i) We say that a sequence $\{f_j(x)\} \subset \mathcal{E}^{\{s\}}(\Omega)$ converges to zero in $\mathcal{E}^{\{s\}}(\Omega)$ ($s \geqq 1$) if for any compact subset K of Ω there is a constant $h > 0$ such that

$$\sup\{|\partial^\alpha f_j(x)|/(h^{|\alpha|}|\alpha|!^s); x \in K, \alpha\} \to 0 \text{ as } j \to \infty$$

(ii) We say that a sequence $\{f_j(x)\} \subset \mathcal{E}^{(s)}(\Omega)$ converges to zero in $\mathcal{E}^{(s)}(\Omega)$ ($s \geqq 1$) if for any compact subset K of Ω and for any $h > 0$ we have the above convergence.

(iii) We say that a sequence $\{f_j(x)\} \subset \mathcal{D}^{\{s\}}(\Omega)$ (resp. $\mathcal{D}^{(s)}(\Omega)$) converges to zero in $\mathcal{D}^{\{s\}}(\Omega)$(resp. in $\mathcal{D}^{(s)}(\Omega)$) for $s > 1$ if there is a compact subset K of Ω such that $\operatorname{supp} f_j \subset K$, $j = 1, 2, \ldots$ and $f_j \to 0$ in $\mathcal{E}^{\{s\}}(\Omega)$(resp. in $\mathcal{E}^{(s)}(\Omega)$).

We denote by $\mathcal{D}^{\{s\}\prime}(\Omega)$ ($\mathcal{D}^{(s)\prime}(\Omega)$, $\mathcal{E}^{\{s\}\prime}(\Omega)$, and $\mathcal{E}^{(s)\prime}(\Omega)$ respectively) the strong dual space of $\mathcal{D}^{\{s\}}(\Omega)$($\mathcal{D}^{(s)}(\Omega)$, $\mathcal{E}^{\{s\}}(\Omega)$, and $\mathcal{E}^{(s)}(\Omega)$ respectively) and call its elements ultradistributions on Ω. We have the inclusion

$$\mathcal{D}'(\Omega) \subset \mathcal{D}^{\{s\}\prime}(\Omega) \subset \mathcal{D}^{(s)\prime}(\Omega) \quad s > 1,$$

$$\mathcal{E}'(\Omega) \subset \mathcal{E}^{\{s\}\prime}(\Omega) \subset \mathcal{E}^{(s)\prime}(\Omega) \quad s > 1,$$

where $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ are the spaces of distributions. It is known that $\mathcal{E}^{\{s\}\prime}(\Omega)$ and $\mathcal{E}^{(s)\prime}(\Omega)$ consist of ultradistributions with compact support in Ω for $s \geqq 1$. Moreover, all the ultradistributions on Ω are in the space of hyperfunctions on Ω

$\mathcal{B}(\Omega)$, which was introduced by M. Sato. For further details we refer the reader to [13].

Definition 1.2. Let $u(x)$ be a hyperfunction on Ω with compact support. We define the Fourier transform $\mathcal{F}[u]$ of u by

$$\mathcal{F}[u](\xi) = \int_\Omega u(x)e^{-ix\xi}\,dx.$$

Let us give a well-known theorem about the estimate of $\mathcal{F}[u](\xi)$ with no proof.

Proposition 1.3. *Let $u(x)$ be a hyperfunction with compact support on Ω. Then $u(x)$ is in $\mathcal{D}^{*\prime}(\Omega)$(resp. in $\mathcal{D}^*(\Omega)$) if and only if*

(i) in case of $* = \{s\}$, for any positive ε there exists a positive number C_ε with

$$|\mathcal{F}[u](\xi)| \leqq C_\varepsilon \exp(\varepsilon|\xi|^{1/s})$$

(resp. there exist positive numbers L and C with

$$|\mathcal{F}[u](\xi)| \leqq C_\varepsilon \exp(-L|\xi|^{1/s})),$$

(ii) in case of $* = (s)$, there exist positive numbers L and C with

$$|\mathcal{F}[u](\xi)| \leqq C_\varepsilon \exp(L|\xi|^{1/s})$$

(resp. for any positive ε there exists a positive number C_ε with

$$|\mathcal{F}[u](\xi)| \leqq C_\varepsilon \exp(-\varepsilon|\xi|^{1/s})).$$

Refer to H. Komatsu [6] for more details. Note that we can also define these functions on a real analytic manifold. Let us introduce new function classes studied by S. Pilipović and others.

Definition 1.4. Let $u \in C^\infty(\mathbb{R}^n)$. Then we say that u is in $\mathcal{S}^{\{s\}}(\mathbb{R}^n)$ (the space of rapidly decreasing ultradifferentiable functions of class $\{s\}$) if for any α and any β there are positive constants C and h such that

$$\| < x >^\alpha \partial^\beta u(x)\|_r \leqq Ch^{|\alpha+\beta|}\alpha!^s\beta!^s$$

where $r \in [1,\infty]$, $s \geqq 1$, $< x >= (1+x^2)^{1/2}$, $\| \quad \|_r$ is the norm of $L^r(\mathbb{R}^n)$. Also we say that u is in $\mathcal{S}^{(s)}(\mathbb{R}^n)$ (the space of rapidly decreasing ultradifferentiable functions of class (s)) if for any positive h there is a constant C such that the above inequality holds. The topologies of the spaces $\mathcal{S}^{\{s\}}(\mathbb{R}^n)$ and $\mathcal{S}^{(s)}(\mathbb{R}^n)$ are defined as follows:

(i) We say that a sequence $\{f_j(x)\} \subset \mathcal{S}^{\{s\}}(\mathbb{R}^n)$ converges to 0 in $\mathcal{S}^{\{s\}}(\mathbb{R}^n)$ if there is a constant $h > 0$ such that

$$\sup\{\| < x >^{\alpha} \partial^{\beta} u(x)\|_r/(h^{|\alpha+\beta|}\alpha!^s\beta!^s); \alpha,\ \beta\} \to 0 \text{ as } j \to \infty.$$

(ii)We say that a sequence $\{f_j(x)\} \subset \mathcal{S}^{(s)}(\mathbb{R}^n)$ converges to 0 in $\mathcal{S}^{(s)}(\mathbb{R}^n)$ if for any $h > 0$ we have the above convergence.

We denote by $\mathcal{S}^{\{s\}\prime}(\mathbb{R}^n)$, $\mathcal{S}^{(s)\prime}(\mathbb{R}^n)$ the strong dual spaces of $\mathcal{S}^{\{s\}}(\mathbb{R}^n)$, $\mathcal{S}^{(s)}(\mathbb{R}^n)$ respectively and call their elements tempered ultradistributions. They can be also called Fourier ultradistributions. It is known that $\mathcal{S}^{\{s\}}(\mathbb{R}^n)$, $\mathcal{S}^{(s)\prime}(\mathbb{R}^n)$ are LN spaces and that $\mathcal{S}^{(s)}(\mathbb{R}^n)$, $\mathcal{S}^{\{s\}\prime}(\mathbb{R}^n)$ are FN spaces. We refer the readers to [8] for more details.

2. Estimates of integrals

Let us prepare some estimates of the integrals which will serve to evaluate kernels. We quote some propositions from G. Lebeau [9] and Melin-Sjöstrand [10].

Definition 2.1. We assume that

$$V_{\eta,a} = \{z \in \mathbb{C}; Rez > a \geqq 0, |Imz| < \eta Rez\}.$$

Let M and M' be real analytic manifolds. We denote by $SR^d(M, M')$ the space of the complex valued analytic functions $f(x, y, \lambda)$ of $M \times M' \times \mathbb{R}_+$ such that for any compact $K \subset M$, $K' \subset M'$ there exist a complex neighborhood K_ε of K, a real neighborhood U of K', constants η, a, C_0, C_1, and analytic functions $a_k(x, y)$ defined on $K_\varepsilon \times U$ such that

(1) f can be prolonged to $K_\varepsilon \times U \times V_{\eta,a}$,

(2) $|f(x, y, \lambda) - \sum_{0\leqq k\leqq N-1} a_k(x, y)\lambda^{d-k}| \leqq C_0 C_1^N N!|\lambda|^{d-N}$ for $\forall N \in \mathbb{N}$ and $\forall(x, y, \lambda) \in K_\varepsilon \times U \times V_{\eta,a}$.

We denote by $A(f) = \sum_{k\geqq 0} a_k(x, y)\lambda^{d-k}$ the asymptotic expansion of f and by $\sigma(f) = a_0(x, y)$ its principal term. If $d\mu(y)$ is a measure with compact support in M' and $f \in SR^d(M, M')$, then

$$\int f(x, y, \lambda)\ d\mu(y) \in SR^d(M, \{pt\})$$

and its asymptotic expansion can be obtained by integration. Here we denote the set of one point by $\{pt\}$ to fix a variable. When X is a complex analytic manifold, we can define $SR^d(X, M)$ in the same way.

Let M be a compact analytic Riemannian manifold. We assume that the manifold is oriented by 1-volume form dx.

Definition 2.2. A function ϕ is called transversally elliptic on a submanifold N if $d\phi = 0$ on N and $rank[Hess\phi] = codimN$ at the points of N.

Proposition 2.3. *Let ϕ be a complex valued analytic function on M such that $Im\phi \geqq 0$. We suppose that*

$$N = \{x \in M; d\phi = 0, Im\phi(x) = 0\}$$

is a connected smooth submanifold with codim$N = q$ *and that* ϕ *is transversally elliptic on* N*. For any* $a \in SR^d(M, \{pt\})$*, put*

$$I_\phi(a) = \int_M e^{i\lambda\phi(x)} a(x, \lambda)\ dx,$$

then $I_\phi(a) = e^{i\lambda c} b(\lambda)$ *where* $b \in SR^{d-q/2}(\{pt\}, \{pt\})$ *and* $c \in \mathbb{R}$ *is the critical value of* ϕ *on* N*.*

Corollary 2.4. *Let* $f : M \longrightarrow [0, 1]$ *be an analytic function which is equal to 1 on* N *and which is transversally elliptic on* N*. Then* $\int_M f^\lambda\ dx$ *is in* $SR^{-q/2}(\{pt\}, \{pt\})$*.*

We next prepare an estimate of another important integral introduced by L. Hörmander [5].

Proposition 2.5. *Let us assume that*

$$I(\xi) = \int_{S^{n-1}} e^{\omega\xi}\ d\omega, \quad \xi \in \mathbb{R}^n.$$

Then we have the estimate of $I(\xi) = I_0(|\xi|)$ *as follows.*

$$I_0(\rho) = (2\pi)^{(n-1)/2} e^\rho \rho^{-(n-1)/2}(1 + O(1/\rho)) \text{ if } \rho \to \infty.$$

Refer to [5] for the precise estimation.

3. Geometry of the spaces

We quote here some results from [9]. We denote by S^n the unit sphere of $\mathbb{R}^{n+1}$, i.e.

$$S^n = \{x \in \mathbb{R}^{n+1}; x^2 = 1\}$$

and by Γ the isotropic complex cone in $\mathbb{R}^{n+1}$:

$$\Gamma = \{z \in \mathbb{C}^{n+1}; z^2 = 0\} = \{z = u + iv; u^2 - v^2 = 0, uv = 0\}.$$

Let ρ be the function on Γ defined by

$$\rho(z) = |z|^2 = u^2 + v^2$$

and DS^n be the strictly pseudoconvex open set of Γ:

$$DS^n = \{z \in \Gamma; \rho(z) < 1/2\}.$$

We have

$$\partial DS^n = \{z \in \Gamma; \rho(z) = 1/2\} = \{u + iv; u^2 = v^2 = 1/4, uv = 0\},$$

so that ∂DS^n can be identified to S^*S^n, the cospherical bundle of S^n. We complexify S^*S^n in the following way:

$$X = \{(z_1, z_2) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}; z_1^2 = 0, z_2^2 = 0, z_1 z_2 = 1/2\}$$

so that the real points of X, i.e. S^*S^n, can be given by $z_2 = z_1^c$. Here z^c means the complex conjugate of z. We also consider the ε neighborhood:

$$X_\varepsilon = \{(z_1, z_2) \in X; |z_1 - z_2^c| < \varepsilon\}.$$

We always provide S^n and S^*S^n with measures inherited from those of $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that

$$\int_{S^n} 1\ dx = \omega_{n+1}, \quad \int_{S^*S^n} 1\ dxdy = \omega_{n+1}\omega_n/2^{2n-1}$$

where ω_n denotes the volume of the unit sphere in $\mathbb{R}^n$. The group $SO(n+1)$ operates transitively on S^n and S^*S^n. The group $U(1)$ operates on S^*S^n by

$$(e^{i\theta}, z) \longrightarrow e^{i\theta} z.$$

The measures are invariant by the actions of these groups.

Note that we identify $S^*\mathbb{R}^n$ to $\{z = u + iv \in \mathbb{C}^n; |v| = 1\}$ or $\mathbb{R}^n \times S^{n-1}$

Definition 3.1. We denote by $H(S^*S^n)$ the set of the hyperfunctional boundary values to $S^*S^n = \partial DS^n$ of the holomorphic functions on DS^n.

Definition 3.2. We denote by $H(S^*\mathbb{R}^n)$ the set of the hyperfunctional boundary values to $S^*\mathbb{R}^n$ of the holomorphic functions on $D\mathbb{R}^n = \{z \in \mathbb{C}^n; |Imz| < 1\}$.

4. The decomposition into irreducible subspaces

We divide this section into two parts for the difference of base spaces. We quote some statements of [9] especially in case of S^n.

(I) In case of S^n

The representation T of $SO(n+1)$ on $L^2(S^n)$ can be defined as follows:

$$T_g f(x) = f(g^{-1}x),$$

for $g \in SO(n+1)$, $x \in S^n$, $f \in L^2(S^n)$. Since the rotation does not change the volume, T is a unitary represention of $SO(n+1)$. In order to decompose T into the irreducible representations, we have to introduce a spherical Laplacian Λ. We know that the eigenvalues of Λ are

$$\mu_k = k(k+n-1), \quad k = 0, 1, 2, \ldots$$

and that their eigenspaces H_k, $k = 0, 1, 2, \ldots$ consist of k-th degree homogeneous harmonic polynomials. Then we have

$$L^2(S^n) = \bigoplus_{k \geqq 0} H_k,$$

$$\nu_k = \dim H_k = N(n+1, k) = (2k+n-1)!(k+n-2)!/(k!(n-1)!).$$

We denote by $S_{k,j}$ an orthonormal base of H_k for $1 \leqq j \leqq \nu_k$. One of the eigenfunctions in H_k is $f_k(x) = (x_1 + ix_2)^k$.

Then we have by Corollary 2.4

$$\|f_k\|^2 = \int_{S^n} (x_1^2 + x_2^2)^k dx \sim Ck^{-((n-1)/2)}$$

because the function $x \to x_1^2 + x_2^2$ is maximal on S^n for $x_1^2 + x_2^2 = 1$, which is of codimension $n-1$. We denote by $S_{k,j}(z)$ the same function restricted on S^*S^n. Let H'_k be the subspace of $L^2(S^*S^n)$ generated by the $S_{k,j}(z)$'s for $1 \leqq j \leqq \nu_k$. The subspaces H'_k's are orthogonal to each other. We can choose the base $S_{k,j}(x)$ of H_k so that the corresponding base $S_{k,j}(z)$ of H'_k may be orthogonal by Schur's lemma. We next have

$$\|S_{k,j}(z)\|_{L^2(S^*S^n)} = 1/\lambda(k)$$

where $\lambda(z)$ is the holomorphic function for $Re z > 0$:

$$\lambda(z) = \left(\int_{S^n} (x_1^2 + x_2^2)^z \, dx\right) / \left(\int_{S^*S^n} (|z_1 + iz_2|^2)^z \, dxdy\right).$$

It follows from Corollary 2.4 that

$$\lambda(k) \sim Ck^{(n-1)/2}.$$

Note that by Poisson's formula, we have

$$(1-z^2)/(\omega_n\{(z-x)^2\}^{(n+1)/2}) = \sum_{j,k} S_{k,j}(z) S_{k,j}(x)^c,$$

$$\sum_{1 \leqq j \leqq \nu_k} S_{k,j}(z) S_{k,j}(x)^c = \lambda_k^0 (2zx)^k,$$

$$\lambda_k^0 \sim Ck^{(n-1)/2}.$$

Now we define some Sobolev spaces.

Definition 4.1. Let $u \in \mathcal{D}'(S^n)$. Then we say that u is in $W^s(S^n)$ (the space of Sobolev functions on S^n of degree s) if for the spherical harmonic decomposition:

$$u(\omega) = \sum_{k,j} a_{k,j} S_{k,j}(\omega), \quad \omega \in S^n$$

the following holds:

$$\sum_{k,j} |a_{k,j}|^2 \{k(k+n-1)\}^s < \infty.$$

Definition 4.2. Let $U \in \mathcal{D}'(S^*S^n)$. Then we say that U is in $W^s(S^*S^n)$ (the space of Sobolev functions on S^*S^n of degree s) if for the spherical harmonic decomposition:

$$U(u+iv) = \sum_{k,j,l,m} A_{k,j,l,m} S_{k,j}(u) S'_{l,m}(v)$$

the next holds:

$$\sum_{k,j,l,m} |A_{k,j,l,m}|^2 \{k(k+n-1) + l(l+n-2)\}^s < \infty.$$

(II) In case of $\mathbb{R}^n$

Let $x \in \mathbb{R}^n$. Let S_x be the linear operator on $L^2(\mathbb{R}^n)$ defined by

$$(S_x \phi)(\xi) = e^{ix\xi} \phi(\xi).$$

S is a unitary representation of $\mathbb{R}^n$. Let $X_\xi(x) = e^{ix\xi}$. Then S can be decomposed by a direct integral

$$S = \int_{\mathbb{R}^n} \bigoplus X_\xi \, d\xi.$$

So we need to study $e^{ix\xi}$. Here we introduce a Poisson kernel:

$$P(x) = 2/\{\omega_{n+1}(x^2+1)^{(n+1)/2}\}.$$

Since we have

$$\int_{\mathbb{R}^n} P(z-x) e^{ix\xi} \, dx = e^{-|\xi|} e^{iz\xi},$$

the correspondence $e^{ix\xi} \longrightarrow e^{-|\xi|} e^{iz\xi}$ is induced by the kernel. Note that $S^*\mathbb{R}^n \cong \mathbb{R}^n \times S^{n-1}$ and $z = u + iv \in \mathbb{R}^n \times S^{n-1}$ $(u, v \in \mathbb{R}^n, \quad |v| = 1)$.

Now we define a Sobolev space on $S^*\mathbb{R}^n$.

Definition 4.3. Let $f \in \mathcal{D}'(S^*\mathbb{R}^n)$. Then we say that f is in $W^s(S^*\mathbb{R}^n)$ (the space of Sobolev functions on $S^*\mathbb{R}^n$ of degree s) if for the spherical harmonic decomposition:

$$f(x+i\omega) = \int_{\mathbb{R}^n} \sum_{k,j} a_{k,j}(\xi) S_{k,j}(\omega) e^{ix\xi}\ d\xi$$

the following holds:

$$\int_{\mathbb{R}^n} \sum_{k,j} |a_{k,j}(\xi)|^2 \{|\xi|^2 + k(k+n-2)\}^s\ d\xi < \infty.$$

Definition 4.4. We denote $\mathcal{S}^*(S^*\mathbb{R}^n)$, $\mathcal{S}^{*\prime}(S^*\mathbb{R}^n)$ by the topological completion of the tensor products $\mathcal{S}^*(\mathbb{R}^n) \otimes \mathcal{E}^*(S^{n-1})$ in $\mathcal{E}^*(S^*\mathbb{R}^n)$, $\mathcal{S}^{*\prime}(\mathbb{R}^n) \otimes \mathcal{D}^{*\prime}(S^{n-1})$ in $\mathcal{D}^{*\prime}(S^*\mathbb{R}^n)$ respectively.

5. Poisson operators and Hörmander operators

Now we can construct the operators as follows.

(I) In case of S^n

Let $f(z) = \sum_{k \geqq 0} \lambda_k^0 z^k$ be the Poisson kernel and u be an ultradistribution of class $*$ (resp. an ultradifferentiable function of class $*$) on the sphere S^n. Then

$$h(z) = \int_{S^n} f(2zx) u(x)\ dx$$

is holomorphic for $|z|^2 < 1/2$ on the cone $z^2 = 0$. Indeed

$$|2zx|^2 = 4\{(ux)^2 + (vx)^2\} < 1$$

considering $uv = 0$ and $u^2 = v^2 < 1/4$. Therefore we define the Poisson operator P:

$$P(u) = bv[\int_{S^n} f(2zx)u(x)\ dx].$$

Then

$$P : \mathcal{D}^{*\prime}(S^n) \longrightarrow \mathcal{D}^{*\prime}(S^*S^n)$$
$$(resp. \quad P : \mathcal{E}^*(S^n) \longrightarrow \mathcal{E}^*(S^*S^n)).$$

Note that

$$P(S_{k,j}(x)) = S_{k,j}(z).$$

When we take $f_1(z) = \sum_{k\geqq 0} \lambda_k^0 c_{k-1} z^k$ as the above f, we define the Hörmander operator H:

$$H(u) = bv[\int_{S^n} f_1(2zx)u(x)\ dx].$$

Note that

$$H(S_{k,j}(x)) = c_{k-1} S_{k,j}(z).$$

Then

$$H : \mathcal{D}^{*\prime}(S^n) \longrightarrow \mathcal{D}^{*\prime}(S^* S^n)$$
$$(resp.\quad H : \mathcal{E}^*(S^n) \longrightarrow \mathcal{E}^*(S^* S^n)).$$

Next we construct the inverse Poisson operator. The function

$$g(2z_2 x) = \sum \lambda_k^0 \lambda(k)(2z_2 x)^k$$

is holomorphic on X_ε for $x \in \mathbb{R}^{n+1}$, $x^2 < 1/(1+2\varepsilon)$; indeed on X_ε we have

$$1/2 = z_1 z_2 = |z_2|^2 + z_2(z_1 - z_2^c),$$

then

$$|z_2|^2 \leqq 1/2 + |z_2|\varepsilon \leqq 1/2 + \varepsilon,$$

therefore

$$|2z_2 x|^2 \leqq 2|x|^2 |z_2|^2 \leqq (1+2\varepsilon)x^2.$$

If U is an ultradistribution of class $*$ (resp. an ultradifferentiable function of class $*$) on $S^* S^n$,

$$h(x) = U(g(2z_2 x))$$

is defined for $x^2 < 1$ and h is harmonic since

$$\Delta_x g(2z_2 x) = 4z_2^2 g''(2z_2 x) = 0 \quad (z_2^2 = 0 \quad \text{on } X).$$

Then we put

$$P'(U) = bv(h).$$

Therefore

$$P' : \mathcal{D}^{*\prime}(S^* S^n) \longrightarrow \mathcal{D}^{*\prime}(S^n)$$
$$(resp.\quad P' : \mathcal{E}^*(S^* S^n) \longrightarrow \mathcal{E}^*(S^n)).$$

Note that

$$P'(S_{k,j}(z)) = S_{k,j}(x).$$

P' is the inverse of P and we have

$$P' \circ P = Id, \quad P \circ P' = \pi$$

where π is the projection : $L^2(S^*S^n) \longrightarrow H'_k$.

When we take $g_1(z) = \sum_{k \geqq 0} \lambda_k^0 \lambda(k) c_k z^k$ as the above g, we can define the inverse Hörmander operator H'. This operator is the integration on the fiber of S^*S^n. Indeed,

$$H'(u) = \int_{|v|=1} U(u+iv)\ dv.$$

Therefore

$$H' : \mathcal{D}^{*\prime}(S^*S^n) \longrightarrow \mathcal{D}^{*\prime}(S^n)$$
$$(resp. \quad H' : \mathcal{E}^*(S^*S^n) \longrightarrow \mathcal{E}^*(S^n)).$$

Note that

$$H'(S_{k,j}(z)) = c_k S_{k,j}(z).$$

Hereafter $K = P$ or H, $K' = P'$ or H'.

Now we have the following two propositions which characterize the singular spectrum SS introduced in [13] and SS_* introduced in [7].

Proposition 5.1. *Let u be a hyperfunction on S^n. If $(x_0, \xi_0) \notin SS(u)$ (resp. $SS_*(u)$), K(u) is analytic (resp. $*$-ultradifferentiable) in the neighborhood of the point $z_0 = u_0 + iv_0 \in S^*S^n$, $u_0 = x_0/2$, $v_0 = (-1/2)(\xi_0/|\xi|)$.*

Proof. In case of SS, the proof was shown by G. Lebeau [9]. In the other cases this proposition is a direct consequence of the next one. But the analytic case is also by [9]. □

Proposition 5.2. *Let u be a hyperfunction on S^n. $P(u)$ is $*$-ultradifferentiable (analytic) in the neighborhood of $z_0 \in S^*S^n$ if and only if $H(u)$ is so.*

Proof. Suppose that $P(u)$ is $*$-ultradifferentiable in the neighborhood of z_0. Take $\psi \in \mathcal{E}^*(S^*S^n)$ which is 1 in a sufficiently small neighborhood of z_0 and which is otherwise 0. Then $\psi P(u) \in \mathcal{E}^*(S^*S^n)$. Let $u_1 = P'\psi P(u)$. Then $Hu_1 \in \mathcal{E}^*(S^*S^n)$

$$Hu - Hu_1 = HP'P(u) - HP'\psi P(u) = HP'(1-\psi)P(u)$$

which is analytic in the neighborhood of z_0. Therefore Hu is $*$-ultradifferentiable in the neighborhood of z_0. We can show the reverse in the same way. □

Proposition 5.3. *Put* $\sigma : T^*S^n \setminus 0 \longrightarrow S^*S^n$, $\sigma(x_0, \xi_0) = x_0/2 - i/2(\xi_0/|\xi_0|)$. *Let* $q \in T^*S^n \setminus 0$ *and* U *be a hyperfunction on* S^*S^n. *If* u *is analytic (resp.* $*$*-ultradifferentiable) in the neighborhood of* $\sigma(q)$, *we have* $q \notin SS(K'(U))$ *iresp.* $SS_*(K'(U))$*).*

Proof. In case of SS, the proof was shown by G. Lebeau [9]. Suppose that U is $*$-ultradifferentiable in the neighborhood of $\sigma(q)$. Take ψ as in the proof of Proposition 5.2 replacing z_0 by $\sigma(q)$. Then $\psi U \in \mathcal{E}^*(S^*S^n)$, $K'(\psi U) \in \mathcal{E}^*(S^n)$,

$$K'(U) - K'(\psi U) = K'((1 - \psi)U),$$

whose SS does not include q. Therefore $q \notin SS_*(K'(U))$. □

From the above propositions the following holds.

Proposition 5.4. *We have the linear isomorphisms:*

$$\begin{aligned} K : \mathcal{D}^{*\prime}(S^n) &\longrightarrow \mathcal{D}^{*\prime}(S^*S^n) \cap H(S^*S^n), \\ K : \mathcal{E}^*(S^n) &\longrightarrow \mathcal{E}^*(S^*S^n) \cap H(S^*S^n). \end{aligned}$$

Proposition 5.5. *We have the following isomorphisms:*

$$\begin{aligned} P : W^s(S^n) &\longrightarrow W^{s+(n-1)/4}(S^*S^n) \cap H(S^*S^n), \\ H : W^s(S^n) &\longrightarrow W^{s-(n-1)/4}(S^*S^n) \cap H(S^*S^n). \end{aligned}$$

Note here that G. Lebeau [9] showed the first isomorphism P.

Definition 5.6. Let M be a real analytic manifold. We denote by $\mathcal{C}_M$ the sheaf of microfunctions on M. We also denote by $\mathcal{C}^*_M$, $\mathcal{C}^{d,*}_M$ the subsheaves of $\mathcal{C}_M$ defined in [4], [7].

The flabbiness of $\mathcal{C}_M$ was first shown by M. Kashiwara.

Proposition 5.7. *The sheaves* $\mathcal{C}^*_{S^n}$, $\mathcal{C}^{d,*}_{S^n}$ *are soft. The sheaf* $\mathcal{C}_{S^n}$ *is flabby.*

Proof. We have only to show the case of $\mathcal{C}^*_{S^n}$, since the other cases are shown in the same way. However the proof is a direct consequence of the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D}^{*\prime}(S^n) & \xrightarrow{K} & \mathcal{D}^{*\prime}(S^*S^n) \cap H(S^*S^n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle sp} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{C}^*(S^*S^n) & \longrightarrow & \mathcal{C}^*(S^*S^n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0. & & \end{array}$$

□

(II) In case of $\mathbb{R}^n$

Let $f(z) = \int_{\mathbb{R}^n} e^{-|\xi|} e^{iz\xi}\ d\xi$ be Poisson kernel. Let u be a tempered ultra-distribution of class $*$ (resp. a rapidly decreasing ultradifferentiable function of class $*$). Then

$$h(z) = \int_{\mathbb{R}^n} f(z-x)u(x)\ dx$$

is holomorphic for $\{z \in \mathbb{C}^n; |Imz| < 1\}$. We define on $S^*\mathbb{R}^n$

$$P(u) = bv[\int f(z-x)u(x)\ dx].$$

Then

$$P : \mathcal{S}^{*\prime}(\mathbb{R}^n) \longrightarrow \mathcal{S}^{*\prime}(S^*\mathbb{R}^n)$$
$$(resp. \quad P : \mathcal{S}^{*}(\mathbb{R}^n) \longrightarrow \mathcal{S}^{*}(S^*\mathbb{R}^n)).$$

Note that

$$P(e^{ix\xi}) = e^{-|\xi|}e^{iz\xi}.$$

Next we construct the inverse Poisson operator. If u is a tempered ultra-distribution of class $*$ iresp. a rapidly decreasing ultradifferentiable function of class $*$j on $S^*\mathbb{R}^n$, we define

$$P'(u) = \int_{S^{n-1}} u(z)\ d\omega * \int_{\mathbb{R}^n} g(\xi)e^{ix\xi}\ d\xi, \quad z = x - i\omega$$

where $g(\xi) = e^{|\xi|}/I(\xi)$.

Therefore

$$P' : \mathcal{S}^{*\prime}(S^*\mathbb{R}^n) \longrightarrow \mathcal{S}^{*\prime}(\mathbb{R}^n)$$
$$(resp. \quad P' : \mathcal{S}^{*}(S^*\mathbb{R}^n) \longrightarrow S^{*}(\mathbb{R}^n)).$$

Note that

$$P'(e^{-|\xi|}e^{iz\xi}) = e^{i\xi x}.$$

Then P' is the inverse of P and we have

$$P' \circ P = Id, \quad P \circ P' = \pi,$$

where π is the projection

$$\mathcal{S}^{*\prime}(S^*\mathbb{R}^n) \longrightarrow \mathcal{S}^{*\prime}(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n)$$
$$(resp. \quad \mathcal{S}^{*}(S^*\mathbb{R}^n) \longrightarrow \mathcal{S}^{*}(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n)).$$

Suppose that $u \in W^s(\mathbb{R}^n)$. We have

$$P(F[u](\xi)e^{ix\xi}) = F[u](\xi)e^{-|\xi|}e^{iz\xi} = F[u](\xi)e^{-|\xi|}e^{ix\xi}e^{\omega\xi}.$$

Considering the spherical harmonic decomposition:

$$e^{\omega\xi} = \sum_{k,j} a_{k,j}(\xi)S_{k,j}(\omega),$$

we check the convergence condition of the integral

$$\int_{\mathbb{R}^n} F[u](\xi)^2 e^{-2|\xi|} \sum_{k,j} |a_{k,j}(\xi)|^2[|\xi|^{2t} + k(k+n-2)^t]\ d\xi.$$

By Proposition 2.5,

$$\sum_{k,j} |a_{k,j}(\xi)|^2 = \int_{|\omega|=1} e^{2\omega\xi}\ d\xi = I(2\xi) = Ce^{2|\xi|}|\xi|^{-(n-1)/2}(1 + O(1/|\xi|)).$$

Then the convergence condition of the ξ-integration is

$$t \leqq s + (n-1)/4.$$

Conversely, suppose that $g \in W^s(S^*\mathbb{R}^n)$ and that

$$g(x - i\omega) = \int_{\mathbb{R}^n} a(\xi)e^{ix\xi}e^{\omega\xi}\ d\xi.$$

We check the convergence condition of the integral

$$\int_{\mathbb{R}^n} |a(\xi)|^2 e^{2\omega\xi}|\xi|^{2t}\ d\xi.$$

Considering again Proposition 2.5

$$I(2\xi) = Ce^{2|\xi|}|\xi|^{-(n-1)/2}(1 + O(1/|\xi|)),$$

the following holds:

$$t \leqq s - (n-1)/4.$$

Therefore we have the following for the Sobolev spaces.

Proposition 5.8. *We have the linear isomorphisms:*

$$P : W^s(\mathbb{R}^n) \longrightarrow W^{s+(n-1)/4}(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n),$$
$$P' : W^s(S^*\mathbb{R}^n) \longrightarrow W^{s-(n-1)/4}(\mathbb{R}^n).$$

Proof. The injectivity of P is evident. The surjectivity is based on the characterization of holomorphic functions in a tubular domain studied by R. D. Carmichael and E. K. Hayashi [3], and V. S. Vladimirov [16]. □

If we take the Hörmander kernel $H(z) = \int_{\mathbb{R}^n} I(\xi)^{-1} e^{iz\xi}\, d\xi$ as the above P, we can define the Hörmander operator

$$H(u) = bv[\int_{\mathbb{R}^n} H(z-x)u(x)\, dx].$$

For the inverse H' of the operator, we have only to integrate on the fiber:

$$H'(u) = \int_{|v|=1} U(u+iv)\, dv.$$

Then by the same argument we have the following two propositions.

Proposition 5.9. *We have the linear isomorphisms:*

$$H : W^s(\mathbb{R}^n) \longrightarrow W^{s-(n-1)/4}(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n),$$

$$H' : W^s(S^*\mathbb{R}^n) \longrightarrow W^{s+(n-1)/4}(\mathbb{R}^n).$$

Hereafter $K = P$ or H, $K' = P'$ or H'.

Proposition 5.10. *We have the linear isomorphisms:*

$$K : \mathcal{S}^*(\mathbb{R}^n) \longrightarrow \mathcal{S}^*(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n),$$

$$K : \mathcal{S}^{*\prime}(\mathbb{R}^n) \longrightarrow \mathcal{S}^{*\prime}(S^*\mathbb{R}^n) \cap H(S^*\mathbb{R}^n).$$

We can characterize the singularity as in the case of S^n.

Proposition 5.11. *If $(x_0, \xi_0) \notin SS(u)$ (resp. $SS_*(u)$) for a tempered ultradistribution u on $\mathbb{R}^n$, $K(u)$ is analytic (resp. $*$-ultradifferentiable) in the neighborhood of $z_0 = x_0 - i\xi_0$ of $S^*\mathbb{R}^n$.*

Proposition 5.12. *Put $\sigma : T^*\mathbb{R}^n \setminus 0 \longrightarrow S^*\mathbb{R}^n$, $\sigma(x_0, \xi_0) = x_0 - i(\xi_0/|\xi_0|)$. Let $q \in T^*\mathbb{R}^n \setminus 0$ and U be a tempered ultradistribution on $S^*\mathbb{R}^n$. If U is analytic (resp. $*$-ultradifferentiable) in the neighborhood of $\sigma(q)$, we have $q \notin SS(K'(U))$ (resp. $SS_*(K'(U))$).*

Proposition 5.11 was proved by S. Pilipović in case of the Hörmander operator. Proposition 5.12 is an easy exercise of the theory of integration in case of the inverse Hörmander operator. We can show the other cases by the following proposition.

Proposition 5.13. *Let u be a tempered ultradistribution on $\mathbb{R}^n$. Pu is analytic (resp. $*$-ultradifferentiable) in the neighborhood of $z_0 = x_0 - i\xi_0$ if and only if Hu is so in the neighborhood of the same point.*

Proof.

$$Pu = Hu * \int I(\xi)e^{-|\xi|}e^{ix\xi}\ d\xi,$$

$$Hu = Pu * \int I(\xi)^{-1}e^{|\xi|}e^{ix\xi}\ d\xi.$$

The functions $I(\xi)e^{-|\xi|}$, $I(\xi)^{-1}e^{|\xi|}$ have no zero points on $\mathbb{R}^n$ and they are temperately increasing. So the second terms of the right sides are tempered distributions. Using the microlocal theory of convolutions, we obtain the proposition. □

Finally, we have reached the next point.

Proposition 5.14. *The sheaves $\mathcal{C}^*_{\mathbb{R}^n}$, $\mathcal{C}^{d,*}_{\mathbb{R}^n}$ are soft. The sheaf $\mathcal{C}_{\mathbb{R}^n}$ is flabby.*

Proof. We can show the case of $\mathcal{C}^*_{\mathbb{R}^n}$ by the following commutative diagram and the following propositions. The other cases are shown in the same way.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}^{*\prime}(\mathbb{R}^n) & \xrightarrow{K} & \mathcal{S}^{*\prime}(S^*\mathbb{R}^n)\cap H(S^*\mathbb{R}^n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle sp} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{C}^*(S^*\mathbb{R}^n) & \longrightarrow & \mathcal{C}^*(S^*\mathbb{R}^n) & \longrightarrow & 0. \end{array}$$

□

Proposition 5.15. *For any $f \in \mathcal{C}^*(S^*\mathbb{R}^n)$ (resp. $\mathcal{C}^{d,*}(S^*\mathbb{R}^n)$, $\mathcal{C}(S^*\mathbb{R}^n)$) with compact support, there exists $F \in \mathcal{S}^{*\prime}(\mathbb{R}^n)$ (resp. $\mathcal{S}^*(\mathbb{R}^n)$, $\mathcal{S}^{\{1\}\prime}(\mathbb{R}^n)$) such that $sp(F) = f$.*

Proof. We show the case of $\mathcal{C}^*(S^*\mathbb{R}^n)$, since the other cases are the same. For $f \in \mathcal{C}^*(S^*\mathbb{R}^n)$ with compact support, there exist $F_1 \in \mathcal{D}^{*\prime}(\mathbb{R}^n)$ and a compact set $K \subset \mathbb{R}^n$ such that $\operatorname{singsupp} F_1 \subset K$. Take a function $G \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{singsupp} G \subset K$, $G = F_1$ outside K. Then by the next lemma of H. Whitney we can find an analytic function $G_1 \in \mathcal{A}(\mathbb{R}^n)$ as close to G as we want and such that $F_1 - G_1 \in \mathcal{S}^{*\prime}(\mathbb{R}^n)$, which is the desired function F. □

Lemma 5.16. *Let $f \in C^k(\mathbb{R}^n)$, $0 \leqq k \leqq \infty$. Let η be a continuous function on $\mathbb{R}^n$ with $\eta(x) > 0$ for any $x \in \mathbb{R}^n$. Then there exists a real analytic function g on $\mathbb{R}^n$ such that we have*

$$|\partial^\alpha f(x) - \partial^\alpha g(x)| < \eta(x) \quad \textit{for } 0 \leqq |\alpha| \leqq \min\{k, 1/\eta(x)\}.$$

This is known as Whitney's approximation theorem.

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